New Experimental Evidences For the Riemann Hypothesis

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Abstract

The zeta function $\boldsymbol{\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}}$ is convergent for $\mathrm{Re}(\boldsymbol{z}) > \boldsymbol{1}$, and the eta function $\boldsymbol{\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}}$ is convergent for $\mathrm{Re}(\boldsymbol{z}) > \boldsymbol{0}$. The eta function and the analytic continuation of zeta function have the same zeros in the critical strip $\boldsymbol{0} < \mathrm{Re}(\boldsymbol{z}) < \boldsymbol{1}$, owing to that $\boldsymbol{\eta(z) = \left(1 - 2^{1-z}\right)\zeta(z)}$. In this paper, we present the new experimental evidences which show that for any $\boldsymbol{a \in (0,1), b \in (-\infty, \infty)}$, there exists a zero $\boldsymbol{\frac{1}{2} + it}$ such that the modulus $\boldsymbol{|\eta(a+ib)| \geq |\eta(a+it)| > |\eta(\frac{1}{2}+it)| = 0}$. These evidences further confirm that all zeros are on the critical line $\mathrm{Re}(\boldsymbol{z}) = \boldsymbol{\frac{1}{2}}$.

Keywords: Riemann zeta function, Dirichlet eta function, partial sum, absolute convergence.

1 Introduction

The Riemann zeta function is represented as [1]

$$\begin{aligned}\zeta(z) &= \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} e^{-z\ln n} \xlongequal[a,b\in\mathbb{R}]{z=a+ib} \sum_{n=1}^{\infty} e^{-(a+ib)\ln n} \\ &= \sum_{n=1}^{\infty} e^{-a\ln n} e^{-ib\ln n} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^a \cos(b\ln n) - i\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^a \sin(b\ln n) \end{aligned} \tag{1}$$

If $a > 1$, both $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^a \cos(b\ln n)$ and $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^a \sin(b\ln n)$ are absolutely convergent. Therefore, $\zeta(z)$ has no zeros for $a > 1$.

By the famous functional equation [2, 3],

$$\zeta(z) = 2^z \pi^{z-1} \sin \frac{z\pi}{2} \Gamma(1-z)\zeta(1-z), \tag{2}$$

we know $\zeta(z)$ *and* $\zeta(1-z)$ *cannot be concurrently convergent*, because at least one of $\mathrm{Re}(z)$ and $\mathrm{Re}(1-z)$ is strictly smaller than 1. So, $\zeta(z)$ and $\zeta(1-z)$ must be two different branches of the analytic continuation of the original series on the complex plane. The famous Riemann zeros are not for the original series, instead for a branch of its analytic continuation.

The general method to test these zeros needs to use the Riemann-Siegel function, which is defined by $Z(t) = e^{i\vartheta(t)}\zeta(\frac{1}{2}+it)$. If $Z(t_1)$ and $Z(t_2)$ have opposite signs, $Z(t)$ vanishes between t_1 and t_2, and so $\zeta(z)$ has a zero on the critical line between $\frac{1}{2}+it_1$ and $\frac{1}{2}+it_2$. We currently know that for the first ten zeros

$$\begin{aligned} t =& 14.134725142, \quad 21.022039639, \quad 25.010857580, \quad 30.424876126, \quad 32.935061588, \\ & 37.586178159, \quad 40.918719012, \quad 43.327073281, \quad 48.005150881, \quad 49.773832478. \end{aligned}$$

The Dirichlet eta function [4] is the alternating series

$$\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}, \quad z \in \mathbb{C}. \tag{3}$$

$\eta(0)$ is defined to be $1/2$. $\eta(1) = \ln 2$, $\eta(2) = \frac{\pi^2}{12}$. Notice that, for $\mathrm{Re}(z) > 1$

$$\begin{aligned} \frac{2}{2^z}\zeta(z) &= \frac{2}{2^z}\left(1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots\right) = \frac{2}{2^z} + \frac{2}{4^z} + \frac{2}{6^z} + \frac{2}{8^z} + \cdots, \\ \left(1 - \frac{2}{2^z}\right)\zeta(z) &= \left(1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots\right) - \left(\frac{2}{2^z} + \frac{2}{4^z} + \frac{2}{6^z} + \frac{2}{8^z} + \cdots\right) \\ &\xlongequal{\text{rearranged}} 1 + \left(\frac{1}{2^z} - \frac{2}{2^z}\right) + \frac{1}{3^z} + \left(\frac{1}{4^z} - \frac{2}{4^z}\right) + \cdots \\ &= 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z} = \eta(z). \end{aligned}$$

