A New Method to Test the Zeros of Riemann Zeta Function

Zhengjun Cao, Lihua Liu

Abstract. The zeta function $\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}$ is convergent only for $\mathrm{Re}(z)>1$. To test its zeros, one needs to use the Riemann-Siegel function $Z(t)$. If $Z(t_1)$ and $Z(t_2)$ have opposite signs, $Z(t)$ vanishes between t_1 and t_2, and $\zeta(z)$ has a zero on the critical line between $\frac{1}{2}+it_1$ and $\frac{1}{2}+it_2$. This method is non-polynomial time, because it has to compute the sum $\sum_{n\le\alpha}\frac{\cos(\vartheta(1/2+it)-t\log n)}{\sqrt{n}}$, where $\alpha=\lfloor\sqrt{t/(2\pi)}\rfloor$. The eta function $\eta(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^z}$ is convergent for $\mathrm{Re}(z)>0$, and $\eta(z)=\left(1-2^{1-z}\right)\zeta(z)$ for the critical strip $0<\mathrm{Re}(z)<1$. The alternating series can be directly used to test the zeros because $\eta(z)$ and the analytic continuation of $\zeta(z)$ have the same zeros in the critical strip. In this paper, we present a polynomial time algorithm to test the zeros based on $\eta(z)$, which is more understandable and suitable for modern computing machines than the general method. Besides, we clarify the actual meaning of logarithm symbol in the Riemann-Siegel formula.

Keywords: Riemann zeta function, Dirichlet eta function, partial sum, absolute convergence.

1 Introduction

The Riemann zeta function $\zeta(z)=\sum\limits_{n=1}^{\infty}\frac{1}{n^z}, z\in\mathbb{C}$, is absolutely convergent in the region $\mathrm{Re}(z)>1$. It is well known [1] that $\zeta(2)=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$. By the famous functional equation [2], $\zeta(z)=2^z\pi^{z-1}\sin\frac{z\pi}{2}\Gamma(1-z)\zeta(1-z)$, we have

$$\zeta(-1)=2^{-1}\pi^{-2}\sin\frac{-\pi}{2}\Gamma(2)\zeta(2)=\frac{1}{2\pi^2}\times(-1)\times 1\times\frac{\pi^2}{6}=-\frac{1}{12}.$$

But by the original series, we have $\zeta(-1)=1+2+3+\cdots\to\infty$.

Actually, in the above functional equation, $\zeta(z)$ *and* $\zeta(1-z)$ *cannot be concurrently convergent*, because at least one of $\mathrm{Re}(z)$ and $\mathrm{Re}(1-z)$ is strictly smaller than 1. So, $\zeta(z)$ and $\zeta(1-z)$ must be two different branches of the analytic continuation of the original series on the complex plane.

The famous Riemann zeros are not for the original series, instead for a branch of its analytic continuation. The general method to test these zeros needs to use the Riemann-Siegel function $Z(t)$. If $Z(t_1)$ and $Z(t_2)$ have opposite signs, $Z(t)$ vanishes between t_1 and t_2, and so $\zeta(z)$ has a zero on the critical line between $\frac{1}{2}+it_1$ and $\frac{1}{2}+it_2$. Clearly, this method is too hard to practice for newcomers, and the mysterious zeros have not been broadly exhibited to the average person.

In this paper, we present a new method to test these famous zeros. The method is based on that $\eta(z)=\left(1-\frac{2}{2^z}\right)\zeta(z)$ in the critical strip $0<\mathrm{Re}(z)<1$, where the Dirichlet eta function $\eta(z)$ is convergent

Z. Cao is with Department of Mathematics, Shanghai University, Shanghai, 200444, China.
L. Liu is with Department of Mathematics, Shanghai Maritime University, Shanghai, 201306, China. liulh@shmtu.edu.cn

for $\text{Re}(z) > 0$. This relationship shows that $\eta(z)$ and the analytic continuation of $\zeta(z)$ have the same zeros in the critical strip. We will make use of the alternating series to test the zeros.

2 Zeta function and Eta function

The Riemann zeta function is further represented as [3]

$$\begin{aligned}\zeta(z) &= \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} e^{-z\ln n} \xlongequal[a,b\in\mathbb{R}]{z=a+ib} \sum_{n=1}^{\infty} e^{-(a+ib)\ln n} \\ &= \sum_{n=1}^{\infty} e^{-a\ln n} e^{-ib\ln n} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^a \cos(b\ln n) - i\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^a \sin(b\ln n)\end{aligned}$$

If $a > 1$, both $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^a \cos(b\ln n)$ and $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^a \sin(b\ln n)$ are absolutely convergent. Therefore, $\zeta(z)$ has no zeros for $a > 1$.

