Don't Trust Setup!
New Directions in Pre-Constrained Cryptography

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Abstract

The recent works of Ananth et al. (ITCS 2022) and Bartusek et al. (Eurocrypt 2023) initiated the study of pre-constrained cryptography which achieves meaningful security even against the system authority, without assuming any trusted setup. In this work we significantly expand this area by defining several new primitives and providing constructions from simple, standard assumptions as follows.

1. *Pre-Constrained Encryption.* We define a weaker notion of pre-constrained encryption (PCE), as compared to the work of Ananth et al. which nevertheless suffices for all known applications. We then provide constructions for *general* constraints, satisfying *malicious* security from a variety of assumptions including DDH, LWE, QR and DCR. Our LWE based construction satisfies *unconditional* security against malicious authorities. In contrast, the construction by Ananth et al. supporting general constraints must rely (inherently) on strong assumptions like indistinguishability obfuscation.

2. *Pre-Constrained Static Functional Encryption.* We provide a new definition for pre-constrained functional encryption in the so-called *static* setting (PCSFE) where the functions to be embedded in secret keys are specified during the setup phase. We provide constructions for PCSFE supporting *general* constraints, with security against *malicious* authorities. As in the case of PCE, our first construction can be instantiated from a variety of assumptions including DDH, LWE, QR and DCR. Our second, LWE based construction satisfies *unconditional* security against malicious authorities.

   We also study *succinctness* in PCSFE, where the public key is sublinear in the number of function keys. We provide the first construction from LWE in the random oracle model. We additionally provide a heuristic construction in the standard model using lattices together with groups.

3. *Pre-Constrained Input Obfuscation.* We define and provide the first construction of pre-constrained input obfuscation from the same assumptions as those used to instantiate PCSFE.

4. *Pre-Constrained Group Signatures.* For pre-constrained group signatures (PCGS), we provide the first construction supporting *general* constraints, achieving *unconditional* security against *malicious* authorities from the LWE assumption. The only other construction by Bartusek et al. supports the restricted set/database membership constraint, and achieves computational security from the DDH assumption (and is therefore quantum insecure).

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1 Introduction

An important balance that the field of cryptography seeks to enable is that between privacy and accountability. As an example, consider the basic primitive of encryption – the key generation procedure for public key encryption (PKE) is typically run by service providers, who are often not trusted by end users but can nevertheless access encrypted user data. This has led to security breaches as well as conflicts between the privacy goals of users and profit/law-enforcement/other considerations of service providers. Anger against mass surveillance and "hidden" trapdoors led to the creation of end-to-end encryption services (E2EE), which guarantees that even the service provider itself cannot access the information that it is storing or transmitting on behalf of the user. Billions of users have adopted E2EE services, happy with the assurance that control of their data is back with themselves.

However, the right to privacy of users must be contrasted with the helplessness of law enforcement agencies, such as the police, in being unable to access conversations that are threatening to society, for example those that incite violence or distribute illegal information such as child sexual abuse material. Is there a way to guarantee user privacy for honest users while enforcing some meaningful notion of accountability for propagating illegal information? The broad question of balancing privacy and accountability has received a lot of attention in recent years, in the context of diverse applications beyond encryption [GP18, FPS+18a, GKL21, AJJM22]. In this work, we further this study by defining several new, arguably meaningful notions for accountable privacy in the setting where there is *no trusted setup*, and by additionally providing constructions for the same. Below, we survey known approaches closest to our approach, and refer the reader to Section 1.4 for a more extensive discussion of related work.