Extending this relationship $\eta(z) = (1-2^{1-z})\zeta(z)$ to the complex plane, we can obtain the functional equation $\zeta(z) = 2^z\pi^{z-1}\sin\frac{z\pi}{2}\Gamma(1-z)\zeta(1-z)$. If $z = -2, -4, -6, \cdots$, $\sin\frac{z\pi}{2} = 0$. These values are called simple zeros of $\zeta(z)$. Since $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, we know $\Gamma(1-z)$ has no zeros. So, $\zeta(z) = 0$ iff $\zeta(1-z) = 0$, which also implies that $\zeta(\bar{z}) = \zeta(1-\bar{z}) = 0$. The famous Riemann hypothesis [5] claims that all the complex zeros of $\zeta(z)$ lie on the critical line $\mathrm{Re}(z) = 1/2$. In history, the zeros of $\zeta(z)$ were very hard to calculate [6]. Nowadays, several million zeros have been obtained [7]. We refer to the table of zeros https://www-users.cse.umn.edu/~odlyzko/zeta_tables/index.html.

In this paper, we present a simple method to test these famous zeros by checking the partial sums of $\eta(z)$. Based on this method, we present the new experimental evidences which show that for any $a \in (0,1), b \in (-\infty, \infty)$, there exists a zero $\frac{1}{2} + it$ such that the modulus $|\eta(a+ib)| \geq |\eta(a+it)| > |\eta(\frac{1}{2}+it)| = 0$. These evidences definitely validate the Riemann hypothesis. To the best of our knowledge, it is first time to invent such a method to confirm the famous hypothesis.

2 The general method to test zeros

Define the functions

$$\chi(z) = 2^{z-1}\pi^z \sec\frac{z\pi}{2} \,/\, \Gamma(z), \quad \vartheta = \vartheta(t) = -\frac{|\chi(\frac{1}{2}+it)|}{2} \arg\chi(\frac{1}{2}+it),$$

where $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\mathrm{d}x$, and the Riemann-Siegel function

$$Z(t) = e^{i\vartheta(t)}\zeta(\frac{1}{2}+it) \tag{4}$$

which is real for real values of t. The Riemann-Siegel theta function appearing above is also defined by

$$\vartheta(t) = \arg[\Gamma(\frac{1}{4}+\frac{1}{2}it)] - \frac{t}{2}\ln\pi. \tag{5}$$

If $Z(t_1)$ and $Z(t_2)$ have opposite signs, $Z(t)$ vanishes between t_1 and t_2, and so $\zeta(z)$ has a zero on the critical line between $\frac{1}{2}+it_1$ and $\frac{1}{2}+it_2$.

To calculate the first nontrivial zero, one needs to determine the sign of $Z(0) = e^{i\vartheta(\frac{1}{2})}\zeta(\frac{1}{2})$. If $z = 1/2$, $\eta(1/2) = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$, which converges to a positive number. Since $\eta(1/2) = (1-2^{1/2})\zeta(1/2)$ and $1-\sqrt{2} < 0$, it claims that $\zeta(1/2) < 0$ (page 388, Ref.[2]). Define

$$\xi(z) = \frac{1}{2}z(z-1)\pi^{-\frac{z}{2}}\Gamma(\frac{z}{2})\zeta(z). \tag{6}$$

Hence, $\xi(1/2) = -\frac{1}{8}\pi^{-\frac{1}{4}}\Gamma(\frac{1}{4})\zeta(1/2)$. Since $\zeta(\frac{1}{2}) < 0$ and $\Gamma(\frac{1}{4}) > 0$, then $\xi(\frac{1}{2}) > 0$, which implies $Z(0) < 0$. By numerical analysis, it shows that $Z(6\pi) > 0$. Therefore, there is one zero at least on the critical line between $t = 0$ and $t = 6\pi$.

3 A new method to test zeros

As we mentioned before, the eta function and the analytic continuation have the same zeros in the critical strip $0 < \mathrm{Re}(z) < 1$. For the first three zeros

$$r_1 = 1/2 + 14.134725\,i, \quad r_2 = 1/2 + 21.0220396\,i, \quad r_3 = 1/2 + 25.01085758\,i,$$

we have the following numerical calculations (see Table 1). With the finite precision, we have the faith in that the three values are really zeros of eta function.