The Dirichlet eta function [4] is the alternating series

$$\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}, \quad z \in \mathbb{C}. \tag{1}$$

$\eta(0)$ is defined to be $1/2$. $\eta(1) = \ln 2$, $\eta(2) = \frac{\pi^2}{12}$. Notice that, for $\text{Re}(z) > 1$

$$\begin{aligned}\frac{2}{2^z}\zeta(z) &= \frac{2}{2^z}\left(1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots\right) = \frac{2}{2^z} + \frac{2}{4^z} + \frac{2}{6^z} + \frac{2}{8^z} + \cdots, \\ \left(1 - \frac{2}{2^z}\right)\zeta(z) &= \left(1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots\right) - \left(\frac{2}{2^z} + \frac{2}{4^z} + \frac{2}{6^z} + \frac{2}{8^z} + \cdots\right) \\ &\xlongequal{\text{rearranged}} 1 + \left(\frac{1}{2^z} - \frac{2}{2^z}\right) + \frac{1}{3^z} + \left(\frac{1}{4^z} - \frac{2}{4^z}\right) + \cdots \\ &= 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z} = \eta(z).\end{aligned}$$

Extending this relationship $\eta(z) = (1 - 2^{1-z})\zeta(z)$ to the complex plane, we can obtain the functional equation $\zeta(z) = 2^z\pi^{z-1}\sin\frac{z\pi}{2}\Gamma(1-z)\zeta(1-z)$. If $z = -2, -4, -6, \cdots$, $\sin\frac{z\pi}{2} = 0$. These values are called simple zeros of $\zeta(z)$. Since $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, we know $\Gamma(1-z)$ has no zeros. So, $\zeta(z) = 0$ iff $\zeta(1-z) = 0$, which also implies that $\zeta(\bar{z}) = \zeta(1-\bar{z}) = 0$. The famous Riemann hypothesis [5] claims that all the complex zeros of $\zeta(z)$ lie on the critical line $\text{Re}(z) = 1/2$.

In history, the zeros of $\zeta(z)$ were very hard to calculate [6]. Nowadays, several million zeros have been obtained [7]. See the table of zeros [8]. It is worth noting that the symbol $\zeta(z)$ didn't refer to the original series, instead its analytic continuation.

3 The general method to test zeros

The general method to test zeros is based on the famous functional equation. Define the functions

$$\chi(z) = 2^{z-1}\pi^z \sec\frac{z\pi}{2} \,/\, \Gamma(z), \quad \vartheta = \vartheta(t) = -\frac{|\chi(\frac{1}{2}+it)|}{2} \arg\chi(\frac{1}{2}+it),$$

where $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\mathrm{d}x$, and the Riemann-Siegel function $Z(t) = e^{i\vartheta(t)}\zeta(\frac{1}{2}+it)$, which is real for real values of t. The theta function appearing above is also defined by $\vartheta(t) = \arg[\Gamma(\frac{1}{4}+\frac{1}{2}it)] - \frac{t}{2}\ln\pi$. If $Z(t_1)$ and $Z(t_2)$ have opposite signs, $Z(t)$ vanishes between t_1 and t_2, and so $\zeta(z)$ has a zero on the critical line between $\frac{1}{2}+it_1$ and $\frac{1}{2}+it_2$.

To calculate the first nontrivial zero, one needs to determine the sign of $Z(0) = e^{i\vartheta(\frac{1}{2})}\zeta(\frac{1}{2})$. If $z = 1/2$, $\eta(1/2) = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$, which converges to a positive number. Since $\eta(1/2) = (1-2^{1/2})\zeta(1/2)$ and $1-\sqrt{2} < 0$, it claims that $\zeta(1/2) < 0$ (page 388, Ref.[2]). Define

$$\xi(z) = \frac{1}{2}z(z-1)\pi^{-\frac{z}{2}}\Gamma(\frac{z}{2})\zeta(z). \tag{2}$$

Hence, $\xi(1/2) = -\frac{1}{8}\pi^{-\frac{1}{4}}\Gamma(\frac{1}{4})\zeta(1/2)$. Since $\zeta(\frac{1}{2}) < 0$ and $\Gamma(\frac{1}{4}) > 0$, then $\xi(\frac{1}{2}) > 0$, which implies $Z(0) < 0$. By numerical analysis, it shows that $Z(6\pi) > 0$. Therefore, there is one zero at least on the critical line between $t = 0$ and $t = 6\pi$. We currently know that the first zero approximates to $1/2 + 14.1347251\,i$.

Theorem 1 (The Riemann-Siegel formula, [9]) *Let $s = 1/2 + it$, $a \in [0,1)$ be the fractional part of $\alpha := \sqrt{t/(2\pi)}$, and*

$$\psi(u) := \frac{\cos(\pi(u^2/2 - u - 1/8))}{\cos(\pi u)} \tag{3}$$

For any $N \in \mathbb{Z}_{>0}$, there exist explicit functions of a alone, $C_0(a), C_1(a), C_2(a), \cdots$, such that

$$Z(t) = 2\sum_{n\le\alpha}\frac{\cos(\vartheta(1/2+it) - t\log n)}{\sqrt{n}} + (-1)^{\lfloor\alpha\rfloor+1}\left(\frac{2\pi}{t}\right)^{1/4}\sum_{m=0}^{N-1}\frac{C_m(a)}{t^{m/2}} + O\left(\frac{1}{t^{N/2+1/4}}\right),$$

