

Efficient IP Masking with Generic Security Guarantees under Minimum Assumptions

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Abstract. Leakage-resilient secret sharing schemes are a fundamental building block for secure computation in the presence of leakage. As a result, there is a strong interest in building secret sharing schemes that combine resilience in practical leakage scenarios with potential for efficient computation. In this work, we revisit the inner-product framework, where a secret y is encoded by two vectors $(\vec{\omega}, \vec{y})$, such that their inner product is equal to y . So far, the most efficient inner-product masking schemes (in which $\vec{\omega}$ is public but random) are provably secure with the same security notions (*i.e.*, in the abstract *probing* model) as additive, Boolean masking, yet at the cost of a slightly more expensive implementation. Hence, their advantage in terms of theoretical security guarantees remains unclear, also raising doubts about their practical relevance. We address this question by showing the leakage resilience of inner-product masking schemes, in the *bounded leakage* threat model. It depicts well implementation contexts where the physical noise is negligible. In this threat model, we show that if m bits are leaked from the d shares \vec{y} of the encoding over an n -bit field, then, with probability at least $1 - 2^{-\lambda}$ over the choice of $\vec{\omega}$, the scheme is $\mathcal{O}\left(\sqrt{2^{-(d-1)\cdot n+m+2\lambda}}\right)$ -leakage resilient. Furthermore, this result holds without assuming independent leakage from the shares, which may be challenging to enforce in practice. We additionally show that in large Mersenne-prime fields, a wise choice of the public coefficients $\vec{\omega}$ can yield leakage resilience up to $\mathcal{O}(n \cdot 2^{-d\cdot n+n+d})$, in the case where one physical bit from each share is revealed to the adversary. The exponential rate of the leakage resilience we put forward significantly improves upon previous bounds in additive masking, where the past literature exhibited a constant exponential rate only.

1 Introduction

Masking is a popular countermeasure against side-channel attacks which works by performing computations over secret-shared data. Various types of encodings can be used for this purpose (*e.g.*, additive [26], multiplicative [25], affine [24], *...*). As a result, research on masked implementations can, at least theoretically, be viewed as a quest towards encodings (and addition/multiplication algorithms)

that provide the best trade-off between physical security and efficiency. Nevertheless, and somewhat biased by standardization and technological constraints, the simplicity of Boolean masking has so far largely dominated the practical scene for at least two reasons. On the one hand, most current algorithms for symmetric encryption operate in binary fields and have efficient bit-level representations. This was already the case of the AES Rijndael [15], and it has been amplified with the recently selected Ascon cipher [16]. Such algorithms naturally favor simple Boolean encodings which became a de facto standard for their masked implementations [30,28,13,29,9,35]. On the other hand, whether an operation is efficient in software depends on the instructions' sets. Here as well, readily available Boolean operations favor bit-level implementations [27].

Over the years, it however also appeared that the secure implementation of Boolean masking in software and hardware is a highly non-trivial task. Theoretically, this is because the security guarantees of Boolean masking, as for example formalized in [49,18,20], strongly rely on the assumptions that (i) the shares' leakages are (sufficiently) independent and (ii) the shares' leakages are (sufficiently) noisy. Concretely, this is because none of these assumptions is easy to fulfill. The independence condition is typically contradicted by physical defaults like *glitches* that can be observed in hardware implementations [43,44] and *transitions* that can be observed in software implementations [14,3]. Dealing with such physical defaults imposes additional constraints on masked implementations [46,22], which in turn implies additional overheads.¹ As for the noise condition, it is typically not fulfilled in lightweight embedded devices [4,8], and there are limited algorithmic approaches to remedy this lack so far.²

Interestingly, the sensitivity to the independence and noise assumptions is actually not disconnected from the choice of the encodings. As far as the independence condition is concerned, it has been observed heuristically that so-called Inner Product (IP) encodings, where the secrets are written as the inner product $\langle \vec{\omega}, \vec{y} \rangle$ between shares' vectors $\vec{\omega}$ and \vec{y} , lead to substantially lower transition leakages [1]. As far as the noise condition is concerned, there is a growing amount of research considering so-called "noise-free" leakages, where it is only required that the leakage function is non-injective. A seed result in this direction is the Asiacrypt 2011 one of Dziembowski and Faust [21], who showed that IP masking can provide strong guarantees in a liberal setting where the leakage function is generic (i.e., can correspond to arbitrary bounded functions), adaptive (i.e., chosen by the adversary) and global (i.e., can leak on many shares jointly). In other words, this paper provides security without strong independence nor noise assumption. Unfortunately, and despite promising from the security viewpoint, such encodings lead to large performance overheads, limiting their concrete interest [2]. This is mostly due to the fact that the IP encodings of [21] are non-linear

¹ This is for example reflected in the above list of masked hardware implementations [30,28,13,29,9,35]. Similar observations hold in the software case.

² More precisely, solutions like shuffling the operations is sometimes considered, but are in fact only effective under some noise assumptions as well [31,52,51].

with respect to $\vec{\omega}$ and \vec{y} , which are both secret, implying that even linear operations (like the AES' MixColumns) require expensive gadgets. Another seed result is the TCC 2016 one of Dziembowski et al., who showed that additive encodings in prime fields can also provide security amplification for noise-free leakages. Such encodings allow efficient multiplications, but their guarantees for so-called local leakages (i.e., again assuming independence) are mostly asymptotic and lead to hardly usable bounds. These seed results gave rise to a rich literature on so-called leakage-resilient secret sharing [5,6,36,41,37,38,39,34,23,40]. We next detail the three most relevant ones for our following discussions.

First, the Crypto 2018 results of Benhamouda et al. consider the security of linear encodings under the independent leakage assumption and for generic bounded leakages [5,6]. Their security claims do not benefit from an increase of the field size, implying large number of shares to reach high security levels. But the fixed and public coefficients of their encodings enable efficient multiplication. Next, the TCC 2022 results of Maji et al. consider the security of linear encodings without independence assumption, but for a very specific class of leakage functions [37]. For this class of leakage functions, they obtain security bounds as strong as the Asiacrypt 2011 ones of Dziembowski and Faust [21]. As soon as the leakage function escapes from this class, which happens already for bit leakages, the combination of global and adaptive leakages they consider implies that security completely collapses (as we discuss in Section 3.1). Finally, the Eurocrypt 2024 results of Faust et al. consider additive prime encodings (which enable efficient multiplications), require the independence assumption, and specialize the Crypto 2018 results of Benhamouda et al. towards (bit leakages and) Hamming weight leakages [23].³ They obtain a security bound that scales with $\log(p)^d$, where p is the field modulus and d the number of shares. While this is a natural first step given the popularity of such leakages in practice [42], it remains a specific result that holds under a strong (independence) assumption, raising the question whether it can be improved, hold under weaker assumptions and be generalized. This state of the art is summarized in Table 1, where N is the number of traces required to perform a successful side-channel attack.

In this context, our first contribution is to answer positively to the three points in the above question. The main tweak we use for this purpose is a conceptual sweet spot regarding a combination of model and assumptions that enables strong theoretical results while being practically-relevant. Namely, we analyze IP encodings under **global** and **generic** (bounded) leakage, but require that the leakage function is **non-adaptive**. Precisely, we require that the coefficients $\vec{\omega}$ used by the IP masking scheme are picked up before specifying the leakage function. This is a weak assumption, as it is widely acknowledged that the leakage function is primarily defined by the target device rather than by the adversary, and the mild adaptations that an adversary could perform with a measurement setup (*e.g.*, by moving a probe) are anyway not feasible at runtime [54].

³ For bit leakages, a similar result was already put forward by Maji et al. in [38], where the authors additionally show that such leakages are worst-case.

	leakage assumptions	security guarantees	encodings	coefficients	leakage function	security bounds	multiplication
Asiacrypt 2011 [21]	global (\perp)	generic	IP	random/secret	adaptive	$N \geq \sqrt{ \mathbb{F} }^d$	inefficient
Crypto 2018 [5,6]	local (\perp)	generic	linear	fixed/public	adaptive	$N \geq (cst)^d$	efficient
TCC 2022 [37]	global (\perp)	very specific (negligible subset)	linear	random/public	adaptive	$N \geq \sqrt{ \mathbb{F} }^d$	efficient
Eurocrypt 2024 [23]	local (\perp)	specific (bits) specific (HW)	additive	\emptyset	non-adaptive	$N \geq \left(\frac{\pi}{2}\right)^d$ $N \geq \log(p)^d$	efficient
Contribution #1	global (\perp)	generic	IP	random/public	non-adaptive	$N \geq \sqrt{ \mathbb{F} }^d$	efficient
Contribution #2	local (\perp)	specific (bits)	IP	fixed/public	non-adaptive	$N \geq \mathbb{F} ^d$	efficient

Table 1: State-of-the-art masking / leakage-resilient secret sharing schemes with guarantees against noise-free leakage.

With this weak assumption, we can obtain major improvements of the Eurocrypt 2024 security bounds, from $\log(p)^d$ for prime fields to $|\mathbb{F}|^{d/2}$ for any field, generalize them to any (field and) sufficiently bounded leakage function, and do not require the independence assumption. Furthermore, our results consider the coefficients $\vec{\omega}$ to be public, which enables leveraging the **efficient** multiplication algorithms from [1]. More precisely, for $|\mathbb{F}| \approx 2^n$ and for an m -bit global leakage on the secret coefficients \vec{y} , the number of attack traces is roughly bounded by $(2^n)^{\frac{d}{2} - \frac{m}{2n}}$. Say for example that the leakage per share is $\frac{m}{d} = \frac{n}{2}$ bits, then the bound is worth $2^{\frac{dn}{4}}$. For $\frac{m}{d} = \frac{n}{4}$ and $\frac{m}{d} = \frac{n}{8}$, it decreases to $2^{\frac{3dn}{8}}$ and $2^{\frac{7dn}{16}}$.

As a comparison, and assuming a $n = 32$ -bit software implementation leaking the Hamming weight of the shares (*i.e.*, $m \approx \log(32) = 5 \approx \frac{n}{6}$), this improves the Eurocrypt 2024 bound from 5^d to $2^{32 * \frac{5}{12}}$, reducing the number of shares to ensure security for 10 million traces from 10 to 2. Further assuming quadratic performance overheads for masking [33], this (roughly) corresponds to performance gains by a factor 25. So our results significantly amplify the interest of masking in larger fields. We refer to Section 3 for a formal treatment.

In Section 3.4, we additionally discuss how to choose the public coefficients $\vec{\omega}$ and the impact of keeping them constant. We also discuss the practicality of the bounded leakage assumption and how to relax it to other (more realistic) settings where the leakage is characterized based on information theoretic metrics.

Our second contribution, presented in Section 4, investigates the complementary question of whether more specific assumptions on the leakage function could lead to significantly improved results. As a theoretical first step in this direction, we consider bit leakages and show that, by choosing the coefficients $\vec{\omega}$ of a (Mersenne prime) IP encoding accordingly, it is possible to improve the previous bound from $|\mathbb{F}|^{d/2}$ to p^d . Technically, this contribution is of independent interest, since it relies on different tools (*i.e.*, FFT analysis *vs.* the leftover hash lemma for the first contribution). The exponential rate of leakage resilience it puts forward also improves significantly upon previous bounds in additive masking, which have only constant exponential rate. Concretely, obtaining this result again requires the independence assumption. It leaves as an interesting open question whether fine-grain characterization for more realistic functions than bit leakages can lead to similar or better gains, in order to fully comprehend the genericity/simplicity *vs.* security tradeoff of leakage-resilient secret sharing.