1.1 Prior Approaches: Definitions

Pre-Constrained (Functional) Encryption. The work of Ananth et al. [AJJM22] proposed the notion of *pre-constrained encryption* (PCE) as a new model for advanced encryption schemes where *even the setup authority* does not have full decryption power, but instead can only decrypt some "authorized" values on private data. Moreover, this model does not assume any trusted setup or CRS generation. In more detail, [AJJM22] define pre-constrained encryption with respect to a constraint family $\mathcal{C}$ and a function family $\mathcal{F}$ where the constraint family $\mathcal{C}$ models the kinds of decryption capabilities permitted to the authority. Now, setup is run with respect to some constraint $C \in \mathcal{C}$ and this produces a public key together with a master secret key which is "constrained" to C, the key generation procedure takes this master key and a function $f \in \mathcal{F}$ and outputs a function key SK_f if and only if $C(f) = 1$. Encryption supports computing a ciphertext CT_x for any input x and decryption of CT_x with SK_f enables recovery of $f(x)$. In particular, note that even the authority holding the master secret can only derive function keys and hence perform computations for authorized functions f, as captured by $C(f) = 1$, and nothing else.

The informed reader must have noticed that the notion of pre-constrained encryption corresponds closely to the advanced notion of functional encryption (FE), where decryption keys correspond to functions instead of users. The syntax and security of FE seem very similar – in FE, the setup procedure outputs a public key and master secret key, the key generator can use the master key to derive function keys SK_f for any function f, the encryptor can compute a ciphertext CT_x for any input x and decryption recovers $f(x)$ and nothing else. Indeed, the notion of "hierarchical" functional encryption [BCG+17] supports delegation capabilities which allow the holder of SK_f to, in turn, generate a functional key $\mathsf{SK}_{g \circ f}$ corresponding to the function $g \circ f$ for any function g. The decryptor can now compute $g(f(x))$ using the delegated functional key applied to the ciphertext. Moreover, hierarchical FE can be constructed generically from a sufficiently expressive FE scheme. Then, it is evident that we can capture the PCE functionality by defining $f'(x) = (f, f(x))$ and $g(f'(x)) = f(x)$ if $C(f) = 1$, $\perp$ otherwise. This may cause concern as to whether PCE is just a special case of hierarchical FE.

However, there is a crucial difference between FE and PCE – the latter permits some *security against authority* while the former does not. In PCE, there does not even *exist* a universal master secret key, which is powerful enough to decrypt any message! This is in stark contrast to FE – indeed, the existence of such a master key has been the cause of much concern in FE schemes, and several works [BF03, Cha07, LW11, Goy07, GLSW08, BGJS16, GHMR18, GHM+19, GV20] have attempted to find solutions to mitigate this so-called "key escrow" problem. That said, aside from this (fundamentally) new security property, the notions of FE and PCE are indeed the same – in particular, PCE also admits a key generation procedure that allows computing function keys for dynamically chosen authorized functions. To underline this connection,

we will refer to the notion of PCE defined by [AJJM22] as PCFE in the remainder of this article.

Self-Detecting Encryption. As noted by Ananth et al. [AJJM22], pre-constrained encryption is meaningful and non-trivial to achieve even without the key delegation feature of functional encryption. Closer to basic public key encryption in the pre-constrained setting, is the primitive of self-detecting encryption defined by Alamati et al. [ABD+21]. A self-detecting encryption scheme is similar to a regular public-key encryption with the key difference that it is possible to detect whether the underlying message of a given ciphertext belongs to a database of certain illegal messages. Moreover, such a check can be performed just by knowing the database values, as opposed to the system's secret key – this enables the feature that illegal contents in encrypted messages can be flagged even without knowing the secret key, without compromising the privacy of honest messages.

Formally, SDE, similar to PKE allows to generate a key pair $(\mathsf{pk}, \mathsf{sk})$. There is a hash algorithm which computes a hash value h_{DB} and state information st from a database DB. A user can then generate a ciphertext ct_m of m by using h_{DB} and pk. The secret key holder can decrypt ct_m. In addition, it has a detection algorithm that can recover m from ct_m and st (without sk) if $m \in \mathsf{DB}$. Observe that DB has no relation to $(\mathsf{pk}, \mathsf{sk})$. The security against the authority of self-detecting encryption guarantees $(\mathsf{pk}, (h_{\mathsf{DB}}, \mathsf{st}), \mathsf{Enc}(\mathsf{pk}, h_{\mathsf{DB}}, m_0)) \overset{\mathsf{c}}{\approx} (\mathsf{pk}, (h_{\mathsf{DB}}, \mathsf{st}), \mathsf{Enc}(\mathsf{pk}, h_{\mathsf{DB}}, m_1))$ if $m_0, m_1 \notin \mathsf{DB}$.