Table 1: Numerical calculations for the first three zeros

Partial-sum	$\eta(1/2+14.134725\,i)=c-di$	$\lvert c\rvert+\lvert d\rvert$
700000	0.00010144 +0.000588673 I	0.000690113
800000	0.00049395 -0.000261985 I	0.000755936
900000	-0.000289405-0.000440756 I	0.000730162
Partial-sum	$\eta(1/2+21.0220396\,i)=c-di$	$\lvert c\rvert+\lvert d\rvert$
700000	-0.000586334+0.000110852 I	0.000697186
800000	0.000553754 +0.0000797987 I	0.000633552
900000	-0.000362282-0.000383504 I	0.000745786
Partial-sum	$\eta(1/2+25.010857\,i)=c-di$	$\lvert c\rvert+\lvert d\rvert$
700000	0.000534317 -0.000268088 I	0.000802405
800000	-0.000439384+0.000345612 I	0.000784996
900000	0.000470723 -0.00023747 I	0.000708193

Theorem 3.1. *Let $z=\frac{1}{2}+bi, b>0$. Denote the partial sum $\sum_{n=1}^{k}\frac{(-1)^{n-1}}{n^z}$ by $c-di$, for some positive integer k. Then the modulus $|c-di|$ is continuous with respect to b.*

Proof. It is easy to see that

$$c-di=\sum_{n=1}^{k}\frac{(-1)^{n-1}}{n^z}=\sum_{n=1}^{K}(-1)^{n-1}e^{-z\ln n}\overset{z=\frac{1}{2}+ib}{=\!=\!=\!=}\sum_{n=1}^{k}(-1)^{n-1}e^{-(\frac{1}{2}+ib)\ln n}$$

$$=\sum_{n=1}^{k}(-1)^{n-1}\sqrt{\frac{1}{n}}\cos(b\ln n)-i\sum_{n=1}^{k}(-1)^{n-1}\sqrt{\frac{1}{n}}\sin(b\ln n),$$

$$|c-di|^2=\left(\sum_{n=1}^{k}(-1)^{n-1}\sqrt{\frac{1}{n}}\cos(b\ln n)\right)^2+\left(\sum_{n=1}^{k}(-1)^{n-1}\sqrt{\frac{1}{n}}\sin(b\ln n)\right)^2.$$

Since all $\cos(b\ln n), \sin(b\ln n), n=1,\cdots,k$, are continuous with respect to b, the above modulus is also continuous with respect to b. □

Likewise, we have the following theorem.

Theorem 3.2. *Let $z=a+ti, 0<a<1$. Denote the partial sum $\sum_{n=1}^{k}\frac{(-1)^{n-1}}{n^z}$ by $c-di$, for some positive integer k. Then the modulus $|c-di|$ is continuous with respect to a.*

Based on these theorems, we now present a new method (see Algorithm 1) to search for a zero in a short interval. Let $s_k:=\sum_{n=1}^{k}(-1)^{n-1}e^{-(\frac{1}{2}+ib)\ln n}$. We compute the mean of partial sums $s_{k_1}, s_{k_2},\cdots,s_{k_\ell}$, so as to partly offset the roundoff errors.

Theorem 3.3. *The computational cost for Algorithm 1 is $O(50k_\ell(3.32p+\log_2(k_\ell))^2)$, where p is the accuracy, i.e., the effective number of these digits which appear to the right of the decimal point.*

Proof. The longest binary length of operands in the procedure is $\log_2(k_\ell)$ (for integer part) plus $\log_2(10^p)$ (for fractional part). The total iteration number is *stepnum* $\times k_\ell$.

Usually, $stepnum = 50$ which suffices to determine the local minimums in a short interval. Note that $\log_2(10) \approx 3.32$. So, the computational cost for a multiplication is $O((3.32p + \log_2(k_\ell))^2)$, and the total cost is $O(50k_\ell(3.32p + \log_2(k_\ell))^2)$. □

Algorithm 1: Testing zeros of Dirichlet eta series in the critical strip

Input: $(b_1, b_2), b_2 > b_1 > 0$, which contains at least one zero of eta series, and a set of positive integers $K = \{k_1, k_2, \cdots, k_\ell\}$, $k_1 < k_2 < \cdots < k_\ell$.