2 Background

2.1 Notation

For any set \mathcal{X} , denoted by calligraphic letters, we denote with $U_{\mathcal{X}}$ the uniform distribution over \mathcal{X} . \mathbb{F} denotes a finite field, and \mathbb{F}_p denotes the finite field of size p , where p is prime. In this context, we also define $\gamma = \exp\left(i\frac{2\pi}{p}\right)$ as the p -th root of unity. In this paper, for a vector (denoted by bold letters) $\vec{y} \in \mathbb{F}^d$, and for some vector $\vec{\omega} \in \mathbb{F}^d \setminus (0, \dots, 0)$, we denote by $\vec{y} + \text{Span}(\vec{\omega})^\perp$ the affine hyperplane

with offset \vec{y} and orthogonal vector $\vec{\omega}$. When the context does not raise any ambiguity, we may replace the vector offset \vec{y} by a scalar offset y , meaning that the actual offset is $\vec{y} = (y, 0, \dots, 0)$.

2.2 Metrics

In this paper, we make use of the *total variation* distance $\text{TV}(\mathbf{p}; \mathbf{q})$ between two probability mass functions \mathbf{p}, \mathbf{q} over the same support \mathcal{X} , defined as the quantity $\frac{1}{2} \sum_{x \in \mathcal{X}} |\mathbf{p}(x) - \mathbf{q}(x)|$. We also make use of other metrics, like the *min-entropy*.

Definition 1 (Min-Entropy). *Let Y be a random variable. The min-entropy of Y is defined as*

$$H_\infty(Y) = \min_{y \in \mathcal{Y}} \{-\log(\Pr(Y = y))\} = -\log \left(\max_{y \in \mathcal{Y}} \Pr(Y = y) \right).$$

We say that Y is a k -source if $H_\infty(Y) \geq k$.

Definition 2 (Average Min-entropy). *The average min-entropy of a random variable Y conditioned on a random variable L , denoted $\tilde{H}_\infty(Y | L)$, is defined as:*

$$\tilde{H}_\infty(Y|L) = -\log_2 \left(\mathbb{E}_\ell \left[2^{-H_\infty(Y|L=\ell)} \right] \right).$$

2.3 Randomness Extractors

Definition 3. *Let \mathcal{X}, \mathcal{S} and \mathcal{Y} be sets. Let $U_{\mathcal{S}}, U_{\mathcal{Y}}$ be the uniform random variables over \mathcal{S} and \mathcal{Y} , respectively. We say that a function $\text{Ext} : \mathcal{X} \times \mathcal{S} \rightarrow \mathcal{Y}$ is a (k, ϵ) strong extractor if, for k -sources X over \mathcal{X} such that $X, U_{\mathcal{S}}, U_{\mathcal{Y}}$ are independent, then*

$$\text{TV}((\text{Ext}(X, U_{\mathcal{S}}), U_{\mathcal{S}}); (U_{\mathcal{Y}}, U_{\mathcal{S}})) \leq \epsilon.$$

Definition 4. *A family \mathcal{H} of hash functions from \mathcal{X} to \mathcal{Y} is called universal if, for every $x, x' \in \mathcal{X}$ with $x \neq x'$, then*

$$\Pr_{h \in \mathcal{H}} [h(x) = h(x')] \leq \frac{1}{|\mathcal{Y}|}.$$

Theorem 1 (Leftover Hash Lemma [32]). *Let X be a k -source over \mathcal{X} . Let $\mathcal{H} = \{h_\omega\}_{\omega \in \mathcal{W}}$ be a universal hash family with output space \mathcal{Y} . For every $x \in \mathcal{X}$ and $\omega \in \mathcal{W}$, define*

$$\text{Ext}(x, \omega) = h_\omega(x).$$

Then Ext is a strong (k, ϵ) -extractor with $\epsilon = |\mathcal{Y}|^{\frac{1}{2}} \cdot 2^{-\frac{k}{2} - 1}$.

For completeness, we include a proof of Theorem 1 in section B.

2.4 Inner-Product Masking

Definition 5 (Inner-Product Masking). Let $\vec{\omega} \in (\mathbb{F}_p^*)^d$. An $\vec{\omega}$ -inner-product encoding of a secret y (or simply $\vec{\omega}$ -encoding for short) is a d -tuple $\llbracket y \rrbracket_{\vec{\omega}} = (y_1, \dots, y_d) \in \mathbb{F}_p^d$, such that

$$y = \langle \vec{\omega}, \llbracket y \rrbracket_{\vec{\omega}} \rangle = \sum_{i=1}^d \omega_i \cdot y_i ,$$

i.e., such that $\llbracket y \rrbracket$ is in $y + \text{Span}(\vec{\omega})^\perp$. The parameter d is called the masking order.

Remark 1. When the context does not raise any ambiguity, we may write $\llbracket y \rrbracket$ instead of $\llbracket y \rrbracket_{\vec{\omega}}$ for short. Moreover, as we are working over a finite field, we may assume without loss of generality that $\omega_1 = 1$ — it suffices to divide any other entry of $\vec{\omega}$ by ω_1 .

2.5 The Leakage-Resilience Threat Model

Following the usual leakage-resilience threat model considered in the literature, an attacker is given access to some leakage function L , adversarially chosen among a given class of leakage [19]. More precisely, in a leakage-resilience context, we assume that for any leakage function of interest, there exists some (possibly randomized) function $\tau : \mathbb{F}^d \rightarrow \mathcal{L}$ such that for any $y \in \mathcal{Y}$:

$$L(y) = \tau(\text{Enc}_{\vec{\omega}}(y)) ,$$

where $\text{Enc}_{\vec{\omega}}(y)$ is a randomized function returning an $\vec{\omega}$ -encoding $\llbracket y \rrbracket_{\vec{\omega}}$ uniformly drawn over $y + \text{Span}(\vec{\omega})^\perp$. We focus on the bounded leakage model, meaning that τ is m -bounded, as defined below.

Definition 6 (Bounded Leakage). A (possibly randomized) function $\tau : \mathbb{F}^d \rightarrow \mathcal{L}$ is m -bounded if $|\mathcal{L}| = 2^m$. We denote with Bnd_m the class of such functions.

This threat model somewhat differs from the so-called *noisy leakage* model usually considered in the literature of provable masking [49], which characterizes the leakage with information theoretic metrics like the statistical distance. On the one hand, any m -bounded function can be seen as a particular case of noisy leakage,⁴ whereas the opposite is not always true [7,47]. On the other hand, the noisy leakage model often requires that each share of the encoding — or each subsequent computation — leak independently from the others. Although commonly accepted in the literature, this so-called *local* leakage assumption is not always realistic, e.g., due to some physical defaults [43,46,14,3,22,10] or due to

⁴ In a binary field, any m -bounded function is δ -noisy, for $\delta = 1 - 2^{-m}$. Note that the name noisy leakage model may be misleading since a function can be δ -noisy without physical noise if it is non-injective.

some correlated noise [11]. Therefore, relaxing this assumption is identified as a major challenge [50, Sec. 4.3.3, 4.6]. We make one step forward in this direction by removing this assumption in the particular case of bounded leakage.

In the leakage-resilient secret sharing literature, an $\vec{\omega}$ -encoding is said to be ϵ -leakage resilient against m -bounded leakage if $\sup_{\tau \in \text{Bnd}_m} \mathcal{M}_{\vec{\omega}}(\mathbf{L}) \leq \epsilon$, where

$$\mathcal{M}_{\vec{\omega}}(\mathbf{L}) = \sup_{y^{(0)}, y^{(1)} \in \mathbb{F}} \text{TV}\left(\Pr\left(\mathbf{L}\left(y^{(0)}\right) \mid \vec{\Omega} = \vec{\omega}\right); \Pr\left(\mathbf{L}\left(y^{(1)}\right) \mid \vec{\Omega} = \vec{\omega}\right)\right).$$

In a side-channel context though, most of the attacks in the state of the art use random plaintexts. This type of adversary is better reflected by a weaker metric, the so-called *statistical bias*:

$$\beta_{\vec{\omega}}(\mathbf{Y} \mid \mathbf{L}) = \mathbb{E}_{\mathbf{Y}} \left[\text{TV}\left(\Pr(\tau(\text{Enc}_{\vec{\omega}}(\mathbf{Y}))); \mathbb{E}_{\mathbf{Y}'} [\Pr(\tau(\text{Enc}_{\vec{\omega}}(\mathbf{Y}')))]\right) \right], \quad (1)$$

where \mathbf{Y}' is an independent copy of \mathbf{Y} . Using Bayes' theorem, the statistical bias may also be rephrased as follows: $\beta_{\vec{\omega}}(\mathbf{Y} \mid \mathbf{L}) = \text{TV}\left(\mathbf{p}_{\mathbf{L}, \mathbf{Y}} \mid_{\vec{\Omega}=\vec{\omega}}; \mathbf{p}_{\mathbf{L}} \mid_{\vec{\Omega}=\vec{\omega}} \otimes \mathbf{p}_{\mathbf{Y}}\right)$.⁵ Unsurprisingly, $\mathcal{M}_{\vec{\omega}}(\mathbf{L})$ upper bounds $\beta_{\vec{\omega}}(\mathbf{Y} \mid \mathbf{L})$, where the secret is uniformly random. Choosing the secret $y \in \mathbb{F}$ can improve the bias by at most a factor $|\mathbb{F}|$. The next proposition (proven in appendix A) formalizes this intuition.

Proposition 1. *Let $\vec{\omega} \in (\mathbb{F}^*)^d$, and \mathbf{L} be any random variable. Assume that \mathbf{Y} is the uniform random variable over \mathbb{F} . Then*

$$\beta_{\vec{\omega}}(\mathbf{Y} \mid \mathbf{L}) \leq \mathcal{M}_{\vec{\omega}}(\mathbf{L}) \leq |\mathbb{F}| \cdot \beta_{\vec{\omega}}(\mathbf{Y} \mid \mathbf{L}).$$

Our goal is therefore to bound $\mathcal{M}_{\vec{\omega}}(\mathbf{L})$, either directly (in section 4), or through a reduction to the statistical bias $\beta_{\vec{\omega}}(\mathbf{Y} \mid \mathbf{L})$ (in section 3).

3 Inner-Product Masking with Randomized Coefficients

3.1 The Dilemma of Adversarially-Chosen Global Leakage

In general, if an adversary can choose the leakage function depending on $\vec{\omega}$, then inner-product masking trivially leads to the very same security as additive masking, as formalized by the following proposition, proven in Appendix C.