The notion of self detecting encryption is philosophically similar to that of public key encryption with a database constraint – the state information st can be seen as a constrained key which only allows the authority to learn m if it belongs to an illegal set. Moreover, this notion supports dynamic updates to the database by changing the hash and satisfies succinctness in that the hash size can be independent of the DB size. However, the constructions of SDE use a CRS generated honestly by a Prm algorithm, hence only achieve security against a semi-honest authority[1], who is moreover, computationally bounded.

Pre-Constrained Group Signatures. Bartusek et al. [BGJP23] further extended the umbrella of pre-constrained cryptography without trusted setup to include signatures. In more detail, they defined the notion of set pre-constrained (SPC) group signatures, which enable tracing of users in messaging systems who sign predefined illegal content while providing security against malicious group managers.

Privacy Preserving Blueprints. The recent work of Kohlweiss et al. [KLN23] also addresses the question of pre-constraining in the context of anonymous credentials. Their primary motivating example is in anonymous e-cash – here, there is a bank that issues e-coins, users who withdraw and spend these coins, and vendors who verify and accept e-coins as payment. Suppose we want an authority to be able to "watch" suspected users for financial fraud. We would like to have a mechanism which will enable an auditor to trace the transactions of these suspected users (on a "watchlist") without revealing the contents of the watchlist or violating the privacy of honest users.

Kohlweiss et al. suggest to enhance the anonymous transaction between the user and verifier with a "privacy preserving blueprint" which allows the user, engaging in a transaction with the verifier, to compute an "escrow" Z which can convince the auditor that s/he is not on the (secret) watchlist without revealing anything else. The watchlist is encoded in a parameter PK_A which is published by the auditor in advance. The verifier verifies that the escrow Z is consistent with the credentials embedded in the e-coins being provided by the user, and rejects the transaction otherwise. At a high level, privacy preserving blueprints can be seen as embedding accountability into a proof system, where there is a user/prover, a vendor/verifier and an auditor who must learn some constrained function about the user's anonymous credentials. On the other hand, PCFE and SDE embed accountability into encryption.

Kohlweiss et al. also consider security against authority, the auditor in this case, and take care to ensure that even a malicious auditor cannot create a blueprint that corresponds to an unauthorized input – for instance, an honest user who is not in the real watchlist. However, their definition relies on trusted parameters generated by an honest setup algorithm – in particular, if the auditor generates the parameters, then security against a malicious auditor cannot be guaranteed[2]. Thus, the reliance on a trusted party is crucial in their notion.

[1] The authors do consider maliciously constructed ciphertexts, but not maliciously generated CRS.

[2] Indeed, there is an attack against the scheme if the auditor generates the trusted parameters. Concretely, they use Pedersen commitment (in Definition 1) to instantiate the commitment scheme and cpar includes group elements for Pedersen commitment. If the auditor knows the discrete log of the group elements, it can easily break the soundness of the blueprint scheme since the adversary can generate a fake commitment that can be opened to an arbitrary value by using the discrete log.

1.2 Prior Approaches: Constructions

Below, we discuss known constructions for pre-constrained cryptography without trusted setup, which is the focus of our work.

Pre-Constrained Functional Encryption. Ananth et al. [AJJM22] (AJJM) provide several constructions for PCFE. Analogously to the literature on FE, they consider special cases of PCFE such as identity-based PCE and attribute-based PCE, denoted as IB-PCE and AB-PCE, respectively. These notions are similar to the celebrated notions of identity-based encryption (IBE) [BF03] and attribute-based encryption (ABE) [GPSW06], except that the functions must now additionally be authorized via a constraint specified during setup. In more detail, an AB-PCE scheme consists of four algorithms – Setup takes as input a constraint C and outputs a public key and master key, KeyGen takes as input a predicate f and outputs SK_f if $C(f) = 1$, Enc takes as input an attribute vector x and a message μ and outputs $(x,\ \mathsf{CT}_x)$ and Dec takes SK_f and $(x,\ \mathsf{CT}_x)$ and outputs μ if $f(x) = 1$. In an IB-PCE, the function f is restricted to be a vector y and we define $f(x) = 1$ iff $x = y$.