Output: $(c, d) \subset (b_1, b_2)$, which contains at least one zero of eta series.

```
steplen ← 1/4 (or 1/32, 1/256, etc), stepnum ← (b_2 − b_1)/steplen
l ← 0, r ← 0, T ← { }                                   // T is the empty set
for j = 0, j ≤ stepnum do
    b ← b_1 + steplen ∗ j, S ← { } for n = 1, n ≤ k_ℓ do
        l ← l + (−1)^(n−1) √(1/n) cos(b ln n)
        r ← r + (−1)^(n−1) √(1/n) sin(b ln n)
        if n ∈ K then
            s ← l − ri                                  // i^2 = −1
            S ← S ∪ {|s|}                               // |s| is the modulus of s
    t ← the mean value of S
    T ← T ∪ {(b, t)}
Find (b̂, t̂) ∈ T, with a local minimum t̂
c ← b̂ − steplen, d ← b̂ + steplen
```

The following Mathematica code can be directly used to test the zeros, in which we take $k_\ell = 5000$.

```
Eta1[a_, b_, k_, mylist_] := Module[{n, l, r, s, t, U, V, precision},
   l = r = 0; U = V = {}; precision = 10;
   For[n = 1, n <= k, n++,
     l = N[l + (-1)^(n - 1)/(n^a)*Cos[b*Log[n]], precision];
     r = N[r + (-1)^(n - 1)/(n^a)*Sin[b*Log[n]], precision];
     If[MemberQ[mylist, n], s = l - r*I; t = Abs[s];
     U = AppendTo[U, {n, s, t}]]]; V = U];
Eta2[a_, b1_, b2_, steplen_, k_, mylist_] :=
 Module[{A, B, stepnum, b, j, W, v, d, precision},
   A = B = W = {}; precision = 10; stepnum = (b2 - b1)/steplen;
   For[j = 0, j <= stepnum, j++,
    b = b1 + steplen*j; A = Eta1[a, b, k, mylist];
    d = N[steplen*j, precision];
    v = N[Mean[A[[All, 3]]], precision];
    B = AppendTo[B, {d, v}] ]; W = B]

k = 5000; mylist = Table[j*10^3, {j, 1, 5}];
```

```
b1 = 10.0; b2 = 20.0; steplen = 1/4;
a = 1/2; A = Eta2[a, b1, b2, steplen, k, mylist]; Print[A];
ListLinePlot[A, Mesh -> Full]
```

```
{{0,1.33823},{0.2500000000,1.61259},{0.5000000000,1.86782},
{0.7500000000,2.09893},{1.000000000,2.30438},{1.250000000,2.46705},
{1.500000000,2.56858},{1.750000000,2.61239},{2.000000000,2.60302},
{2.250000000,2.52423},{2.500000000,2.36813},{2.750000000,2.14915},
{3.000000000,1.87241},{3.250000000,1.5319},{3.500000000,1.13841},
{3.750000000,0.709616},{4.000000000,0.253182},{4.250000000,0.2182},
{4.500000000,0.683107},{4.750000000,1.13002},{5.000000000,1.54547},
{5.250000000,1.90438},{5.500000000,2.19277},{5.750000000,2.41041},
{6.000000000,2.54455},{6.250000000,2.57632},{6.500000000,2.51031},
{6.750000000,2.36476},{7.000000000,2.14311},{7.250000000,1.84833},
{7.500000000,1.51511},{7.750000000,1.20657},{8.000000000,1.00278},
{8.250000000,0.988215},{8.500000000,1.15136},{8.750000000,1.39071},
{9.000000000,1.63032},{9.250000000,1.81398},{9.500000000,1.89421},
{9.750000000,1.85839},{10.00000000,1.71362}}
```

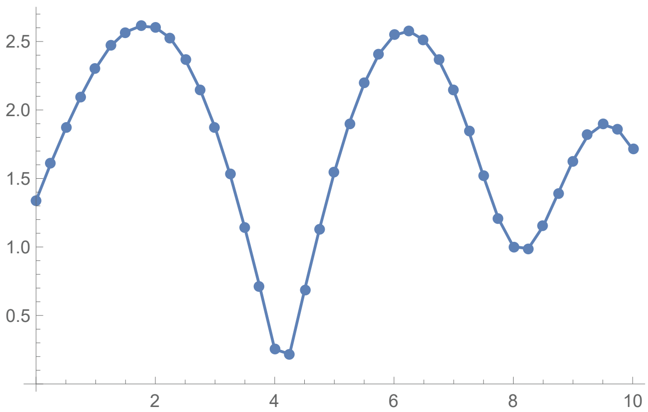

Fig. 1 Local minima for the interval (10, 20)

By Fig.1, we see there are two local minimums of modulus, corresponding to the tuples (4.25, 0.2182), (8.25, 0.988215). So, the two possible intervals are (14.0, 14.5), (18.0, 18.5).