Proposition 2. *Let $\vec{\omega}, \vec{\omega}' \in (\mathbb{F}^*)^d$. Then, for any m -bounded leakage function τ ,⁶ there exist some m -bounded leakage function τ' such that for all observed leakage $\vec{\ell}$:*

$$\Pr\left(\tau(\text{Enc}_{\vec{\omega}}(y)) = \vec{\ell}\right) = \Pr\left(\tau'(\text{Enc}_{\vec{\omega}'}(y)) = \vec{\ell}\right).$$

⁵ Notice that since \mathbf{Y} is taken uniform in the definition of the statistical bias, \mathbf{Y} is independent of $\vec{\omega}$.

⁶ Although beyond the scope of this paper, the result also applies to δ -noisy leakage functions [19].

Even worse, in our context where we do not assume the shares to leak independently from each other, we would allow the adversary to choose the function $\text{Dec}_{\vec{\omega}} : \vec{x} \mapsto \langle \vec{\omega}, \vec{x} \rangle$ as a leakage. $\text{Dec}_{\vec{\omega}}$ is m -bounded, for $m = \log |\mathbb{F}|$. Yet, by definition, $\text{Dec}_{\vec{\omega}}(\llbracket y \rrbracket_{\vec{\omega}}) = y$. In other words, leaking more than $\log |\mathbb{F}|$ bits in this model would lead to a successful attack, regardless of the number of shares d . Maji et al. circumvent this issue by restricting the adversary to a sub-class of m -bounded leakage functions, whose choice is let to the adversary, yet whose size is reflected in their security bound [37]. Hence, in order to get a non-trivial bound, the sub-class must have a negligible size compared to the class Bnd_m .

As argued in introduction of the paper, an adversarially-chosen leakage function is nevertheless unlikely to happen in practice, since the leakage function is usually more a property of the target device than a property of the adversary. So to circumvent this dilemma, we may simply assume that the adversary still selects the leakage function, yet without knowing the $\vec{\omega}$ coefficients in advance. Conceptually, this means now that $\vec{\omega}$ is drawn uniformly over $(\mathbb{F}^*)^d$ once the leakage function $\tau \in \text{Bnd}_m$ is chosen by the adversary. Observe then that the metrics $\beta_{\vec{\omega}}(Y | L)$ and $M_{\vec{\omega}}(L)$ are now random variables, whose randomness is taken over the uniform choice of $\vec{\omega} \in (\mathbb{F}^*)^d$. As a result, we want to upper bound them with a high probability independent of the choice of $\tau \in \text{Bnd}_m$. This motivates the following definition.

Definition 7 (Stochastic Leakage Resilience). *For $\delta > 0$, a d -share inner-product encoding is $(\delta, \epsilon_{m,d})$ -leakage resilient against m -bounded leakage if for any $\delta > 0$, it is $\epsilon_{m,d}$ -leakage resilient with probability at least $1 - \delta$, or equivalently, if*

$$\sup_{\tau \in \text{Bnd}_m} \Pr(M_{\vec{\omega}}(L) \geq \epsilon_{d,m}) \leq \delta ,$$

where the randomness is taken over the uniform choice of $\vec{\omega} \in (\mathbb{F}^*)^d$.

Concretely, we will rather try to upper-bound the expectation of the statistical bias, and then to apply concentration inequalities, *e.g.*, Markov, and Proposition 1. This justifies the introduction of the following metric.

Definition 8 (Bias for Random Coefficients). *Let $Y, L, \vec{\Omega}$ be random variables. The statistical bias for random coefficients induced by a leakage L over the $\vec{\Omega}$ -encoding of a secret Y is defined by*

$$\beta_{\vec{\Omega}}(Y|L) = \mathbb{E}_{\vec{\omega}} [\beta_{\vec{\omega}}(Y | L)] . \quad (2)$$

3.2 Limitations of Previous Techniques

So far, we have an objective, namely upper-bounding the – statistical or worst-case – bias for random coefficients. A naive approach to tackle this challenge would be to leverage some of the tools used in the leakage-resilient secret sharing literature, like the Cauchy-Schwarz trick, or Poisson summation formula [6]. Unfortunately, they would suffer from two drawbacks. Firstly, they rely on Fourier

analysis, which implicitly requires the independence assumption. As discussed in the introduction, ensuring this assumption may be challenging and generally implies implementation overheads — but still remains an option. Secondly, they would require to naively compute Equation 2 by upper-bounding each term inside the expectation. For the same reason developed in subsection 3.1, each bound would then not be able to depend on $\vec{\omega}$, unless further characterizing the leakage function.⁷ This would lead to the very same security bounds as obtained for additive masking, which would not be helpful. Therefore, an alternative approach is needed to tackle the issue of the expectation in Equation 2. Fortunately, this is what the following statement proposes.

Proposition 3. *The averaged statistical bias may be equivalently defined as follows:*

$$\beta_{\vec{\Omega}}(\mathbf{Y}|\mathbf{L}) = \mathbb{E}_{\vec{\ell}} \left[\text{TV} \left(\mathbf{p}_{\mathbf{Y}, \vec{\Omega}} |_{\mathbf{L}=\vec{\ell}}; \mathbf{p}_{\mathbf{Y}} \otimes \mathbf{p}_{\vec{\Omega}} \right) \right] . \quad (3)$$

Intuitively, Equation 3 “swaps” the expectations over $\vec{\Omega}$ and \mathbf{L} respectively. We will leverage this equivalent formulation of $\beta_{\vec{\Omega}}(\mathbf{Y}|\mathbf{L})$ in the next subsection.

Proof. In order to prove the proposition, we will show that the average statistical bias is equal to $\text{TV} \left(\mathbf{p}_{\mathbf{Y}, \mathbf{L}, \vec{\Omega}}; \mathbf{p}_{\mathbf{Y}} \otimes \mathbf{p}_{\mathbf{L}} \otimes \mathbf{p}_{\vec{\Omega}} \right)$. Leveraging the symmetry of the roles played by \mathbf{L} , \mathbf{Y} and $\vec{\Omega}$ in the latter quantity, it will imply that we may permute \mathbf{L} and $\vec{\omega}$ in the definition of the average statistical bias, hence the result.

By definition of the statistical bias given by Equation 1, we have

$$\beta_{\vec{\omega}}(\mathbf{Y} | \mathbf{L}) = \mathbb{E}_y \left[\text{TV} \left(\mathbf{p}_{\mathbf{L} | \mathbf{Y}=y}; \mathbb{E}_{y'} \left[\mathbf{p}_{\mathbf{L} | \mathbf{Y}=y'} \right] \right) \right] = \mathbb{E}_y \left[\text{TV} \left(\mathbf{p}_{\mathbf{L} | \mathbf{Y}=y}; \mathbf{p}_{\mathbf{L}} \right) \right]$$

Using the definition of conditional probability, we may rephrase the latter right-hand side as follows:

$$\beta_{\vec{\omega}}(\mathbf{Y} | \mathbf{L}) = \text{TV} \left(\mathbf{p}_{\mathbf{L}, \mathbf{Y}}; \mathbf{p}_{\mathbf{L}} \otimes \mathbf{p}_{\mathbf{Y}} \right) .$$

Actually, the probability distributions $\mathbf{p}_{\mathbf{L}, \mathbf{Y}}$ and $\mathbf{p}_{\mathbf{L}} \otimes \mathbf{p}_{\mathbf{Y}}$ implicitly depend on the value of $\vec{\omega}$, so they may be rephrased as $\mathbf{p}_{\mathbf{L}, \mathbf{Y}} |_{\vec{\Omega}=\vec{\omega}}$ and $\mathbf{p}_{\mathbf{L}, \mathbf{Y}'} |_{\vec{\Omega}=\vec{\omega}}$ respectively, where \mathbf{Y}' is an independent copy of \mathbf{Y} . So by definition of the average statistical bias,

$$\begin{aligned} \beta_{\vec{\Omega}}(\mathbf{Y}|\mathbf{L}) &= \mathbb{E}_{\vec{\omega}} [\beta_{\vec{\omega}}(\mathbf{Y} | \mathbf{L})] \\ &= \mathbb{E}_{\vec{\omega}} \left[\text{TV} \left(\mathbf{p}_{\mathbf{L}, \mathbf{Y}} |_{\vec{\Omega}=\vec{\omega}}; \mathbf{p}_{\mathbf{L}, \mathbf{Y}'} |_{\vec{\Omega}=\vec{\omega}} \right) \right] \\ &= \text{TV} \left(\mathbf{p}_{\mathbf{L}, \mathbf{Y}, \vec{\Omega}}; \mathbf{p}_{\mathbf{L}, \mathbf{Y}', \vec{\Omega}'} \right) , \end{aligned}$$

where $\vec{\Omega}'$ is an independent copy of $\vec{\Omega}$. Proposition 3 follows by observing that $\mathbf{p}_{\mathbf{L}, \mathbf{Y}', \vec{\Omega}'}$ is equal to $\mathbf{p}_{\mathbf{L}} \otimes \mathbf{p}_{\mathbf{Y}} \otimes \mathbf{p}_{\vec{\Omega}}$, by virtue of the independence of \mathbf{Y}' and $\vec{\Omega}'$, and by using again the definition of conditional probability. \square

⁷ As we will see in section 4.

3.3 The Leftover Hash Lemma to the Rescue

We now present the main result of this section, which estimates the amount of information given to an adversary which chooses the leakage function independently of the inner product coefficients, and gets the leakage information together with the inner product coefficients in full.

Theorem 2. *Let \mathbb{F} be a field, and $d \geq 1$ be an integer. Let $Y \in \mathbb{F}$ and $\vec{\Omega} \in (\mathbb{F}^*)^d$ be two independent uniformly distributed random variables, with $d \geq 1$. Furthermore, let $L(Y) = \tau(\text{Enc}_{\vec{\Omega}}(Y))$, where $\tau \in \text{Bnd}_m$, for some $m \geq 1$. Assume that the internal randomness of τ is independent of Y and $\vec{\Omega}$. Then, it holds*

$$\beta_{\vec{\Omega}}(Y|L) \leq |\mathbb{F}|^{\frac{1-d}{2}} \cdot 2^{\frac{m-2}{2}} \cdot \left(\frac{|\mathbb{F}|}{|\mathbb{F}-1} \right)^d.$$

Despite Theorem 2 applies only to uniformly distributed secrets, its combination with Proposition 1 provides a bound for arbitrary distributed secrets.

Corollary 1. *For any $\delta > 0$, the d -share inner-product encoding is $(\delta, \epsilon_{m,d})$ -leakage resilient against m -bounded leakage for*

$$\epsilon_{m,d} = \delta^{-1} \cdot |\mathbb{F}|^{\frac{3-d}{2}} \cdot 2^{\frac{m-2}{2}} \cdot \left(\frac{|\mathbb{F}|}{|\mathbb{F}-1} \right)^d.$$

Proof. Let $\delta > 0$. By applying successively Proposition 1, Theorem 2, and Markov's inequality, we have that

$$\begin{aligned} & \Pr \left(M_{\vec{\omega}}(L) \geq \delta^{-1} \cdot |\mathbb{F}|^{\frac{3-d}{2}} \cdot 2^{\frac{m-2}{2}} \cdot \left(\frac{|\mathbb{F}|}{|\mathbb{F}-1} \right)^d \right) \\ & \leq \Pr \left(\beta_{\vec{\omega}}(Y | L) \geq \delta^{-1} \cdot |\mathbb{F}|^{\frac{1-d}{2}} \cdot 2^{\frac{m-2}{2}} \cdot \left(\frac{|\mathbb{F}|}{|\mathbb{F}-1} \right)^d \right) \\ & \leq \Pr(\beta_{\vec{\omega}}(Y | L) \geq \delta^{-1} \cdot \beta_{\vec{\Omega}}(Y|L)) \\ & \leq \delta, \end{aligned}$$

where the probability is taken over the choice of $\vec{\omega}$ over $(\mathbb{F}^*)^d$. □

We prove Theorem 2 by connecting the theory of inner product masking to the one of extractors, following an approach similar to the work of Dziembowski and Faust [21]. First, we present the lemmata involved in the proof. In the end of this section, we prove Theorem 2 based on the lemmas.