AJJM provide the first constructions of PCE – they build AB-PCE for point constraints (i.e. $C_{x^*}(f) = 1$ iff $f(x^*) = 0$) and IB-PCE for general constraints C from the Learning With Errors (LWE) assumption. Both these constructions cleverly use the "punctured" proof technique of the ABE scheme by Boneh et al. [BGG+14] to puncture the master key in the constructions.

To construct AB-PCE for general circuit constraints, they rely on the strong primitive of witness encryption (WE) together with NIZK proofs with perfect soundness. Additionally, they construct PCE for general constraints from the powerful hammer of indistinguishability obfuscation (iO) and NIZKs with perfect soundness. Moreover, they show that the usage of strong primitives like iO and WE is inherent since AB-PCE for general circuit constraints implies WE for NP while PCE for general circuit constraints implies iO for $\mathsf{P/poly}$.

In the context of general constraints, the equivalence of PCFE to strong notions like witness encryption and obfuscation is discouraging – real world applications may require support for arbitrary constraints and the inherent reliance on such strong primitives creates barriers to real world deployment, at least in the near future. As an example, consider their own motivating example of spam filtering, where the secret key should open ciphertexts containing spam – here the functionality "is-spam?" is formalized by some constraint C, which could be arbitrary.

We also note that all their constructions are natively secure against a semi-malicious authority, who can choose the randomness used during setup but cannot arbitrarily deviate from the honest setup procedure. To obtain security against a full-fledged malicious authority who can deviate arbitrarily during setup, they construct a compiler that "bootstraps" semi-malicious to malicious security. While such a compiler has the advantage of being generally applicable, a disadvantage is that it uses non-interactive witness indistinguishable proofs (NIWI), which can be instantiated using pairings or iO [GOS12, BP15] – this induces an additional reliance on these assumptions and is particularly dissatisfying if the underlying construction is based on (conjectured) post-quantum assumptions. On the positive side, their formalism supports dynamic key delegation which could be desirable in some settings.

Pre-Constrained Group Signatures. Bartusek et al. [BGJP23] provided concretely efficient protocols for Set Pre-Constrained (SPC) group signatures for the Decision Diffie Hellman (DDH) assumption, and also provide an implementation to demonstrate practical efficiency. Along the way to constructing SPC group signatures, they also provide a construction for SPC encryption from DDH.

1.3 Our Approach: Untrusted Setup and Unbounded Security

In this work, we study broad primitives under the umbrella of pre-constrained cryptography, with a focus on *untrusted setup* and *unbounded security* against the authority. We believe that unbounded security against authority is much more meaningful than computational security in this setting, since in the real world, authorities are typically much more powerful than users and can possess effectively unbounded computational resources[3]. Note that prior constructions of PCE [AJJM22] and SPC group signatures [BGJP23] satisfy security only against efficient (i.e., PPT) adversaries. In more detail, we ask:

[3]While a similar effect can be achieved by making the security parameter very large, this will negatively impact the performance of the scheme.

Can we construct pre-constrained cryptography that is secure against unbounded authorities? Moreover, can we support general constraints and rely on quantum safe assumptions?

We answer the above question in the affirmative by definining several meaningful notions of pre-constrained cryptography and providing constructions for the same. Some (but not all) of our constructions satisfy security against unbounded authority and can be conjectured post-quantum secure. We discuss our suggested new primitives next.

Pre-Constrained Encryption. Our first contribution is to define a notion of pre-constrained encryption (PCE) which interpolates the notions of self-detecting encryption and pre-constrained functional encryption in terms of functionality, but, similarly to PCFE, does not rely on trusted setup. In more detail, our notion generalizes the database functionality considered by SDE but removes the dynamic key delegation feature considered by PCFE. We adapt the formalism of PCFE for our definition, but emphasize that our definition is not a special case of the AJJM PCFE definition.