where $i\vartheta(1/2+it) = \frac{it}{2}\log\frac{|t|}{2\pi} - \frac{it}{2} - \mathrm{sgn}(t)\frac{i\pi}{8} + O(1/|t|)$,

$$C_0(a) = \psi(2a), \quad C_1(a) = -\frac{1}{3}(2\pi)^{-3/2}\psi^{(3)}(2a), \quad C_2(a) = \frac{1}{18}(2\pi)^{-3}\psi^{(6)}(2a) + \frac{1}{4}(2\pi)^{-1}\psi^{(2)}(2a),$$

$$C_3(a) = -\frac{1}{162}(2\pi)^{-9/2}\psi^{(9)}(2a) - \frac{2}{15}(2\pi)^{-5/2}\psi^{(5)}(2a) - \frac{1}{8}(2\pi)^{-1/2}\psi^{(1)}(2a), \cdots$$

Riemann and Siegel gave recursive procedures for calculating the coefficients $C_m(a)$. Furthermore, Siegel proved that only derivatives $\psi^{(r)}(2a)$ for $r \equiv 3m \bmod 4$ appear in $C_m(a)$. We will use the sophisticated software Mathematica to test the general method for $N = 4$ in the above Riemann-Siegel formula.

```
Z[t_] := Module[{\[Alpha], a, k, s, \[CurlyTheta], \[Psi], u, QD, j,
       n, c0, c1, c2, c3, A, B, w},
       \[Psi][u_] := Cos[Pi (u^2/2 - u - 1/8)]/Cos[Pi*u];
       QD[j_, u_] := Derivative[j][\[Psi]][u];
       \[Alpha] = Sqrt[t/(2*Pi)]; a = FractionalPart[\[Alpha]];
       c0 = \[Psi][2*a]; c1 = -1/3*(2*Pi)^(-3/2)*QD[3, 2*a];
       c2 = 1/18*(2*Pi)^(-3)*QD[6, 2*a] + 1/4*(2*Pi)^(-1) QD[2, 2*a];
       c3 = -1/162*(2*Pi)^(-9/2)*QD[9, 2*a] -
            2/15*(2*Pi)^(-5/2) QD[5, 2*a] - 1/8*(2*Pi)^(-1/2) QD[1, 2*a];
       k = IntegerPart[\[Alpha]];
       s = Floor[\[Alpha]]; \[CurlyTheta] = t/2+Pi/8-t/2*Log[t/(2*Pi)];
       A = 2*Sum[Cos[ \[CurlyTheta] - t*Log[n]]/n^(1/2), {n, 1, k}];
       B = (-1)^(s + 1)*(2*Pi/t)^(1/4)*(c0+c1/t^(1/2)+c2/t+c3/t^(3/2));
       w = N[A + B, 10]]
```

```
W = Table[{i, Z[i]}, {i, 10, 20, 0.25}]; Print[W]; ListLinePlot[W, Mesh -> Full]
```

```
{{10.,-1.53122},{10.25,-1.55299},{10.5,-1.54687},{10.75,-1.53031},
{11.,-1.5024},{11.25,-1.46231},{11.5,-1.4093},{11.75,-1.34271},
{12.,-1.26203},{12.25,-1.1669},{12.5,-1.05711},{12.75,-0.932695},
{13.,-0.793889},{13.25,-0.641181},{13.5,-0.475326},{13.75,-0.297361},
{14.,-0.108615},{14.25,0.0892823},{14.5,0.294403},{14.75,0.504527},
{15.,0.717155},{15.25,0.929529},{15.5,1.13867},{15.75,1.3414},
{16.,1.53442},{16.25,1.71433},{16.5,1.87772},{16.75,2.02124},
{17.,2.14164},{17.25,2.23593},{17.5,2.30138},{17.75,2.33567},
{18.,2.33695},{18.25,2.30392},{18.5,2.2359},{18.75,2.13295},
{19.,2.00889},{19.25,4.19801*10^12},{19.5,1.64405},{19.75,1.39951},
{20.,1.14976}}
```

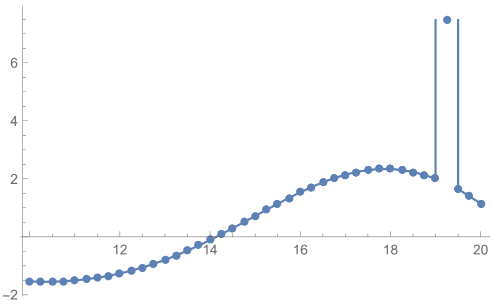

Fig. 1 Z(t) sample values for (10, 20), with Log[e, n]

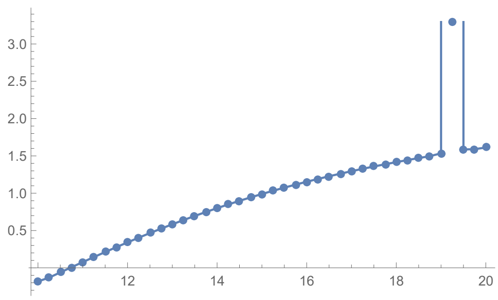

Fig. 2 Z(t) sample values for (10, 20), with Log[10, n]