We start by connecting the statistical distance for coefficients in $(\mathbb{F}^*)^d$ to the one for coefficients in \mathbb{F}^d . While the first set corresponds to the coefficients where our encoding operates, the latter guarantees universality of the inner product hash function, as we will see later in Lemma 2.

Lemma 1. *Let $Y, L, \vec{\Omega}$ and $\vec{\Omega}'$ be random variables. Assume that $\vec{\Omega}$ and $\vec{\Omega}'$ are uniformly distributed, respectively, over $(\mathbb{F}^*)^d$ and \mathbb{F}^d . Then,*

$$\beta_{\vec{\Omega}}(Y|L) \leq \left(\frac{|\mathbb{F}|}{|\mathbb{F}| - 1} \right)^d \beta_{\vec{\Omega}'}(Y|L).$$

Proof. Using Definition 8 and the assumption that $\vec{\Omega}$ and $\vec{\Omega}'$ are uniformly distributed over $(\mathbb{F}^*)^d$ and \mathbb{F}^d , we can rewrite

$$\begin{aligned} \beta_{\vec{\Omega}'}(Y|L) &= \frac{1}{|\mathbb{F}|^d} \sum_{\vec{\omega} \in \mathbb{F}^d} \beta_{\vec{\omega}}(Y|L) \\ &= \frac{1}{|\mathbb{F}|^d} \sum_{\vec{\omega} \in (\mathbb{F}^*)^d} \beta_{\vec{\omega}}(Y|L) + \frac{1}{|\mathbb{F}|^d} \sum_{\vec{\omega} \in \mathbb{F}^d \setminus (\mathbb{F}^*)^d} \beta_{\vec{\omega}}(Y|L) \\ &\geq \frac{1}{|\mathbb{F}|^d} \sum_{\vec{\omega} \in (\mathbb{F}^*)^d} \beta_{\vec{\omega}}(Y|L) \\ &= \left(\frac{(|\mathbb{F}| - 1)}{|\mathbb{F}|} \right)^d \cdot \frac{1}{(|\mathbb{F}| - 1)^d} \sum_{\vec{\omega} \in (\mathbb{F}^*)^d} \beta_{\vec{\omega}}(Y|L) \\ &= \left(\frac{(|\mathbb{F}| - 1)}{|\mathbb{F}|} \right)^d \cdot \beta_{\vec{\Omega}}(Y|L) \end{aligned}$$

This concludes the proof. \square

For every $\vec{\omega} \in \mathbb{F}^d$, let $h_{\vec{\omega}} : \mathbb{F}^d \rightarrow \mathbb{F}$ be the function that, on input any $\vec{x} \in \mathbb{F}^d$, outputs $h_{\vec{\omega}}(\vec{x}) = \langle \vec{\omega}, \vec{x} \rangle$. Define $\mathcal{H}_{\text{IP}} = \{h_{\vec{\omega}}\}_{\vec{\omega} \in \mathbb{F}^d}$. The following claim holds.

Lemma 2. *\mathcal{H}_{IP} is a universal family of hash functions.*

Proof. By the definition of universal family of hash functions, we will prove that

$$\Pr_{\vec{\omega} \xleftarrow{\$} \mathbb{F}^d} [h_{\vec{\omega}}(\vec{x}) = h_{\vec{\omega}}(\vec{x}')] \leq \frac{1}{|\mathbb{F}|} \quad (4)$$

holds for any couple of distinct elements $\vec{x}, \vec{x}' \in \mathbb{F}^d$. As $h_{\vec{\omega}}(\vec{x}) = \langle \vec{\omega}, \vec{x} \rangle$, we get $[h_{\vec{\omega}}(\vec{x}) = h_{\vec{\omega}}(\vec{x}')] \Leftrightarrow [\langle \vec{\omega}, \vec{x} \rangle = \langle \vec{\omega}, \vec{x}' \rangle] \Leftrightarrow [\langle \vec{\omega}, \vec{x} - \vec{x}' \rangle = 0]$. Due to the uniform distribution of $\vec{\omega}$, we have

$$\Pr_{\vec{\omega} \xleftarrow{\$} \mathbb{F}^d} [h_{\vec{\omega}}(\vec{x}) = h_{\vec{\omega}}(\vec{x}')] = \frac{|\{\vec{\omega} \in \mathbb{F}^d \text{ with } \langle \vec{\omega}, \vec{x} - \vec{x}' \rangle = 0\}|}{|\mathbb{F}|^d}.$$

Next, we can transform the equation by replacing $|\{\vec{\omega} \in \mathbb{F}^d \setminus \{0\}^d : \langle \vec{\omega}, \vec{x} - \vec{x}' \rangle = 0\}|$ with

$$|\{\vec{\omega} \in \mathbb{F}^d \setminus \{0\}^d : \langle \vec{\omega}, \vec{x} - \vec{x}' \rangle = 0\}| = |\mathbb{F}|^{d-1} \quad (5)$$

Equation 5 holds because for every $\vec{x}, \vec{x}' \in \mathbb{F}^d$ there is at least one i such that $x_i - x'_i \neq 0$. Consequently, for every $(d-1)$ -tuple $\omega_1, \dots, \omega_{i-1}, \omega_{i+1}, \dots, \omega_d$ there exists a unique ω_i such that $\langle \vec{\omega}, \vec{x} - \vec{x}' \rangle = 0$. This results in the claim – Equation 4 – as it holds $\frac{|\mathbb{F}|^{d-1}}{|\mathbb{F}|^d} = \frac{1}{|\mathbb{F}|}$. \square

Lemma 3. *Let $\vec{\Omega}'$ and Y be two independent uniform random variables over \mathbb{F}^d and \mathbb{F} , respectively. Furthermore, let $L(Y) = \tau(\text{Enc}_{\vec{\Omega}}(Y))$, where $\tau \in \text{Bnd}_m$. Assume that the internal randomness of τ is independent of Y and $\vec{\Omega}'$. Let $\vec{\ell} \in \mathcal{L}$ such that $\Pr[L = \vec{\ell}] \neq 0$, and let $\vec{X}_{\vec{\ell}} = (\text{Enc}_{\vec{\Omega}}(Y) \mid L = \vec{\ell})$. Then, it holds that $\vec{X}_{\vec{\ell}}$ and $\vec{\Omega}'$ are independent.*

Proof. For any $\vec{x} \in \mathbb{F}^d$, and any $\vec{\omega} \in \mathbb{F}^d$ we need to prove that the probability

$$p_{\vec{\ell}, \vec{\omega}} = \Pr(\text{Enc}_{\vec{\Omega}}(Y) = \vec{x} \mid \vec{\Omega}' = \vec{\omega}, L = \vec{\ell})$$

does not depend on $\vec{\omega}$. Using Bayes' theorem, we have

$$p_{\vec{\ell}, \vec{\omega}} = \frac{\Pr(L = \vec{\ell} \mid \text{Enc}_{\vec{\Omega}}(Y) = \vec{x}, \vec{\Omega}' = \vec{\omega}) \cdot \Pr(\text{Enc}_{\vec{\Omega}}(Y) = \vec{x} \mid \vec{\Omega}' = \vec{\omega})}{\Pr(L = \vec{\ell} \mid \vec{\Omega}' = \vec{\omega})}.$$

Let us inspect every factor in the right-hand side.

- The first factor, *i.e.* $\Pr(L = \vec{\ell} \mid \text{Enc}_{\vec{\Omega}}(Y) = \vec{x}, \vec{\Omega}' = \vec{\omega})$, can be rephrased as $\Pr(\tau(\vec{x}) = \vec{\ell})$. Since we assumed the internal randomness of τ to be independent of Y and $\vec{\Omega}'$, this probability is also independent of $\vec{\omega}$.
- We rewrite the second factor as

$$\Pr(\text{Enc}_{\vec{\Omega}}(Y) \mid \vec{\Omega}' = \vec{\omega}) = \Pr(\text{Enc}_{\vec{\omega}}(Y) = \vec{x}).$$

Given that Y is assumed uniform, so is $\text{Enc}_{\vec{\omega}}(Y)$, *i.e.*, it does not depend on $\vec{\omega}$. This holds because the first $d - 1$ components X_1, \dots, X_{d-1} of $\text{Enc}_{\vec{\omega}}(Y)$ are sampled at random independently of $\vec{\Omega}'$, and the last one is computed as $X_d = Y - \sum_{j=1}^{d-1} \Omega_j X_j$. Since Y is uniformly random and independent of both $\vec{\Omega}'$ and X_1, \dots, X_{d-1} , this guarantees that also X_d , and thus the whole encoding $\text{Enc}_{\vec{\omega}}(Y) = (X_1, \dots, X_d)$, are independent of $\vec{\Omega}'$.

- Using the total probability formula, the third factor $\Pr(L = \vec{\ell} \mid \vec{\Omega}' = \vec{\omega})$ can be expressed as the sum over $\vec{x} \in \mathbb{F}^d$ of the product of the first two ones. Since they do not depend on $\vec{\omega}$, their sum does not either.

Thus, none of the factors depend on $\vec{\omega}$, hence $p_{\vec{\ell}, \vec{\omega}}$ is independent of $\vec{\omega}$. \square

Similarly to what is done by Dziembowski and Faust [21, Lemma 20], we need to estimate the entropy of the encoding after leakage. For sake of tightness we use a better approximation that follows from the works of Dodis [17].

Lemma 4. *Let Y be uniformly distributed over \mathbb{F} , let $\tau \in \text{Bnd}_m$, and define $L = \tau(\text{Enc}_{\vec{\Omega}}(Y))$. Then*

$$\tilde{H}_{\infty}(\text{Enc}_{\vec{\Omega}}(Y) \mid L) \geq d \log_2 |\mathbb{F}| - m.$$

Proof. Dodis' lemma [17, Lemma 2.2, item b] guarantees that for every couple of random variables A, B such that B takes values in a space \mathcal{B} of 2^m elements, then

$$\tilde{H}_\infty(A | B) \geq H_\infty(A) - m.$$

By setting $A = \text{Enc}_{\vec{\Omega}}(Y)$ and $B = L$, the lemma follows with $H_\infty(\text{Enc}_{\vec{\Omega}}(Y)) = d \log_2 |\mathbb{F}|$. \square

Putting all these lemmas together, Theorem 2 follows.