In more detail, we define PCE as a PKE scheme where a constraint C is embedded in the secret key created during setup, the encryptor may compute a ciphertext for any message x and decryption succeeds to recover x if and only if $C(x) = 1$. We ask that the public key does not reveal information about C – *constraint-hiding* and that the secret key holder cannot learn any leakage on x if $C(x) = 0$ – *security against authority*. For some applications, it will be meaningful to divide the input into a public part and private part, referred to as attribute and plaintext respectively. Similar to [AJJM22], we envisage the deployment of pre-constrained encryption in conjunction with an end-to-end encryption scheme so that user data is encrypted twice, once under each scheme. The former is used for accountability to the authority, while the latter is used for regular communication. Standard cryptographic tools are used to ensure that the data encrypted under both schemes is the same – please see [AJJM22] for a discussion.

Why is PCE meaningful? The above notion of PCE is arguably natural – indeed, it is closely connected with reusable 2 round 2 party secure computation, which has been studied extensively in the literature [BL20, AJJM20, BGMM20, BJKL21, AJJM21, BGSZ22, IKSS23]. In more detail, PCE can be seen as a special case of reusable 2PC by collapsing the setup and decrypt algorithms of PCE into the same (first) party with input C and by considering encrypt as the second party with input x. However, we believe that it is meaningful to study PCE as a separate notion for the following reasons:

1. The fundamental security property in PCE is against authority without relying on trusted setup – this renders 2PC protocols with trusted setup (such as CRS) or satisfying only semi-malicious security, ill-suited for our setting.

2. Since reusable 2PC security definitions are simulation based, 4 rounds are optimal for malicious security in the plain model [KO04]. However, PCE generalizes PKE so we cannot admit protocols which incur more than 2 rounds. To the best of our knowledge, 2 round maliciously secure reusable MPC in the plain model relies on super-polynomial-time simulation and strong assumptions such as iO [FJK23]. In contrast, our definitions are game based and admit constructions from standard assumptions.

In terms of applications, our notion of PCE enables several new applications:

1. *Checking Data Sanitization:* As our first example, consider a constraint C which encodes some program that checks the content for illegal or undesirable attributes such as violence or racial biases. Now, the setup provides a secret key that encodes C, the encryptor computes a ciphertext for some input x and the authority can recover x if and only if $C(x) = 1$.

2. *Crime Investigation:* During a crime investigation, it is desirable for the authority to have a key encoding some constraint that checks for names of suspects (or such other material) in encrypted chat conversations, and allows them to only decrypt the matching chat messages. Here, the user's message x is encrypted in ciphertext CT_x and recovered by the key if and only if $C(x) = 1$.

3. *Contest:* As a third, more fun example, consider a newspaper company that publishes a crossword puzzle in its newspaper and gives a reward for solving the puzzle. To send the reward, the company needs to know the personal information of the puzzle solvers, but it should be able not to learn personal information of people who send the

wrong answers. Here, we can encode the puzzle answer in the constraint that is embedded in the public and secret keys. A reader computes a ciphertext where the answer and his/her personal information are the attribute and plaintext, respectively. By the constraint-hiding property, the public key does not reveal information about the answer. If the answer is correct, the newspaper has the required information while if it is wrong, readers' personal information is not revealed.

Our notion of PCE also provides a simpler way to realize some applications that were studied by [AJJM22]. For instance, they motivate their notion of identity-based PCE via the example of a "no-fly list" where the public key contains a list (say S) of suspected individuals who should not be allowed to fly. The encryption and function keys are with respect to identities (possibly user public keys). The master key which encodes S can be used to derive the secret key for any identity in S via a key derivation procedure, which in turn can be used to decrypt any ciphertext associated with an identity in S. In our framework too, the public key can encode a no-fly list as a constraint C and the ciphertext can encode the identity x. If $C(x) = 1$, the ciphertext can be decrypted. Thus, the difference is that we do not explicitly provide a key derivation procedure, but the constrained master secret key suffices for decryption.