Remark 1. By Fig.1, we see there is a zero near 14. But by Fig.2, there is a zero near 11. In fact, the target zero is 14.134725. So, we want to clarify that the logarithm symbol $\log n$ in the representation of $Z(t)$, actually means Log$[e, n]$, not Log$[10, n]$. By the way, the logarithm symbol in the masterwork [2] (first appeared on page 2, Chap. I) relates to the calculation $\log\left(1 - \frac{1}{(n+1)^s}\right) - \log\left(1 - \frac{1}{n^s}\right) = \int_n^{n+1} \frac{s}{x(x^s-1)} dx$, which also means Log$[e, n]$.

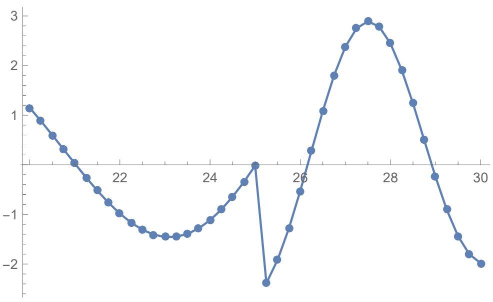

Fig. 3 Z(t) sample values for (20, 30)

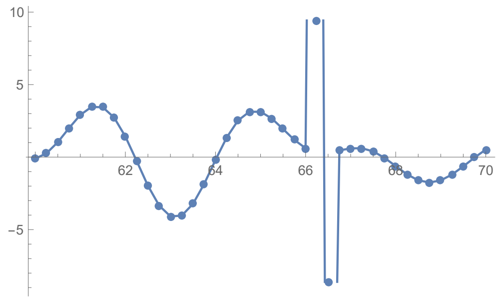

Fig. 4 Z(t) sample values for (60, 70)

Notice that $Z(t)$ is discontinuous in the whole interval (10, 20), because $C_0(a), C_1(a), C_2(a), \cdots$, have poles. Likewise, we have the following results (see Fig.3, Fig.4). There are some poles, too. In fact, there are two zeros 21.0220396, 25.01085758, in the interval (20, 30), and four zeros 60.8317785, 65.112544, 67.07981, 69.546401 in the interval (60, 70). For a shorter interval, we have the following results.

```
j=1132489; W=Table[{i, Z[j+i]}, {i, 0, 1, 1/32}]; ListLinePlot[W,Mesh->Full]
j=1132490; W=Table[{i, Z[j+i]}, {i, 0, 1, 1/32}]; ListLinePlot[W,Mesh->Full]
```

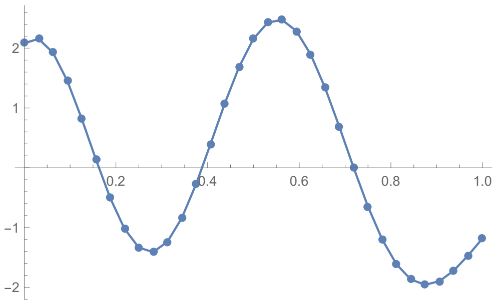

Fig. 5 Z(t) sample values for (1132489, 1132490)

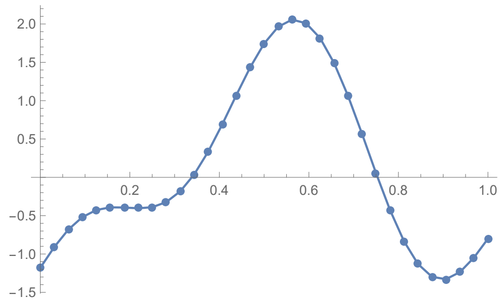

Fig. 6 Z(t) sample values for (1132490, 1132491)

By Fig.5 and Fig.6, we see there should be three zeros in the interval (1132489, 1132490), and two zeros in the interval (1132490, 1132491). Actually, there are four zeros

$$1132489.162779754, \quad 1132489.585214591, \quad 1132490.165304883, \quad 1132490.658714411.$$

See [8] for details. The inconsistence is due to the roundoff errors and systematic errors, or the ListLinePlot function fails to detect the poles in the interval (1132489, 1132490). In short, the general method becomes inaccurate with the increasing of t.

4 A new method to test zeros

For the first three zeros$r_1 = 1/2 + 14.134725\,i, r_2 = 1/2 + 21.0220396\,i, r_3 = 1/2 + 25.01085758\,i$, we have the following numerical calculations (see Table 1).