For the first interval (14.0, 14.5), we have the following experimental results.

```
b1 = 14.0; b2 = 14.5; steplen = 1/32;
a = 1/2; A = Eta2[a, b1, b2, steplen, k, mylist]; Print[A];
ListLinePlot[A, Mesh -> Full]
```

```
{{0,0.253182},{0.03125000000,0.194737},{0.06250000000,0.136141},
{0.09375000000,0.0775431},{0.1250000000,0.0202913},{0.1562500000,0.0421528},
```

```
{0.1875000000,0.100516},{0.2187500000,0.159346},{0.2500000000,0.2182},
{0.2812500000,0.276977},{0.3125000000,0.335624},{0.3437500000,0.394103},
{0.3750000000,0.452384},{0.4062500000,0.510443},{0.4375000000,0.568259},
{0.4687500000,0.625818},{0.5000000000,0.683107}}
```

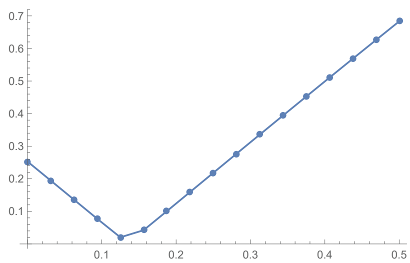

Fig. 2 Local minima for (14.0, 14.5)

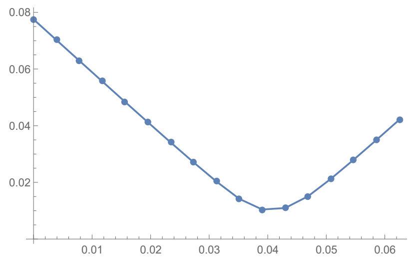

Fig. 3 Local minima for (14.09375, 14.15625)

By Fig.2, it is easy to see that the local minimum of modulus is 0.0202913, corresponding to the tuple $(0.125, 0.0202913)$. So, the shorter interval is (14.09375, 14.15625), which still contains the target zero 14.134725142.

```
b1 = 14.09375; b2 = 14.15625; steplen = 1/256;
a = 1/2; A = Eta2[a, b1, b2, steplen, k, mylist]; Print[A];
ListLinePlot[A, Mesh -> Full]

{{0,0.0775431},{0.003906250000,0.070239},{0.007812500000,0.0629472},
{0.01171875000,0.0556731},{0.01562500000,0.0484252},{0.01953125000,0.0412181},
{0.02343750000,0.0340781},{0.02734375000,0.0270592},{0.03125000000,0.0202913},
{0.03515625000,0.0141802},{0.03906250000,0.0104262},{0.04296875000,0.0109457},
{0.04687500000,0.0150783},{0.05078125000,0.0212469},{0.05468750000,0.0280083},
{0.05859375000,0.0350174},{0.06250000000,0.0421528}}
```

With the shorter step length 1/256, we find the local minimum of modulus is 0.0104262, corresponding to the tuple (0.0390625, 0.0104262). So, the shorter interval is (14.1289, 14.1367), which still contains the target zero 14.134725142. By the similar procedure, we obtain the strictly decreasing modulus chain and the nested intervals

$$\begin{aligned} \text{modulus}: &\quad 0.2182 > 0.0202913 > 0.0104262 > \cdots \\ \text{intervals}: &\quad (14.0, 14.5) \supset (14.09375, 14.15625) \supset (14.1289, 14.1367) \supset \cdots \end{aligned}$$

Finally, we can obtain a more accurate approximation of the target zero.