Proof of Theorem 2.

$$\begin{aligned} \beta_{\vec{\Omega}}(Y|L) &\leq \left(\frac{|\mathbb{F}|}{|\mathbb{F}|-1}\right)^d \beta_{\vec{\Omega}'}(Y|L) && \text{(Lemma 1)} \\ &= \left(\frac{|\mathbb{F}|}{|\mathbb{F}|-1}\right)^d \sum_{\vec{\ell} \in \mathcal{L}} \Pr(L = \vec{\ell}) \cdot \text{TV}(\mathbf{p}_{Y, \vec{\Omega}'} |_{\{L=\vec{\ell}\}}; \mathbf{p}_Y \otimes \mathbf{p}_{\vec{\Omega}'}) \\ &\leq \left(\frac{|\mathbb{F}|}{|\mathbb{F}|-1}\right)^d \sum_{\vec{\ell} \in \mathcal{L}} \Pr(L = \vec{\ell}) \cdot |\mathbb{F}|^{\frac{1}{2}} \cdot 2^{-1-H_\infty(\text{Enc}_{\vec{\Omega}}(Y) | L=\vec{\ell})/2} \\ & && \text{(LHL)} \\ &= \frac{1}{2} \left(\frac{|\mathbb{F}|}{|\mathbb{F}|-1}\right)^d \cdot |\mathbb{F}|^{\frac{1}{2}} \cdot \sum_{\vec{\ell} \in \mathcal{L}} \Pr(L = \vec{\ell}) \cdot 2^{-H_\infty(\text{Enc}_{\vec{\Omega}}(Y) | L=\vec{\ell})/2} \\ &\leq \frac{1}{2} \left(\frac{|\mathbb{F}|}{|\mathbb{F}|-1}\right)^d \cdot |\mathbb{F}|^{\frac{1}{2}} \cdot \left(\sum_{\vec{\ell} \in \mathcal{L}} \Pr(L = \vec{\ell})\right)^{1/2} \\ &\quad \cdot \left(\sum_{\vec{\ell} \in \mathcal{L}} \Pr(L = \vec{\ell}) \cdot 2^{-H_\infty(\text{Enc}_{\vec{\Omega}}(Y) | L=\vec{\ell})/2}\right)^{1/2} && \text{(Cauchy-Schwarz)} \\ &\leq \frac{1}{2} \left(\frac{|\mathbb{F}|}{|\mathbb{F}|-1}\right)^d \cdot |\mathbb{F}|^{\frac{1}{2}} \cdot 2^{-\tilde{H}_\infty(\text{Enc}_{\vec{\Omega}}(Y) | L)/2} \\ &\leq \frac{1}{2} \left(\frac{|\mathbb{F}|}{|\mathbb{F}|-1}\right)^d \cdot |\mathbb{F}|^{\frac{1}{2}} \cdot 2^{(-d \log_2 |\mathbb{F}| + m)/2} && \text{(Lemma 4)} \\ &= |\mathbb{F}|^{\frac{1-d}{2}} \cdot 2^{\frac{m-2}{2}} \cdot \left(\frac{|\mathbb{F}|}{|\mathbb{F}|-1}\right)^d, \end{aligned}$$

which completes the proof. It is important to note that the approximation step involving the leftover hash lemma (LHL) requires independence of the inputs to the inner product extractor, namely $\vec{X}_{\vec{\ell}} = (\text{Enc}_{\vec{\Omega}}(Y) | L = \vec{\ell})$ and $\vec{\Omega}'$. The independence of these inputs is shown in Lemma 3. \square

3.4 Discussion

We conclude this section by discussing two aspects of our theoretical results, to put them in perspective with practical considerations.

Drawing $\vec{\omega}$ once for all. So far, our threat model assumes that the public coefficients $\vec{\omega}$ are drawn independently of the physical leakage behaviour of the device, *e.g.*, after it comes out of the foundry, but before the encryption. This means that if we now consider a threat scenario in which the device operates several encryptions, the public coefficients $\vec{\omega}$ must be refreshed before every new encryption, in order to stick to the theoretical assumptions of Theorem 2 and Corollary 1. Despite this may be achievable with limited overheads, it questions what would happen in the even more convenient situation where $\vec{\omega}$ is not refreshed, *i.e.*, $\vec{\omega}$ is drawn once for all before the very first encryption. This would naturally represent a gap with our security model but, as we argue next, may not significantly impact the concrete security of an IP encoding.

First, the probability that such a coefficient leads to weak security by chance is negligible, as proven above. So the only thing that could happen is that the adversary now tries to exploit the possibility to tweak the leakage function to fall in one of the statistically unlikely worst-cases. Concretely, this would mean adapting the setup in order to shape the leakage function as required.

While this is theoretically feasible, we believe that the extent of adversarial control on side-channel measurement setups is in general insufficient to lead to significant security degradations. The practical investigation of such attacks is an interesting scope for further research. Note that in case these adaptive attacks are possible, it is likely that adapting the leakage function to $\vec{\omega}$ is not instantaneous and various tradeoffs could be considered, between updating these coefficients for every new encryption and keeping them constant all the time.

From Bounded to Noisy Leakages. So far, we have established our results in a threat model where the leakage function has a bounded range. This is aligned with similar assumptions in the leakage-resilient secret sharing literature [6]. Yet, the difficulty of setting a reasonable value m for the function's range is a frequent criticism of the bounded leakage model, because it is essentially determined by the attacker's measurement ability — *e.g.*, the resolution on the oscilloscope — which can improve with reasonable effort and budget.

Interestingly, we can relax the bounded leakage assumption to the noisy setting — and for free. The only part of the proof of Theorem 2 and Corollary 1 relying on the bounded leakage assumption is Lemma 4, which lower bounds the average min-entropy left after leakage. In the noisy setting, the output of the leakage function can be arbitrarily long but must retain some noise-hiding sensitive information. Therefore, this becomes more a property of the leakage, rather than a property of the adversary's measurement ability. If we quantify

this information as ℓ in the U-noisy model (see Definition 6 in [7]), it means that we can replace m with ℓ in all the results of this section.⁸

4 Beyond the Square-Root Frontier

The main result of section 3, namely Theorem 2, resolves the dilemma mentioned in subsection 3.1. It allows to circumvent the case where the attacker is able to adversarially choose the leakage function, depending on $\vec{\omega}$.

Interestingly, the literature on inner-product masking schemes has thoroughly investigated the opposite threat scenario: with enough characterization, the leakage function might be considered as fixed and known, so that designers may choose the coefficients constructively, *i.e.*, to maximize the side-channel security. This is for example at the core of the line of research on *security order amplification* for IP masking, provided that the scheme operates over a binary field and that the leakage function is linear, in a high-noise regime [53,48,12].

In this section, we draw our interest on a similar threat scenario, where the leakage function is fixed, and where the designer may choose the coefficients accordingly. Our intuition follows from the probabilistic argument in section 3: if we get good bounds on average for random coefficients, there must be some vector $\vec{\omega}_{\text{bad}}$ for which the security bound is worse, as well as some vector $\vec{\omega}_{\text{good}}$ for which the security guarantee is better. It is not that hard to find instances of bad coefficients in the literature. Take the Least Significant Bit (LSB) as an example. It has been shown in the literature that $\vec{\omega}_{\text{bad}} = (1, 1, \dots, 1)$ results in *much* worse security bounds than what could expect from Theorem 2, whether it is in a binary field [45], or in a prime field [38]. Does that mean that we could also get *much* better coefficients? We answer positively to this question, by investigating the use-case of physical bit leakage applied to each share of an $\vec{\omega}$ -encoding in a prime field. To this end, we need to introduce some Fourier analysis tools that will serve to derive a more refined security bound than the generic one from Theorem 2, for some well-chosen coefficients.

4.1 Background on Fourier Analysis

We first recall the definition of the discrete Fourier transform of a function. Then, we recall the Poisson summation formula that has been already used in many related works [6].

Definition 9 (Discrete Fourier Transform). *Let $f : \mathbb{F}_p \rightarrow \mathbb{C}$ be a function over \mathbb{F}_p seen as a cyclic group. The discrete Fourier transform of f is the mapping*

$$\alpha \in \mathbb{F}_p \mapsto \widehat{f}(\alpha) = \frac{1}{p} \sum_{x \in \mathbb{F}_p} f(x) \gamma^{\alpha \cdot x} \ ,$$

⁸ Generic reductions proposed in [7,47] could also be considered.

Equivalently, for any $x \in \mathbb{F}_p$, the inverse discrete Fourier transform is the mapping

$$x \in \mathbb{F}_p \mapsto f(x) = \sum_{\alpha \in \mathbb{F}_p} \widehat{f}(\alpha) \cdot \gamma^{-\alpha \cdot x} .$$

Lemma 5 (Poisson Summation Formula). *Let f_1, \dots, f_d be functions $\mathbb{F}_p \rightarrow \mathbb{C}$, and let $\vec{y} + C \subseteq \mathbb{F}_p^d$ be an affine subspace of \mathbb{F}_p^d , with offset $\vec{y} \in \mathbb{F}_p^d$ and kernel C . Then the following equality holds:*

$$\mathbb{E}_{\vec{x} \stackrel{\$}{\leftarrow} \vec{y} + C} \left[\prod_{i=1}^d f_i(x_i) \right] = \sum_{\vec{\alpha} \in C^\perp} \left(\prod_{i=1}^d \widehat{f}_i(\alpha_i) \right) \cdot \gamma^{-\langle \vec{\alpha}, \vec{y} \rangle} .$$

Proof. Expressing every function f_i through its inverse Fourier transform, we get:

$$\begin{aligned} \mathbb{E}_{\vec{x} \stackrel{\$}{\leftarrow} \vec{y} + C} \left[\prod_{i=1}^d f_i(x_i) \right] &= \mathbb{E}_{\vec{x} \stackrel{\$}{\leftarrow} \vec{y} + C} \left[\prod_{i=1}^d \sum_{\alpha_i \in \mathbb{F}_p} \widehat{f}_i(\alpha_i) \cdot \gamma^{-\alpha_i \cdot x_i} \right] \\ &= \mathbb{E}_{\vec{x} \stackrel{\$}{\leftarrow} \vec{y} + C} \left[\sum_{\vec{\alpha} \in \mathbb{F}_p^d} \left(\prod_{i=1}^d \widehat{f}_i(\alpha_i) \cdot \gamma^{-\alpha_i \cdot x_i} \right) \right] \\ &= \sum_{\vec{\alpha} \in \mathbb{F}_p^d} \left(\prod_{i=1}^d \widehat{f}_i(\alpha_i) \right) \cdot \mathbb{E}_{\vec{x} \stackrel{\$}{\leftarrow} \vec{y} + C} \left[\gamma^{-\langle \vec{\alpha}, \vec{x} \rangle} \right] , \end{aligned}$$

where the second equality comes from distributing the product of sums, and the third equality holds thanks to the linearity of the expectation. Let us now focus on the term $\mathbb{E}_{\vec{x} \stackrel{\$}{\leftarrow} \vec{y} + C} \left[\gamma^{-\langle \vec{\alpha}, \vec{x} \rangle} \right]$, for a fixed $\vec{\alpha} \in \mathbb{F}_p^d$, we can express any $\vec{x} \in \vec{y} + C$ as the sum $\vec{y} + (\vec{x} - \vec{y})$, where the term inside the parenthesis belongs to C . Hence:

$$\begin{aligned} \mathbb{E}_{\vec{x} \stackrel{\$}{\leftarrow} \vec{y} + C} \left[\gamma^{-\langle \vec{\alpha}, \vec{x} \rangle} \right] &= \mathbb{E}_{\vec{x} \stackrel{\$}{\leftarrow} \vec{y} + C} \left[\gamma^{-\langle \vec{\alpha}, \vec{y} + (\vec{x} - \vec{y}) \rangle} \right] \\ &= \gamma^{-\langle \vec{\alpha}, \vec{y} \rangle} \cdot \mathbb{E}_{\vec{x} \stackrel{\$}{\leftarrow} C} \left[\gamma^{-\langle \vec{\alpha}, \vec{x} \rangle} \right] , \end{aligned}$$

where the second equality is obtained by a change of variable $\vec{x} - \vec{y} \mapsto \vec{x}$. If $\vec{x} \perp \vec{\alpha}$, then $\gamma^{-\langle \vec{\alpha}, \vec{x} \rangle} = 1$, otherwise, the expectation equals zero. As a consequence, we may restrict the sum over the orthogonal subspace of C . \square

4.2 Inner-Product Masking in the Independence Assumption

In this subsection, and for the remaining of the paper, we focus on a particular class of leakage functions occurring on a single encoding, verifying the so-called *independence assumption*.