Table 1: Numerical calculations for the first three zeros

Partial-sum	$\eta(1/2 + 14.134725\,i) = c - di$	$\lvert c\rvert + \lvert d\rvert$
700000	0.00010144 +0.000588673 I	0.000690113
800000	0.00049395 -0.000261985 I	0.000755936
900000	-0.000289405-0.000440756 I	0.000730162
Partial-sum	$\eta(1/2 + 21.0220396\,i) = c - di$	$\lvert c\rvert + \lvert d\rvert$
700000	-0.000586334+0.000110852 I	0.000697186
800000	0.000553754 +0.0000797987 I	0.000633552
900000	-0.000362282-0.000383504 I	0.000745786
Partial-sum	$\eta(1/2 + 25.010857\,i) = c - di$	$\lvert c\rvert + \lvert d\rvert$
700000	0.000534317 -0.000268088 I	0.000802405
800000	-0.000439384+0.000345612 I	0.000784996
900000	0.000470723 -0.00023747 I	0.000708193

With the finite precision, we have the faith in that the three values are really zeros of eta function. Of course, they are not for the original zeta series. Practically, the three series $\zeta(r_1), \zeta(r_2), \zeta(r_3)$ are divergent, not convergent.

Theorem 2. *Let $z = \frac{1}{2} + bi, b > 0$. Denote the partial sum $\sum_{n=1}^{k} \frac{(-1)^{n-1}}{n^z}$ by $c - di$, for some positive integer k. Then the modulus $|c - di|$ is continuous with respect to b.*

Proof. It is easy to see that

$$\begin{aligned} c - di &= \sum_{n=1}^{k} \frac{(-1)^{n-1}}{n^z} = \sum_{n=1}^{K} (-1)^{n-1} e^{-z \ln n} \overset{z=\frac{1}{2}+ib}{=\!=\!=} \sum_{n=1}^{k} (-1)^{n-1} e^{-(\frac{1}{2}+ib)\ln n} \\ &= \sum_{n=1}^{k} (-1)^{n-1} \sqrt{\frac{1}{n}} \cos(b \ln n) - i \sum_{n=1}^{k} (-1)^{n-1} \sqrt{\frac{1}{n}} \sin(b \ln n), \\ |c - di|^2 &= \left(\sum_{n=1}^{k} (-1)^{n-1} \sqrt{\frac{1}{n}} \cos(b \ln n) \right)^2 + \left(\sum_{n=1}^{k} (-1)^{n-1} \sqrt{\frac{1}{n}} \sin(b \ln n) \right)^2 . \end{aligned}$$

Since all $\cos(b \ln n), \sin(b \ln n), n = 1, \cdots, k$, are continuous with respect to b, the above modulus is also continuous with respect to b. □

Based on this theorem, we now present a new method (see Algorithm 1) to search for a zero in a short interval. Let $s_k := \sum_{n=1}^{k} (-1)^{n-1} e^{-(\frac{1}{2}+ib)\ln n}$. We compute the mean of partial sums $s_{k_1}, s_{k_2}, \cdots, s_{k_\ell}$, so as to partly offset the roundoff errors.

Algorithm 1: Testing zeros of Dirichlet eta series in the critical strip

---

Input: $(b_1, b_2), b_2 > b_1 > 0$, which contains at least one zero of eta series, and a set of positive integers $K = \{k_1, k_2, \cdots, k_\ell\}$, $k_1 < k_2 < \cdots < k_\ell$.

Output: $(c, d) \subset (b_1, b_2)$, which contains at least one zero of eta series.

$steplen \leftarrow 1/4$ (or 1/32, 1/256, etc), $stepnum \leftarrow (b_2 - b_1)/steplen$

$l \leftarrow 0,\ r \leftarrow 0,\ T \leftarrow \{\ \}$ // T is the empty set

for $j = 0, j \le stepnum$ **do**

  $b \leftarrow b_1 + steplen * j,\ S \leftarrow \{\ \}$ **for** $n = 1, n \le k_\ell$ **do**

    $l \leftarrow l + (-1)^{n-1} \sqrt{\frac{1}{n}} \cos(b \ln n)$

    $r \leftarrow r + (-1)^{n-1} \sqrt{\frac{1}{n}} \sin(b \ln n)$

    if $n \in K$ **then**

      $s \leftarrow l - ri$ // $i^2 = -1$

      $S \leftarrow S \cup \{|s|\}$ // $|s|$ is the modulus of s

  $t \leftarrow$ the mean value of S

  $T \leftarrow T \cup \{(b, t)\}$

Find $(\hat{b}, \hat{t}) \in T$, with a local minimum $\hat{t}$

$c \leftarrow \hat{b} - steplen, d \leftarrow \hat{b} + steplen$

Theorem 3. *The computational cost for Algorithm 1 is $O(50k_\ell(3.32p + \log k_\ell)^2)$, where p is the accuracy, i.e., the effective number of these digits which appear to the right of the decimal point.*

Proof. The longest binary length of operands in the procedure is $\log k_\ell$ (for integer part) plus $\log 10^p$ (for fractional part). The total iteration number is $stepnum \times k_\ell$. Usually, $stepnum = 50$ which suffices to determine the local minimums in a short interval. Note that $\log_2(10) \approx 3.32$. So, the computational cost for a multiplication is $O((3.32p + \log k_\ell)^2)$, and the total cost is $O(50k_\ell(3.32p + \log k_\ell)^2)$. □