Pre-Constrained Static Functional Encryption: We extend pre-constrained encryption to encompass a weaker form of functional encryption which we call "static" functional encryption. In this primitive, a set of functions $f_1, \ldots, f_Q$ is specified during setup, which outputs a public key and secret keys for f_i for $i \in [Q]$. These secret keys can only be used to compute the specified function and nothing else just as in functional encryption, except that now even the authority cannot compute anything beyond the specified functions. We refer to this notion as pre-constrained "static" FE because unlike in FE, there is no "on-the-fly" key derivation procedure for functions, and functions must be specified during setup. In contrast, [AJJM22] study a dynamic notion of pre-constrained FE which supports key derivation.

Our notion of static PCFE can also be instantiated using reusable 2PC as discussed above, if the receiver in reusable 2PC prepares many independent first messages. However, as in PCE, we desire a 2 round protocol without trusted setup which satisfies malicious security, hence we believe there is merit in defining this notion separately. Note that our notion also achieves a form of function hiding where the public key does not reveal the embedded functions. However, we emphasize that this notion does not imply indistinguishability obfuscation since the function hiding considered here is much weaker than that considered in FE – in particular, the adversary is not given functional decryption keys.

We believe that our notion suffices for several applications of FE, provides a meaningful guarantee for security against authority, and can satisfy a meaningful notion of function privacy. Additionally, it admits constructions from weaker assumptions than [AJJM22] even for general constraints. To see the usefulness of our primitive, consider one of the standard motivating scenarios of functional encryption – running medical research algorithms on genomic data. In this context, it is customary to assume that a user is only willing to contribute their genomic data for authorized medical studies which reveal the output of the computation but keep the input hidden. Now, let us say that the authorized medical research algorithms are represented by circuits $f_1, \ldots, f_Q$. These functions are fixed in advance, and moreover, must be kept confidential. The user encrypts data and the authority can only use the function keys to learn the output.

Pre-Constrained Input Obfuscation. We also define the notion of pre-constrained input obfuscation where the setup algorithm hides a set of inputs in the public key, the obfuscate algorithm produces an obfuscated program and the evaluation algorithm allows to compute the output of the program on the hidden inputs. To see the motivation for this notion, say that a user/prover claims they have an algorithm for a difficult problem. To check this without leaking the algorithm, the verifier chooses several large random inputs and programs them into the public key. The prover obfuscates her code using this key (note that the inputs remain hidden) and sends this to the verifier, who can test whether the correct output is produced in reasonable time.

Constructions. We provide the following new constructions:

1. *Pre-Constrained Encryption*

   - As a warm-up, we provide a simple construction of PCE for *general* constraints, which relies on two-message statistically sender-private oblivious transfer (SSP-OT) as well as garbled circuits and achieves security against a *malicious* authority. Note that SSP-OT can be based on diverse assumptions such as DDH

[AIR01, NP01], QR and DCR [HK12], LWE [BD18, DGI+19, ADD+23], LPN and Nisan-Wigderson style derandomization [BF22]. This construction bears similarities to constructions of reusable 2PC that have appeared in the literature [GSW23], though the details are quite different.

- Next we provide a construction which supports *general* constraints and relies on malicious circuit-private fully homomorphic encryption (FHE)[OPP14], which can be instantiated from the LWE assumption. This construction satisfies *unconditional* security against a *malicious* authority, and can be conjectured post-quantum secure.
- Our third construction supports the *database* checking constraint, relies on two-message statistically sender-private oblivious transfer, and achieves *unconditional* security against a *malicious* authority. Thus, our construction can be instantiated using DDH, QR, DCR, LWE, LPN and Nisan-Wigderson style derandomization, as discussed above. In contrast, the only other construction by Bartusek et al. [BGJP23] relies on the DDH assumption in the random oracle model. Thus, our construction has the advantage of being secure in the standard model and plausibly post-quantum secure.

2. *Pre-Constrained Static Functional Encryption.* We provide the first constructions for pre-constrained static functional encryption (PCSFE) for *general* constraints, with security against *malicious* authorities. First, we show that natural adaptations of the PCE constructions described above, from two-message SSP-OT and garbled circuits and from malicious circuit-private FHE can be extended to this setting.