Definition 10 (Independence Assumption). A (possibly randomized) function $\tau : \mathbb{F}^d \rightarrow \mathcal{L}$ verifies the independence assumption if there exist some (possibly randomized) functions $\varphi_1, \dots, \varphi_d$ such that for any input $\vec{x} \in \mathbb{F}^d$, (1) the random variables $\varphi_i(x_i)$ are mutually independent; and (2) $\tau(\vec{x})$ and $(\varphi_1(x_1), \dots, \varphi_d(x_d))$, are identically distributed.

The independence assumption is a condition required in the *noisy leakage* model [49,19]. When applied to an encoding generated from inner-product masking, one may rephrase the security metric, as stated hereafter.⁹

Proposition 4. Let $\vec{\omega} \in (\mathbb{F}^*)^d$. Let $\tau : \mathbb{F}^d \rightarrow \mathcal{L}^d$ verifying the independence assumption (Definition 10), and denote by $\varphi_1, \dots, \varphi_d$ the functions $\mathbb{F} \rightarrow \mathcal{L}$ such that for any $\vec{x} \in \mathbb{F}^d$, $\tau(\vec{x}) = (\varphi_1(x_1), \dots, \varphi_d(x_d))$. Let $\mathbf{p}_{\varphi_i, \ell_i}$ be the mapping $x \mapsto \Pr(\varphi_i(x) = \ell_i)$.¹⁰ For any $y^{(0)}, y^{(1)} \in \mathbb{F}_p$, let $L(y^{(i)}) = \tau(\text{Enc}_{\vec{\omega}}(y^{(i)}))$, for $i \in \{0, 1\}$, and denote its probability distribution by \mathbf{p}_i . Then,

$$\text{TV}(\mathbf{p}_0; \mathbf{p}_1) = \frac{1}{2} \sum_{\vec{\ell} \in \mathcal{L}^d} \left| \sum_{\alpha \in \mathbb{F}_p^*} \left(\prod_{i=1}^d \widehat{\mathbf{p}_{\varphi_i, \ell_i}}(\omega_i \cdot \alpha) \right) \cdot \left(\gamma^{-\alpha \cdot y^{(0)}} - \gamma^{-\alpha \cdot y^{(1)}} \right) \right|.$$

Proof. By definition of the total variation,

$$\text{TV}(\mathbf{p}_0; \mathbf{p}_1) = \frac{1}{2} \sum_{\vec{\ell} \in \mathcal{L}^d} \left| \Pr(L(y^{(0)}) = \vec{\ell}) - \Pr(L(y^{(1)}) = \vec{\ell}) \right|.$$

Denote the latter quantity by Δ for short. The independence assumption (see Definition 10) implies that for any $\vec{\ell} \in \mathcal{L}$, and any $\vec{x} \in \mathbb{F}_p^d$,

$$\Pr(\tau(\vec{x}) = \vec{\ell}) = \prod_{i=1}^d \Pr(\varphi_i(x_i) = \ell_i).$$

Injecting this into Δ gives

$$\Delta = \frac{1}{2} \sum_{\vec{\ell} \in \mathcal{L}^d} \left| \mathbb{E}_{\llbracket y \rrbracket_{\vec{\omega}} \stackrel{\$}{\leftarrow} y^{(0)} + \text{Span}(\vec{\omega})^\perp} \left[\prod_{i=1}^d \Pr(\varphi_i(y_i) = \ell_i) \right] - \mathbb{E}_{\llbracket y \rrbracket_{\vec{\omega}} \stackrel{\$}{\leftarrow} y^{(1)} + \text{Span}(\vec{\omega})^\perp} \left[\prod_{i=1}^d \Pr(\varphi_i(y_i) = \ell_i) \right] \right|.$$

Applying Poisson summation formula of Lemma 5 to both terms inside the absolute value, respectively with $\vec{y} = (y^{(0)}, 0, \dots, 0)$, $C = \text{Span}(\vec{\omega})$ for the first term, and $\vec{y} = (y^{(1)}, 0, \dots, 0)$, $C = \text{Span}(\vec{\omega})$ for the second term. It follows that:

$$\Delta = \frac{1}{2} \sum_{\vec{\ell} \in \mathcal{L}^d} \left| \sum_{\vec{\alpha} \in \text{Span}(\vec{\omega})^\perp} \left(\prod_{i=1}^d \widehat{\mathbf{p}_{\varphi_i, \ell_i}}(\alpha_i) \right) \cdot \left(\gamma^{-\langle \vec{\alpha}, \llbracket y^{(0)} \rrbracket \rangle} - \gamma^{-\langle \vec{\alpha}, \llbracket y^{(1)} \rrbracket \rangle} \right) \right|.$$

⁹ Proposition 4 adapts the statement of Faust *et al.* [23, Prop. 3] to IP masking.

¹⁰ This is not a probability mass function as it is a function of y instead of ℓ .

We then notice that the sum over $\text{Span}(\vec{\omega})^\perp$ may be replaced by a simple sum over \mathbb{F}_p :

$$\Delta = \frac{1}{2} \sum_{\vec{\ell} \in \mathcal{L}^d} \left| \sum_{\alpha \in \mathbb{F}_p} \left(\prod_{i=1}^d \widehat{\mathfrak{p}}_{\varphi_i, \ell_i}(\omega_i \cdot \alpha) \right) \cdot \left(\gamma^{-\alpha \cdot \omega_1 \cdot y^{(0)}} - \gamma^{-\alpha \cdot \omega_1 \cdot y^{(1)}} \right) \right| .$$

The proof ends by simplifying $\omega_1 = 1$, and observing that the summand with $\alpha = 0$ equals 0. \square

Proposition 5 (Upper-Bound Lemma). *Let $\vec{\omega} \in (\mathbb{F}_p^*)^d$, and let L be defined as in Proposition 4. Then the following inequality holds:*

$$M_{\vec{\omega}}(L) \leq \sum_{\vec{\ell} \in \mathcal{L}^d} \sum_{\alpha \in \mathbb{F}_p^*} \prod_{i=1}^d |\widehat{\mathfrak{p}}_{\varphi_i, \ell_i}(\omega_i \cdot \alpha)| .$$

Proof. Apply the triangle inequality in the right-hand side of Proposition 4 and observe that $|\gamma^{-\alpha \cdot y^{(0)}} - \gamma^{-\alpha \cdot y^{(1)}}| \leq 2$ for every $y^{(0)}, y^{(1)} \in \mathbb{F}_p$. \square

4.3 Application: the Physical Bit Leakage Model

The statements introduced in the previous subsection hold for any leakage model verifying the independence assumption. We now focus on a specific sub-class of such leakages, namely the *physical-bit leakage* model. In this model, each share is encoded and stored as a binary string in a register, whose one bit is revealed to the adversary. This leakage model is already considered an object of interest in the leakage-resilient secret sharing literature [45,38].

Our goal is to find good coefficients $\vec{\omega}$ such that the worst-case bias can be much better bounded than when the coefficients are chosen randomly. To this end, we will leverage Proposition 5. Hereupon, we need to recall two facts on the Fourier spectrum of the physical-bit leakage.

Proposition 6 (Fourier spectrum of the LSB model [38, Claim 15]).

The Fourier spectrum of the lsb leakage function is given by:

$$\widehat{\mathbf{1}}_{\text{lsb},0}(\alpha) = -\widehat{\mathbf{1}}_{\text{lsb},1}(\alpha) = \frac{1}{2p} \cdot \left(\cos\left(\frac{\alpha \cdot \pi}{p}\right) \right)^{-1} \cdot e^{\frac{i\pi\alpha}{p}} .$$

Proposition 7 (Reduction from ksb to lsb [23, p.15]). *Let ksb be the function that maps a value $x \in \mathbb{F}_p$ to its $(k+1)$ -th significant bit — i.e., the bit of weight 2^k — where p is a Mersenne number. Then, the Fourier spectrum of the ksb function verifies*

$$\widehat{\mathbf{1}}_{\text{ksb},0}(\alpha) = \widehat{\mathbf{1}}_{\text{lsb},0}(2^{-k} \cdot \alpha) .$$

By injecting the Fourier transform of the ksb into Proposition 5, a product of cosine appears. It turns out that for some well-chosen coefficient $\vec{\omega}$, this product can be simplified, as stated by the following lemma.