The following Mathematica code can be directly used to test the zeros, in which we take $k_\ell = 8000$.

```
Eta[b_,k_,mylist_]:= Module[{a,n,l,r,s,t,U,V,precision},
   l = r = 0; U = V = {}; a = 1/2; precision = 10;
   For[n = 1, n <=k, n++,
      l = N[l + (-1)^(n - 1)/(n^a)*Cos[b*Log[n]], precision];
      r = N[r + (-1)^(n - 1)/(n^a)*Sin[b*Log[n]], precision];
      If[MemberQ[mylist, n], s = l - r*I; t = Abs[s];
        U = AppendTo[U, {n, s, t}]]];
   V = U];
Eta2[b1_, b2_, steplen_, k_, mylist_] :=
  Module[{A, B, stepnum, b, j, W, v, precision},
  A = B = W = {}; precision = 10; stepnum = (b2 - b1)/steplen;
  For[j = 0, j <= stepnum, j++, b = b1 + steplen*j;
       A = Eta[b, k, mylist]; v = N[Mean[A[[All, 3]]], precision];
       B = AppendTo[B, {b, v}]];
  W = B]
```

```
 k = 8000; mylist = Table[j*10^3, {j, 3, 8}];
 b1 = 60.0; b2 = 70.0; steplen = 1/4;
 A = Eta2[b1, b2, steplen, k, mylist]; Print[A];
 ListLinePlot[A, Mesh -> Full]
```

```
{{60.,1.32321},{60.25,1.26717},{60.5,0.888527},{60.75,0.249642},
{61.,0.558794},{61.25,1.43329},{61.5,2.25682},{61.75,2.91184},
{62.,3.31517},{62.25,3.43194},{62.5,3.26288},{62.75,2.85191},
{63.,2.31346},{63.25,1.83446},{63.5,1.61748},{63.75,1.68183},
{64.,1.80219},{64.25,1.77718},{64.5,1.5243},{64.75,1.03149},
{65.,0.343867},{65.25,0.43392},{65.5,1.17642},{65.75,1.77527},
{66.,2.13253},{66.25,2.16633},{66.5,1.84827},{66.75,1.21098},
{67.,0.323144},{67.25,0.714845},{67.5,1.76714},{67.75,2.69032},
{68.,3.36282},{68.25,3.69281},{68.5,3.62851},{68.75,3.1727},
{69.,2.38064},{69.25,1.34991},{69.5,0.211323},{69.75,0.893024},
{70.,1.83385}}
```

By Fig.7, we see there are four local minimums of modulus, corresponding to the tuples

$$(60.75, 0.249642), \quad (65., 0.343867), \quad (67., 0.323144), \quad (69.5, 0.211323).$$

So, the four possible intervals are (60.5, 61), (64.75, 65.25), (66.75, 67.25), (69.25, 69.75). In fact, the target zeros are 60.8317785, 65.112544, 67.07981, 69.546401.

For the first interval (60.5, 61), we have the following results.

```
 b1 = 60.5; b2 = 61.0; steplen=1/32;
 A = Eta2[b1, b2, steplen, k, mylist]; Print[A]; ListLinePlot[A, Mesh -> Full]
```

```
{{60.5,0.888527},{60.5313,0.821196},{60.5625,0.74993},{60.5938,0.674901},
{60.625,0.596287},{60.6563,0.514269},{60.6875,0.429029},{60.7188,0.340756},
```

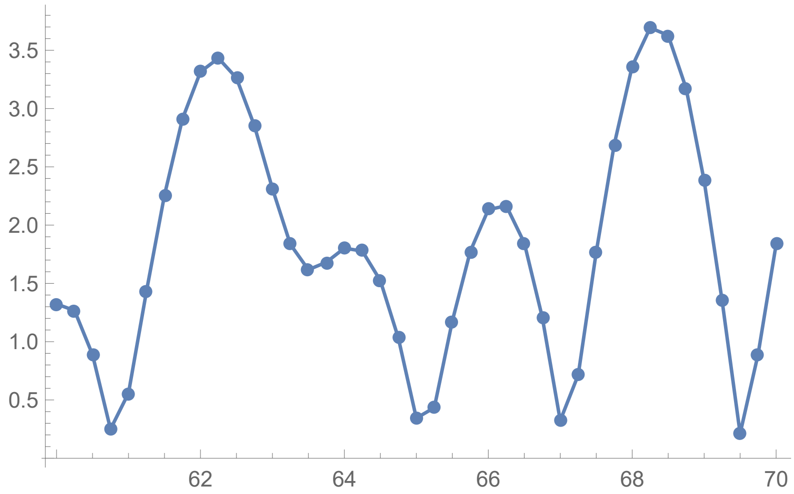

Fig. 7 The local minimums for the interval (60, 70)

```
{60.75,0.249642},{60.7813,0.155887},{60.8125,0.0597751},{60.8438,0.0393607},
{60.875,0.139692},{60.9063,0.242131},{60.9375,0.346274},{60.9688,0.451899},
{61.,0.558794}}
```