For the other local minimum (8.25, 0.988215), we have the below results.

```
b1 = 18.0; b2 = 18.5; steplen = 1/32;
a = 1/2; A = Eta2[a, b1, b2, steplen, k, mylist]; Print[A];
ListLinePlot[A, Mesh -> Full]
```

```
{{0,1.00278},{0.03125000000,0.989387},{0.06250000000,0.979231},
{0.09375000000,0.972401},{0.1250000000,0.968948},{0.1562500000,0.968873},
{0.1875000000,0.972129},{0.2187500000,0.978623},{0.2500000000,0.988215},
{0.2812500000,1.00073},{0.3125000000,1.01596},{0.3437500000,1.03369},
{0.3750000000,1.05367},{0.4062500000,1.07566},{0.4375000000,1.09943},
{0.4687500000,1.12473},{0.5000000000,1.15136}}

b1 = 18.09375; b2 = 18.21875; steplen = 1/256;
a = 1/2; A = Eta2[a, b1, b2, steplen, k, mylist]; Print[A];
ListLinePlot[A, Mesh -> Full]

{{0,0.972401},{0.003906250000,0.971785},{0.007812500000,0.971221},
{0.01171875000,0.97071},{0.01562500000,0.970251},{0.01953125000,0.969846},
{0.02343750000,0.969494},{0.02734375000,0.969194},{0.03125000000,0.968948},
{0.03515625000,0.968754},{0.03906250000,0.968613},{0.04296875000,0.968525},
{0.04687500000,0.96849},{0.05078125000,0.968507},{0.05468750000,0.968577},
{0.05859375000,0.968699},{0.06250000000,0.968873},{0.06640625000,0.969099},
{0.07031250000,0.969377},{0.07421875000,0.969708},{0.07812500000,0.970089},
{0.08203125000,0.970523},{0.08593750000,0.971007},{0.08984375000,0.971543},
{0.09375000000,0.972129},{0.09765625000,0.972766},{0.1015625000,0.973454},
{0.1054687500,0.974192},{0.1093750000,0.974979},{0.1132812500,0.975816},
{0.1171875000,0.976703},{0.1210937500,0.977638},{0.1250000000,0.978623}}
```

It is easy to see that there does not exist a strictly decreasing modulus chain. So, it does not correspond to a zero.

4 Local minima on vertical lines

In the above experiments, we always take $a = 1/2$. Now, we take any $a \in (0, 1)$.

```
k = 5000; mylist = Table[j*10^3, {j, 1, 5}];
b1 = 10.0; b2 = 20.0; steplen = 1/4;
a = 1/3; A = Eta2[a, b1, b2, steplen, k, mylist];
a = 1/2; B = Eta2[a, b1, b2, steplen, k, mylist];
a = 2/3; U = Eta2[a, b1, b2, steplen, k, mylist];
ListLinePlot[{A, B, U}, Mesh -> Full,
  PlotLabels -> {Callout["a=1/3", {Scaled[0.21], Above}],
     Callout["a=1/2", {Scaled[0.78], Below}],
     Callout["a=2/3", {Scaled[0.95], Below}]}]

a = 1/4; A = Eta2[a, b1, b2, steplen, k, mylist];
a = 1/2; B = Eta2[a, b1, b2, steplen, k, mylist];
a = 3/4; U = Eta2[a, b1, b2, steplen, k, mylist];
ListLinePlot[{A, B, U}, Mesh -> Full,
  PlotLabels -> {Callout["a=1/4", {Scaled[0.25], Above}],
     Callout["a=1/2", {Scaled[0.75], Below}],
     Callout["a=3/4", {Scaled[0.95], Below}]}]
```

The results are plotted as below (see Fig.4–9). It is easy to find that $b = 14.134725142$ locally minimizes the modulus $|\eta(a+ib)|$ for $a = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$.

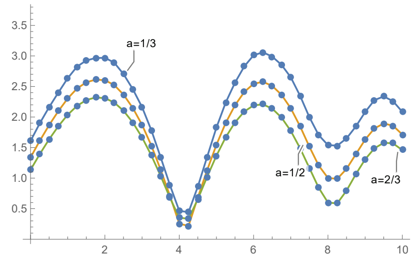

Fig. 4 Local minima for $a = 1/3, 1/2, 2/3$, and interval (10, 20)

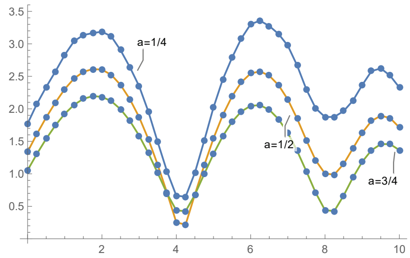

Fig. 5 Local minima for $a = 1/4, 1/2, 3/4$, and interval (10, 20)

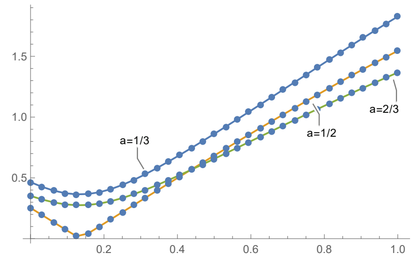

Fig. 6 Local minima for $a = 1/3, 1/2, 2/3$, and interval (14, 15)