Lemma 6. *Let p be an odd integer, let $\alpha \in \mathbb{Z}_p^*$, and let d be an integer. Then the following equality holds:*

$$\prod_{i=1}^d \cos\left(\frac{2^{i-1} \cdot \alpha \cdot \pi}{p}\right) = \frac{1}{2^d} \cdot \frac{\sin\left(\frac{2^d \cdot \alpha \cdot \pi}{p}\right)}{\sin\left(\frac{\alpha \cdot \pi}{p}\right)}.$$

Proof. Let

$$\mathcal{C} = \prod_{i=1}^d \cos\left(\frac{2^{i-1} \cdot \alpha \cdot \pi}{p}\right), \text{ and } \mathcal{S} = \prod_{i=1}^d \sin\left(\frac{2^{i-1} \cdot \alpha \cdot \pi}{p}\right).$$

By rearranging the factors in the product $\mathcal{C} \cdot \mathcal{S}$, and leveraging the identity $\sin(2x) = 2 \cos(x) \sin(x)$, we get that

$$\mathcal{C} \cdot \mathcal{S} = \frac{1}{2^d} \cdot \prod_{i=1}^d \sin\left(\frac{2^i \cdot \alpha \cdot \pi}{p}\right) = \frac{1}{2^d} \cdot \prod_{i=2}^{d+1} \sin\left(\frac{2^{i-1} \cdot \alpha \cdot \pi}{p}\right).$$

Introducing an extra factor for $i = 1$, and extracting the factor of index $i = d+1$ in the product, we get that

$$\mathcal{C} \cdot \mathcal{S} = \frac{1}{2^d} \cdot \sin\left(\frac{2^d \alpha \pi}{p}\right) \cdot \sin\left(\frac{\alpha \pi}{p}\right)^{-1} \cdot \prod_{i=1}^d \sin\left(\frac{2^{i-1} \cdot \alpha \cdot \pi}{p}\right).$$

We conclude by identifying the product in the latter right-hand side as \mathcal{S} . Since p is prime, there is no value of $\alpha \in \mathbb{F}_p^*$ such that $\mathcal{S} = 0$. Hence, we may divide each side by \mathcal{S} . \square

Proposition 8 (Good even coefficients). *Let $\text{ksb}_1, \dots, \text{ksb}_d$ be respectively the $(k_1 + 1)$ -th, \dots , $(k_d + 1)$ -th significant-bit leakage functions, for $k_1, \dots, k_d \in \llbracket 0, n - 1 \rrbracket$.*

Let $\vec{\omega} = (2^{k_1}, \dots, 2^{k_1+i-1}, \dots, 2^{k_d+d-1})$, and let $L : y \mapsto \tau(\text{Enc}_{\vec{\omega}}(y))$, where $\tau(\vec{x}) = (\text{ksb}_1(x_1), \dots, \text{ksb}_d(x_d))$. Then,

$$M_{\vec{\omega}}(L) \leq \left(\frac{2}{p}\right)^d \cdot p \cdot \left(\log \left\lfloor \frac{p-1}{2} \right\rfloor + 1\right).$$

Proof. Applying Proposition 5 to the ksb leakage model, we have

$$M_{\vec{\omega}}(L) \leq \sum_{\vec{\ell} \in \mathcal{L}^d} \sum_{\alpha \in \mathbb{F}_p^*} \prod_{i=1}^d |\widehat{\text{psb}}_{\text{ksb}_i, \ell_i}(\omega_i \cdot \alpha)|.$$

Proposition 7 and Proposition 6 imply that for any $\vec{\ell}$ we have

$$\sum_{\alpha \in \mathbb{F}_p^*} \prod_{i=1}^d |\widehat{\text{psb}}_{\text{ksb}_i, \ell_i}(\omega_i \cdot \alpha)| = \sum_{\alpha \in \mathbb{F}_p^*} \prod_{i=1}^d \frac{1}{2p} \cdot \left| \cos\left(\frac{2^{i-1} \cdot \alpha \cdot \pi}{p}\right) \right|^{-1}, \quad (6)$$

hence,

$$\begin{aligned} M_{\vec{\omega}}(\mathbf{L}) &\leq 2^d \cdot \sum_{\alpha \in \mathbb{F}_p^*} \prod_{i=1}^d \frac{1}{2p} \cdot \left| \cos\left(\frac{2^{i-1} \cdot \alpha \cdot \pi}{p}\right) \right|^{-1} \\ &= \frac{1}{p^d} \cdot \sum_{\alpha \in \mathbb{F}_p^*} \prod_{i=1}^d \left| \cos\left(\frac{2^{i-1} \cdot \alpha \cdot \pi}{p}\right) \right|^{-1}. \end{aligned}$$

Invoking Lemma 6, we get that:

$$\mathrm{TV}\left(\mathbf{L}(y^{(0)}); \mathbf{L}(y^{(1)})\right) \leq \left(\frac{2}{p}\right)^d \cdot \sum_{\alpha \in \mathbb{F}_p^*} \left| \frac{\sin\left(\frac{\alpha \cdot \pi}{p}\right)}{\sin\left(\frac{2^d \cdot \alpha \cdot \pi}{p}\right)} \right|.$$

We conclude the proof by using Lemma 7. \square

Lemma 7. *Let p be a prime number. Then, for any integer d ,*

$$\sum_{\alpha \in \mathbb{F}_p^*} \left| \frac{\sin\left(\frac{\alpha \cdot \pi}{p}\right)}{\sin\left(\frac{2^d \cdot \alpha \cdot \pi}{p}\right)} \right| \leq p \cdot \left(\log \left\lfloor \frac{p-1}{2} \right\rfloor + 1 \right).$$

In order to prove the lemma, we need the following identity.

Claim. Let p be a prime number, and let $x, y \in \mathbb{F}_p$. Then,

$$\left| \sin\left(x \cdot y \cdot \frac{\pi}{p}\right) \right| = \sin\left((x \otimes y) \cdot \frac{\pi}{p}\right),$$

where \otimes denotes the field multiplication in \mathbb{F}_p .

Proof. Observe that the function $t \mapsto |\sin(t)|$ is π -periodic, so for any $x, y \in \mathbb{Z}_p$,

$$\begin{aligned} \left| \sin\left(\frac{x \cdot y}{p} \cdot \pi\right) \right| &= \sin\left(\left(\frac{x \cdot y}{p} \cdot \pi\right) [\pi]\right) \\ &= \sin\left(\frac{x \cdot y}{p} [1] \cdot \pi\right) \\ &= \sin\left((x \cdot y) [p] \cdot \frac{\pi}{p}\right) \\ &= \sin\left((x \otimes y) \cdot \frac{\pi}{p}\right). \end{aligned}$$

\square

Proof of Lemma 7. Let S be the sum to bound. By upper bounding the numerator of each summand by one, we get that

$$S \leq \sum_{\alpha \in \mathbb{F}_p^*} \left| \sin\left(\alpha \cdot 2^d \cdot \frac{\pi}{p}\right) \right|^{-1}.$$

Now we use the previous claim to make the change of variable $\alpha \leftarrow \alpha \otimes 2^d$ in the latter sum. As a result,

$$S \leq \sum_{\alpha \in \mathbb{F}_p^*} \sin\left(\alpha \otimes 2^d \cdot \frac{\pi}{p}\right)^{-1} = \sum_{\alpha \in \mathbb{F}_p^*} \sin\left(\frac{\alpha \cdot \pi}{p}\right)^{-1} .$$

We now use the symmetry of the sine with respect to $\frac{\pi}{2}$:

$$S \leq 2 \cdot \sum_{\alpha=1}^{\lfloor \frac{p-1}{2} \rfloor} \sin\left(\frac{\alpha \cdot \pi}{p}\right)^{-1} .$$

Observe now that for any $x \in [0, \frac{\pi}{2}]$, $\sin(x) \geq \frac{2x}{\pi}$, *i.e.*, $\sin(x)^{-1} \leq \frac{\pi}{2x}$. Hence,

$$S \leq 2 \cdot \sum_{\alpha=1}^{\lfloor \frac{p-1}{2} \rfloor} \frac{\pi}{2 \cdot \left(\frac{\alpha\pi}{p}\right)} = p \cdot \sum_{\alpha=1}^{\lfloor \frac{p-1}{2} \rfloor} \frac{1}{\alpha} \leq p \cdot \left(\log \left\lfloor \frac{p-1}{2} \right\rfloor + 1\right) \leq p \cdot (\log(p) + 1) .$$

□

We have emphasized one good combination of even inner-product coefficients, using powers of two. The following proposition allows to derive a combination of odd coefficients leading to the same result.

Proposition 9. *If $\vec{\omega}$ is a good choice of even coefficients, then $-\vec{\omega}$ is an as good choice of odd coefficients.*

Proof. Observe that since p is prime, $-\omega$ is odd if and only if ω is even. Then, leverage the parity of the cosine function in Equation 6. □

Corollary 2 (Good odd coefficients). *Let $\text{ksb}_1, \dots, \text{ksb}_d$ be respectively the $(k_1 + 1)$ -th, \dots , $(k_d + 1)$ -th significant-bit leakage functions, for $k_1, \dots, k_d \in \llbracket 0, n - 1 \rrbracket$.*

Let $\vec{\omega} = -(2^{k_1}, \dots, 2^{k_1+i-1}, \dots, 2^{k_d+d-1})$, and let $L : y \mapsto \tau(\text{Enc}_{\vec{\omega}}(y))$, where $\tau(\vec{x}) = (\text{ksb}_1(x_1), \dots, \text{ksb}_d(x_d))$. Then,

$$M_{\vec{\omega}}(L) \leq \left(\frac{2}{p}\right)^d \cdot p \cdot \left(\log \left\lfloor \frac{p-1}{2} \right\rfloor + 1\right) .$$

In other words, upon a good choice of $\vec{\omega}$, we may improve the security bound from $\mathcal{O}\left(2^n \cdot \sqrt{2^{-nd+n+d}}\right)$ with Theorem 2 to $\mathcal{O}(n \cdot 2^{-dn+n+d})$ with a finer-grain analysis. In particular, these asymptotic bounds suggest that the designer has an almost similar incentive in increasing the bit-size as increasing the masking order. This may be helpful if one is cheaper than the other.

Interestingly, such an improvement of the security bound by wisely choosing the coefficients may happen because the bound obtained with additive masking was much worse. It has been emphasized that the amplitude of the Fourier spectrum of the `lsb` leakage function is extremely concentrated over two points [38,23],

which was the root cause of a tight security bound independent of the field size. By carefully choosing the coefficients in Proposition 8 and Corollary 2, we somehow leverage Proposition 4 to break this concentration. This suggests that a similar analysis may be applied to other leakage functions suffering from bad security bounds for some given inner-product coefficients. It is likely that they would suffer from the concentration of the Fourier spectrum as well. If so, a careful Fourier analysis may give some hints to derive good inner-product coefficients. Overall, this case study depicts how a designer could take profit from leakage characterization to derive better security bounds (holding with probability one) than that of Theorem 2, for some well-chosen coefficients.

A Reduction from Non-Uniform to Uniform Secrets

Proof of Proposition 1. The first inequality is proven by using Jensen’s inequality, thanks to the convexity of the total variation, and by bounding the expectation by the supremum. The second inequality is proven as follows: let $\vec{\ell} \in \mathcal{L}$, and denote $\Delta_{\vec{\ell}} = \left| \Pr\left(\mathbf{L}(y^{(0)}) = \vec{\ell} \mid \vec{\Omega} = \vec{\omega}\right) - \Pr\left(\mathbf{L}(y^{(1)}) = \vec{\ell} \mid \vec{\Omega} = \vec{\omega}\right) \right|$ for short. Then we have

$$\begin{aligned} \Delta_{\vec{\ell}} &= \left| \Pr\left(\mathbf{L} = \vec{\ell} \mid \mathbf{Y} = y^{(0)}, \vec{\Omega} = \vec{\omega}\right) - \Pr\left(\mathbf{L} = \vec{\ell} \mid \mathbf{Y} = y^{(1)}, \vec{\Omega} = \vec{\omega}\right) \right| \\ &= |\mathbb{F}| \cdot \left| \Pr\left(\mathbf{L} = \vec{\ell}, \mathbf{Y} = y^{(0)} \mid \vec{\Omega} = \vec{\omega}\right) - \Pr\left(\mathbf{L} = \vec{\ell}, \mathbf{Y} = y^{(1)} \mid \vec{\Omega} = \vec{\omega}\right) \right| \\ &\leq |\mathbb{F}| \cdot \left| \Pr\left(\mathbf{L} = \vec{\ell}, \mathbf{Y} = y^{(0)} \mid \vec{\Omega} = \vec{\omega}\right) - \Pr\left(\mathbf{L} = \vec{\ell} \mid \vec{\Omega} = \vec{\omega}\right) \cdot \Pr\left(\mathbf{Y} = y^{(0)}\right) \right| \\ &\quad + |\mathbb{F}| \cdot \left| \Pr\left(\mathbf{L} = \vec{\ell} \mid \vec{\Omega} = \vec{\omega}\right) \cdot \Pr\left(\mathbf{Y} = y^{(1)}\right) - \Pr\left(\mathbf{L} = \vec{\ell}, \mathbf{Y} = y^{(1)} \mid \vec{\Omega} = \vec{\omega}\right) \right| \\ &\leq |\mathbb{F}| \cdot \sum_{y \in \mathbb{F}} \left| \Pr\left(\mathbf{L} = \vec{\ell}, \mathbf{Y} = y \mid \vec{\Omega} = \vec{\omega}\right) - \Pr\left(\mathbf{L} = \vec{\ell} \mid \vec{\Omega} = \vec{\omega}\right) \cdot \Pr(\mathbf{Y} = y) \right| \end{aligned}$$