By Fig.8, it is easy to see that the local minimum of modulus is 0.0393607, corresponding to the tuple (60.8438, 0.0393607). So, the shorter interval is (60.8125, 60.875).

```
b1 = 60.8125; b2 = 60.875; steplen=1/256;
A = Eta2[b1, b2, steplen, k, mylist]; Print[A]; ListLinePlot[A, Mesh -> Full]
```

```
{{60.8125,0.0597751},{60.8164,0.0476377},{60.8203,0.0355041},
{60.8242,0.0234348},{60.8281,0.0117589},{60.832,0.0071543},
{60.8359,0.0152062},{60.8398,0.0271091},{60.8438,0.0393607},
{60.8477,0.0517337},{60.8516,0.0641754},{60.8555,0.0766675},
{60.8594,0.0892015},{60.8633,0.101773},{60.8672,0.11438},
{60.8711,0.12702},{60.875,0.139692}}
```

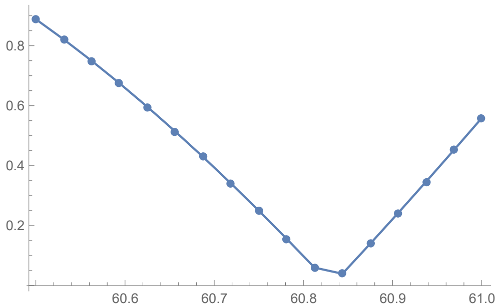

Fig. 8 The local minimum for (60.5, 61)

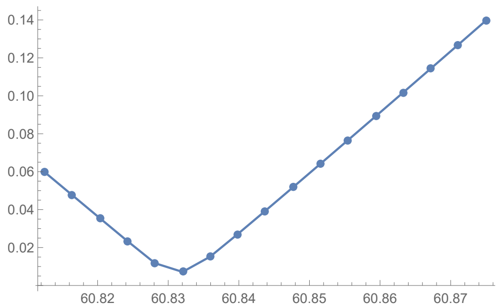

Fig. 9 The local minimum for (60.8125, 60.875)

With the shorter step length 1/256, we find the local minimum of modulus (see Fig.9) is 0.0071543, corresponding to the tuple (60.832, 0.0071543). So, the shorter interval is (60.8281, 60.8359), which still contains the target zero 60.8317785. By the similar procedure, we obtain the strictly decreasing modulus chain corresponding to the nested intervals,

$$\begin{aligned} &\text{modulus}: && 0.249642 > 0.0393607 > 0.0071543 > \cdots \\ &\text{intervals}: && (60.5, 61) \supset (60.8125, 60.875) \supset (60.8281, 60.8359) \supset \cdots \end{aligned}$$

Finally, we can obtain a more accurate approximation of the target zero.

5 Comparisons

The computational cost for the general method based on the Riemann-Siegel formula is dominated by that for computing the sum

$$\sum_{n \le \alpha} \frac{\cos(\vartheta(1/2+it) - t\log n)}{\sqrt{n}}.$$

If t is very large, then $O\left(\frac{1}{t^{N/2+1/4}}\right)$ is negligible, and N can be taken very small, say $N = 2$ or 3. But $\alpha = \sqrt{t/(2\pi)}$ becomes big enough. So, its computational cost is $O(\sqrt{t/2\pi}(3.32p + \log\, t)^2)$, where p is the accuracy. In this case, the new method takes $O(50k_\ell(3.32p + \log\tau)^2)$ cost, where $\tau = \max\{\log\, t, \log\, k_\ell\}$. We refer to Table 2 for the comparisons between the general method and the new method.

Table 2 Comparisons between the general method and the new method

	General method	New method
Principle	Riemann-Siegel formula Z(t), discontinuous, very complicated, far beyond the average person	partial sums of $\eta(z)$, continuous, straightforward, easily understandable
Manual computation	possible for $t \le 40$	impossible ($k_\ell \ge 3000$)
Cost	$O(\sqrt{t/2\pi}(3.32p + \log\, t)^2)$, depending on t, tremendous cost for a big integer t, non-polynomial time	$O(50k_\ell(3.32p + \log\tau)^2)$, depending mainly on the index k_ℓ for partial sums, polynomial time

6 Further discussions

Let b = 1132488.8, or 1132489.8 (the four highest zeros in the table [8], are 1132489.162779754, 1132489.585214591, 1132490.165304883, 1132490.658714411). To facilitate the ListLinePlot function, we revise the programming code as below.

```
Eta2[b1_, b2_, steplen_, k_, mylist_] :=
    Module[{A, B, stepnum, b, j, d, W, v, precision},
     A = B = W = {}; precision = 10; stepnum = (b2 - b1)/steplen;
     For[j = 0, j <= stepnum, j++,
       b = b1 + steplen*j; A = Eta[b, k, mylist];
       v = N[Mean[A[[All, 3]]], precision];
       d = N[steplen*j, precision]; B = AppendTo[B, {d, v}]]; W = B]
```