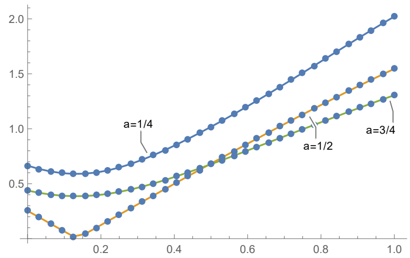

Fig. 7 Local minima for $a = 1/4, 1/2, 3/4$, and interval (14, 15)

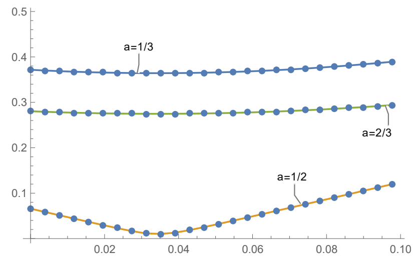

Fig. 8 Local minima for $a = 1/3, 1/2, 2/3$, and interval (14.10, 14.20)

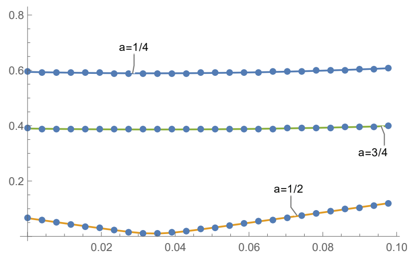

Fig. 9 Local minima for $a = 1/4, 1/2, 3/4$, and interval (14.10, 14.20)

Notice that $(\zeta(a+ib))'_b = \sum_{n=1}^{\infty}(-1)^n i e^{-a\ln n} e^{-ib\ln n}\ln n$, $(\zeta(a+ib))''_b = \sum_{n=1}^{\infty}(-1)^{n+2} e^{-a\ln n} e^{-ib\ln n}\ln^2 n$. Both are more complicated than the original series. So, the general methods to find local minima using First Derivative Test or Second Derivative Test are not applicable. But by numerical calculation and induction, we have the following result.

Theorem 4.1 (Local minima on vertical lines)**.** *If $\frac{1}{2}+it$ is a zero of $\eta(z)$, then for any $a \in (0,1)$, $a+it$ is a local minima of $|\eta(a+ib)|$. Namely, there exists an interval $[b_1, b_2]$ and $t \in (b_1, b_2)$ such that $|\eta(a+it)| = \min_{b\in(b_1,b_2)}\{|\eta(a+ib)|, a \in (0,1)|\}$.*

For example,

$$|\eta(a + 14.134725142\, i)| = \min_{b\in[10,20]}\{|\eta(a+ib)|, a \in (0,1)\},$$
$$|\eta(a + 21.022039639\, i)| = \min_{b\in[20,23]}\{|\eta(a+ib)|, a \in (0,1)\},$$
$$|\eta(a + 25.01085758\, i)| = \min_{b\in[23,30]}\{|\eta(a+ib)|, a \in (0,1)\}, \cdots$$

By Theorem 4.1, for any $a+bi, a \in (0,1)$, we have

$$|\eta(a+ib)| \geq |\eta(a+it)| \tag{7}$$

where $|\eta(\frac{1}{2}+it)| = 0$, and t is the nearest to b.

5 Monotonicity on any horizontal line with a zero

To investigate the monotonicity on any horizontal line with a zero, we revise the original programming code as below.

```
Eta3[b_,steplen_,k_,mylist_]:=Module[{A,B,a,stepnum,j,W,v,precision},
   A = B = W = {}; precision = 10; stepnum = 1/steplen;
   For[j=0, j<=stepnum, j++, a=steplen*j; A=Eta1[a, b, k, mylist];
   v=N[Mean[A[[All, 3]]], precision]; B=AppendTo[B, {a, v}]]; W=B]

k = 5000; mylist = Table[j*10^3, {j, 1, 5}]; steplen = 1/32;
b = 14.134725142;  A = Eta3[b, steplen, k, mylist];
b = 21.022039639;  B = Eta3[b, steplen, k, mylist];
b = 25.010857580;  U = Eta3[b, steplen, k, mylist];
ListLinePlot[{A, B, U}, Mesh -> Full,
  PlotLabels -> {Callout["b=14.13", {Scaled[0.35], Below}],
   Callout["b=21.02", {Scaled[0.08], Above}],
   Callout["b=25.01", {Scaled[0.95], Above}]}]

b = 30.424876126;  A = Eta3[b, steplen, k, mylist];
b = 32.935061588;  B = Eta3[b, steplen, k, mylist];
b = 37.586178159;  U = Eta3[b, steplen, k, mylist];
ListLinePlot[{A, B, U}, Mesh -> Full,
  PlotLabels -> {Callout["b=30.42", {Scaled[0.35], Below}],
   Callout["b=32.93", {Scaled[0.08], Above}],
   Callout["b=37.58", {Scaled[0.95], Above}]}]
```

The experimental results are plotted as above (see Fig.10, 11). Here is the relationship between the three modulus (see Fig.12).