Here, the first equality holds by making the knowledge of $y^{(0)}, y^{(1)}$ explicit. The second equality holds by definition of the conditional probability, and using the fact that \mathbf{Y} is considered uniform here. The first inequality comes from the triangle inequality, while the second inequality holds since we upper-bound two positive terms of a sum by the whole sum over \mathbb{F} . We conclude the proof by summing $\Delta_{\vec{\ell}}$ over \mathcal{L} . \square

B Leftover Hash Lemma

Definition 11. *The collision probability $\text{CP}(X)$ of a random variable X is defined as the probability that two independent samples of X are equal, i.e.,*

$$\text{CP}(X) = \Pr(X = X')$$

where X' is an independent copy of X .

Proposition 10 (Simple Properties of the Collision Probability). *Let X be a random variable over \mathcal{X} .*

1. *Let \mathbf{p}_X be the probability mass function of X . Then $\text{CP}(X) = \|\mathbf{p}_X\|_2^2$.*
2. *$\text{CP}(X) \leq 2^{-H_\infty(X)}$.*

Proof. item 1 follows from the chain of equalities below.

$$\text{CP}(X) = \mathbb{P}_{X \sim X'}[X = X'] = \sum_{x \in \mathcal{X}} \mathbb{P}[X = x]^2 = \sum_{x \in \mathcal{X}} \mathbf{p}_X(x)^2 = \|\mathbf{p}_X\|_2^2.$$

We prove item 2 as follows

$$\text{CP}(X) = \sum_{x \in \mathcal{X}} \mathbb{P}[X = x]^2 \leq \max_{x \in \mathcal{X}} \mathbb{P}[X = x] \cdot \sum_{x \in \mathcal{X}} \mathbb{P}[X = x] = 2^{-H_\infty(X)}.$$

□

Proof of Theorem 1. The proof¹¹ is divided into three main parts. First, we upper bound the collision probability of the extractor. Next, we use this to bound the euclidean distance between the extractor's output and true randomness. Finally, we convert the euclidean distance into total variation.

Part 1. The collision probability for the joint random variable $(h_\Omega(X), \Omega)$ is

$$\begin{aligned} \text{CP}(h_\Omega(X), \Omega) &= \Pr(h_\Omega(X) = h_{\Omega'}(X'), \Omega = \Omega') \\ &= \Pr(\Omega = \Omega') \cdot \Pr(h_\Omega(X) = h_{\Omega'}(X') \mid \Omega = \Omega') \end{aligned}$$

Let us elaborate on the second factor of the right-hand side, that we rephrase as $\Pr(h_\Omega(X) = h_\Omega(X'))$. The total probability formula tells us that

$$\begin{aligned} \Pr(h_\Omega(X) = h_\Omega(X')) &= \Pr(h_\Omega(X) = h_\Omega(X') \mid X \neq X') \cdot \Pr(X \neq X') \\ &\quad + \Pr(h_\Omega(X) = h_\Omega(X') \mid X = X') \cdot \Pr(X = X') . \end{aligned}$$

Since \mathcal{H} is universal, we may bound the first term in the previous sum as follows:

$$\Pr(h_\Omega(X) = h_\Omega(X') \mid X \neq X') \cdot \Pr(X' \neq X) \leq \frac{1}{|\mathcal{Y}|} .$$

As per the second term of the sum, it can be trivially upper-bounded by the collision probability $\text{CP}(X)$, which in turn can be upper bounded using its min entropy (Proposition 10, item 2). As a result,

$$\text{CP}(h_\Omega(X), \Omega) \leq \frac{1}{|\mathcal{W}|} \left(\frac{1}{|\mathcal{Y}|} + \frac{1}{2^k} \right) = \frac{1 + 4\epsilon^2}{|\mathcal{W}| \cdot |\mathcal{Y}|} .$$

Here, the equality comes from defining $\epsilon = |\mathcal{Y}|^{\frac{1}{2}} \cdot 2^{-\frac{k}{2}-1}$.

¹¹ The proof comes from Reyzin's lecture notes, we have just adapted the notation.

Part 2. Let ν be the vector corresponding to the difference between the probability distributions of $(h_\Omega(X), \Omega)$ and $(U_{\mathcal{Y}}, U_{\mathcal{W}})$. This means that

$$\begin{aligned}
\|\nu\|_2^2 &= \|\Pr(h_\Omega(X), \Omega) - \Pr(U_{\mathcal{Y}}, U_{\mathcal{W}})\|_2^2 \\
&= \sum_{y, \omega} \left(\Pr(h_\Omega(X) = y, \Omega = \vec{\omega}) - \frac{1}{|\mathcal{Y}|} \cdot \frac{1}{|\mathcal{W}|} \right)^2 \\
&= \sum_{y, \omega} \Pr(h_\Omega(X) = y, \Omega = \vec{\omega})^2 \\
&\quad - 2 \cdot \sum_{y, \omega} \Pr(h_\Omega(X) = y, \Omega = \vec{\omega}) \cdot \frac{1}{|\mathcal{Y}|} \cdot \frac{1}{|\mathcal{W}|} \\
&\quad + \sum_{y, \omega} \left(\frac{1}{|\mathcal{Y}|} \cdot \frac{1}{|\mathcal{W}|} \right)^2 \\
&= \text{CP}(h_\Omega(X), \Omega) - \frac{2}{|\mathcal{Y}| \cdot |\mathcal{W}|} + \frac{1}{|\mathcal{Y}| \cdot |\mathcal{W}|} \\
&\leq \frac{4 \cdot \epsilon^2}{|\mathcal{Y}| \cdot |\mathcal{W}|} ,
\end{aligned}$$

where the first two equalities hold by definition of norm 2, the third equality is obtained by distributing the square for each term of the sum, the fourth equality holds by Proposition 10, item 1, and the last inequality follows from part 1.

Part 3. Let u_i be the sign of the i -th entry of the vector ν , *i.e.*, such that ν can be expressed as $\nu = (\dots, u_i \cdot |\nu_i|, \dots)$. Then Cauchy-Schwarz inequality states that

$$\|\nu\|_1 = \langle \nu, u \rangle \leq \|\nu\|_2 \cdot \|u\|_2 .$$

Given that $\|\nu\|_2 \leq \frac{2 \cdot \epsilon}{\sqrt{|\mathcal{Y}| \cdot |\mathcal{W}|}}$, and $\|u\|_2 = \sqrt{|\mathcal{Y}| \cdot |\mathcal{W}|}$, we get that

$$\text{TV}(\Pr(h_\Omega(X), \Omega); \Pr(U_{\mathcal{Y}}, U_{\mathcal{W}})) = \frac{1}{2} \|\nu\|_1 \leq \epsilon ,$$

which completes the proof. \square

C Invariance of Inner-Product Masking

In order to prove Proposition 2, we leverage the following results. The first one allows to convert an $\vec{\omega}$ -encoding into a $\vec{1}$ -encoding, and inversely.

Proposition 11. *For any fixed $y \in \mathbb{F}$, and any $\vec{\omega} \in (\mathbb{F}_p^*)^d$ the vector $\llbracket y \rrbracket_{\vec{1}} = (y_1, \dots, y_d)$ is a $\vec{1}$ -encoding of y if and only if the vector $\llbracket y \rrbracket_{\vec{\omega}} = (\omega_1^{-1} \cdot y_1, \dots, \omega_d^{-1} \cdot y_d)$ is an $\vec{\omega}$ -encoding of y .*

Proposition 11 straightforwardly implies the following corollary.

Corollary 3. *For any $y \in \mathbb{F}$, the random vector $\llbracket y \rrbracket_{\vec{\omega}}$ is uniformly distributed over $y + \text{Span}(\vec{\omega})^\perp$ if and only if $\llbracket y \rrbracket_{\vec{1}}$ is uniformly distributed over $y + \text{Span}(\vec{1})^\perp$.*

We are now ready to prove Proposition 2.

Proof of Proposition 2. Define the pointwise product $\psi_{\vec{\omega}} : \vec{x} \mapsto \vec{\omega} \odot \vec{x}$.¹² Let $\llbracket y \rrbracket_{\vec{\omega}}$ be the uniform random vector over $y + \text{Span}(\vec{\omega})^\perp$. Then, by applying Corollary 3 twice, the random vector $\llbracket y \rrbracket_{\vec{\omega}'} = \psi_{\vec{\omega}'}^{-1} \circ \psi_{\vec{\omega}}(\llbracket y \rrbracket_{\vec{\omega}})$ is uniform over $y + \text{Span}(\vec{\omega}')^\perp$. Moreover, for any function $\tau : \mathbb{F}^d \rightarrow \mathcal{L}$, we have

$$\tau(\llbracket y \rrbracket_{\vec{\omega}}) = \tau(\psi_{\vec{\omega}}^{-1} \circ \psi_{\vec{\omega}'}(\llbracket y \rrbracket_{\vec{\omega}'})) .$$

Define therefore $\tau' : \vec{x} \mapsto \tau \circ \psi_{\vec{\omega}}^{-1} \circ \psi_{\vec{\omega}'}(\vec{x})$. It follows that for any $\vec{\ell} \in \mathcal{L}$,

$$\mathbb{E}_{\llbracket y \rrbracket_{\vec{\omega}} \leftarrow y + \text{Span}(\vec{\omega})^\perp} \left[\Pr\left(\tau(\llbracket y \rrbracket_{\vec{\omega}}) = \vec{\ell}\right) \right] = \mathbb{E}_{\llbracket y \rrbracket_{\vec{\omega}'} \leftarrow y + \text{Span}(\vec{\omega}')^\perp} \left[\Pr\left(\tau'(\llbracket y \rrbracket_{\vec{\omega}'}) = \vec{\ell}\right) \right] .$$

It remains to prove that τ' is m -bounded. This follows from the fact that τ and τ' have the same image space. Interestingly, it is clear from the definition of τ' that if τ verifies the independence assumption, then so does τ' . \square

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¹² Notice that $\psi_{\vec{\omega}}$ is invertible, and its inverse corresponds to $\psi_{\vec{\omega}^{-1}}$, where $\vec{\omega}^{-1} = (\omega_1^{-1}, \dots, \omega_d^{-1})$.

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