We now have the following results.

```
k = 8000; mylist = Table[j*10^3, {j, 5, 8}]; b1 = 1132488.8; b2 = 1132490.0;
steplen = 1/32;  A = Eta2[b1, b2, steplen, k, mylist];
Print[A];  ListLinePlot[A, Mesh -> Full]

k = 8000; mylist = Table[j*10^3, {j, 5, 8}]; b1 = 1132489.8; b2 = 1132491.0;
steplen = 1/64; A = Eta2[b1, b2, steplen, k, mylist];
Print[A]; ListLinePlot[A, Mesh -> Full]
```

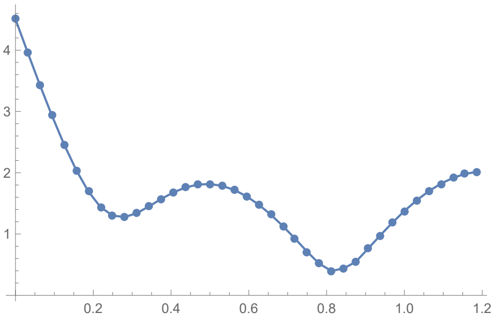

Fig. 10 The local minimum for (1132488.8, 1132490.0)

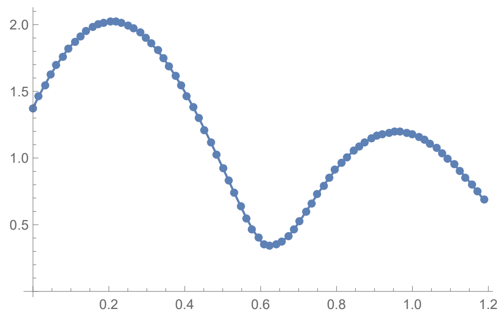

Fig. 11 The local minimum for (1132489.8, 1132491.0)

We find there are only three zeros (see Fig.10, Fig.11), near 1132489.08125, 1132489.6125, 1132490.425. The inconsistence is due to the loss of significant digits. In this case, the loss of accuracy becomes so serious as to invalidate the results.

The general method is also facing the challenge. For example, taking the highest two zeros $\mu = 1/2 + 1132490.165304883\, i$, $\nu = 1/2 + 1132490.658714411\, i$, we find the partial sums of $\eta(\mu), \eta(\nu)$ converge very slowly (see Table 3), compared with that of $\eta(1/2 + 14.134725142\, i)$.

Table 3: Convergence rates for three zeros

Partial-sum	$\eta(1/2 + 14.134725142\, i) = c - di$	$\lvert c - di\rvert$
70000	-0.0015479+0.00108417 I	0.00188982
80000	0.00141456 +0.00106019 I	0.00176776
Partial-sum	$\eta(1/2 + 1132490.165304883\, i) = c - di$	$\lvert c - di\rvert$
70000	0.860207 -0.294609 I	0.909258
80000	0.368013 -0.683709 I	0.776461
Partial-sum	$\eta(1/2 + 1132490.658714411\, i) = c - di$	$\lvert c - di\rvert$
70000	0.00572052 +0.0259613 I	0.0265841
80000	-0.0803842-0.595348 I	0.60075

The difference originates from that the fractional part 0.658714411 is far less than the integral part 1132490 in the direct computations of

$$\cos(1132490.658714411 \times \ln n), \quad \sin(1132490.658714411 \times \ln n), \quad \cdots,$$

So, its contribution to the fractional part, $\{\frac{1132490.658714411 \times \ln n}{2\pi}\}$, becomes negligible, and the usual calculating methods for sine and cosine functions will fall flat.

7 Conclusion

We show that the Dirichlet eta function and the analytic continuation of Riemann zeta function have the same zeros in the critical strip. Based on this relationship and that the partial sum of eta series is continuous, we present a simple method to test zeros. The programming code is also presented, which is very easy to execute. To the best of our knowledge, it is the first time to invent such a simple method for newcomers to test the famous zeros.

References

[1] Lang S.: Complex Analysis, 4-th ed., Graduate Texts in Mathematics, vol.103, Springer, New York (1999).

[2] Titchmarsh E.: The Theory of the Riemann Zeta-Function, Oxford University Press (1986).

[3] Pérez-Marco R.: Notes on the Riemann Hypothesis, arXiv:1707.01770v2 (2018).

[4] Spanier J. and Oldham K.: The Zeta Numbers and Related Functions, An Atlas of Functions, Washington, DC, Hemisphere, 25-33 (1987).

[5] Nuttall J.: Wronskians, Cumulants, and the Riemann hypothesis. Constructive Approximation, 38, 193-212 (2013).

[6] Rubinstein M.: Elliptic curves of high rank and the Riemann zeta function on the one line, Experimental Mathematics, 22(4), 465-480 (2013).

[7] Wolf M.: Evidence in favor of the Baez-Duarte criterion for the Riemann hypothesis, arXiv:math/0605485v4 (2006).

[8] Odlyzko A.: Tables of zeros of the Riemann zeta function, `https://www-users.cse.umn.edu/~odlyzko /zeta_tables/index.html.`

[9] O'Sullivan C.: A generalization of the Riemann-Siegel formula, arxiv:1811.01130v2 (2019).