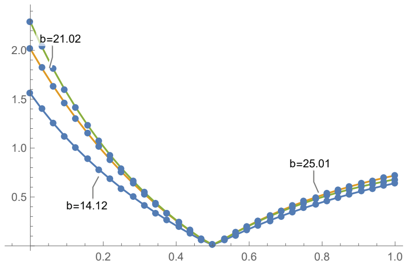

Fig. 10 The monotonicity for $t = 14.134725142, 21.022039639, 25.01085758$

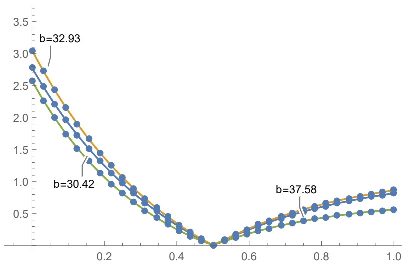

Fig. 11 The monotonicity for $t = 30.424876126, 32.935061588, 37.586178159$

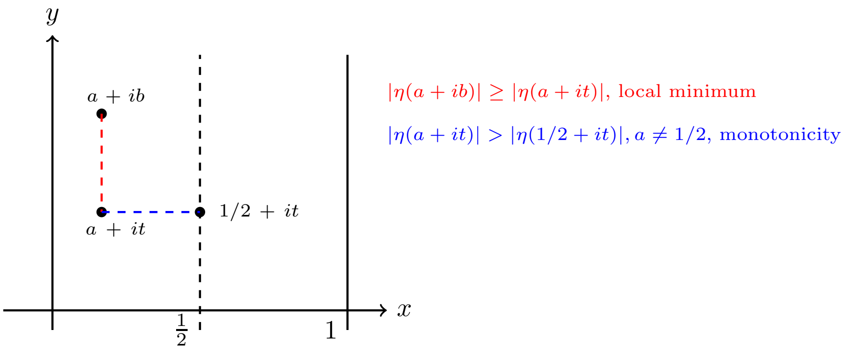

Fig. 12 The relationship for three modulus

Note that the first derivative $(\zeta(a+ib))'_a = \sum_{n=1}^{\infty}(-1)^n e^{-a\ln n} e^{-ib\ln n}\ln n$, which is more complicated than the original series. So, it is impossible to theoretically validate the monotonicity. But by numerical calculation and induction, we have the below theorem.

Theorem 5.1 (Monotonicity on any horizontal line with a zero)**.** *If $\frac{1}{2}+it$ is a zero of $\eta(z)$, then $|\eta(a+it)|$ is a strictly decreasing function for $a \in (0, 1/2)$. But $|\eta(a+it)|$ is a strictly increasing function for $a \in (1/2, 1)$.*

By Theorem 5.1, it is easy to see that

$$|\eta(a+it)| > |\eta(\frac{1}{2}+it)| = 0 \tag{8}$$

if $a \in (0,1) \setminus \{1/2\}$. Combining (7) and (8), we have the following corollary.

Corollary 5.2. *For any $a+bi, a \in (0,1)$, we have*

$$|\eta(a+ib)| \geq |\eta(a+it)| > |\eta(\frac{1}{2}+it)| = 0$$

where $\frac{1}{2}+it$ is a zero and t is the nearest to b.

6 Conclusion

The local minima on vertical lines and the monotonicity on any horizontal line with a zero in the critical strip, play a key role in validating the Riemann hypothesis. But the general methods to find local minima using First Derivative Test or Second Derivative Test, are not applicable for the eta series because its first derivative and second derivative are more complicated than the original series. Besides, the usual methods to validate the monotonicity are not applicable. The numerical calculation method proposed in this paper could be a good choice for the intractable hypothesis. It could be helpful for the future works on this topic.

Declarations

Ethical Approval: Not applicable.
Funding: Not applicable.
Authors' contributions: ZJ Cao wrote this article solely.
Competing interests: The author declares that he has no conflict of interest.
Availability of data and materials: All data generated or analysed during this study are included in this published article.

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