   Next, we study *succinctness* in PCSFE, where the public key is sublinear in the number of function keys supported by the scheme. We show that such a notion is *impossible* to achieve against a malicious authority. Relaxing the security to semi-malicious, we provide the first construction from LWE in the random oracle model. We additionally provide a construction that uses a weaker heuristic than ROM from lattices and groups. Providing a construction from standard assumptions in the standard model is a fascinating open problem.

3. *Pre-Constrained Input Obfuscation.* We provide a construction of pre-constrained input obfuscation (PCIO) from pre-constrained static functional encryption. This lets us instantiate PCIO from the same assumptions as those used to instantiate PCSFE.

4. *Pre-Constrained Group Signatures.* For pre-constrained group signatures (PCGS), we provide the first construction supporting *general* constraints, achieving *unconditional* security against *malicious* authorities. The only other construction by Bartusek et al. [BGJP23] supports the restricted set/database membership constraint, and achieves computational security. Our construction relies on the LWE assumption while theirs relies on DDH – thus, our construction has the advantage of being plausibly post-quantum secure. Additionally, our construction achieves a stronger security notion, namely *unlinkability* as compared to theirs. On the other hand, their construction enjoys concrete efficiency and comes with an implementation, whereas ours does not.

1.4 Technical Overview

We proceed to outline the technical ideas used in our constructions.

Pre-Constrained Encryption. We define PCE to have three algorithms – $\mathsf{Setup}, \mathsf{Enc}, \mathsf{Dec}$. Here, Enc and Dec are standard encryption and decryption algorithms of PKE. Setup takes as input a security parameter and a boolean circuit C, and outputs a public key and a secret key. The secret key can recover a plaintext x from encryption of x if $C(x) = 1$. As discussed above, this notion simplifies the notion of PC(F)E defined by Ananth et al. [AJJM22] and can also be seen as a natural generalization of the notion of set pre-constrained encryption by Bartusek et al. [BGJP23]. We remark that our security notions are game-based, similar to Ananth et al. [AJJM22] in contrast to the ideal-functionality-based security notions of Bartusek et al. [BGJP23]. We make this choice to separate the authentication of constraints from the schemes.

We define three security requirements for PCE. One is the standard indistinguishability against adversaries who do not have secret keys. Another is constraint-hiding, which ensures public keys do not reveal information about the circuit embedded during the setup phase. The third is security against authority – this can be semi-honest, semi-malicious or

malicious, which are increasingly stronger. We focus on malicious authority in this section, which allows the authority to behave arbitrarily during the setup phase. The requirement imposed by this notion of security is very strong – a plaintext x must remain hidden from a malicious authority when $C(x) = 0$ even if the authority itself generates a possibly malformed public key $\widetilde{\mathsf{pk}}$ under which the message is encrypted! Here, we let $C \leftarrow \mathsf{Ext}(1^\lambda, \widetilde{\mathsf{pk}})$ where Ext is a possibly inefficient extractor. In our work, we consider unconditional security against the authority – namely, if $C(x) = 0$, even a computationally *unbounded* authority cannot obtain information about x.

It is easy to show that security against semi-honest/semi-malicious/malicious authority and constraint-hiding imply the standard indistinguishability – we do not discuss this in the remainder of this overview.

Pre-Constrained Static Functional Encryption. The notion of PCE admits a natural generalization to the more advanced notion of PCSFE, just as PKE generalizes to FE. A PCSFE consists of three algorithms $(\mathsf{Setup}, \mathsf{Enc}, \mathsf{Dec})$. Here, Enc and Dec are encryption and decryption algorithms of standard FE. Setup takes as input a security parameter and functions $(f_1, \ldots, f_Q)$, and outputs a public key and functional decryption keys $(\mathsf{sk}_{f_1}, \ldots, \mathsf{sk}_{f_Q})$.

There are three security requirements for PCSFE. One is the standard indistinguishability against adversaries who do not have functional decryption keys. Another is function-hiding, which ensures that public keys do not reveal information about the functions embedded during the setup phase. The third is security against malicious authority, which guarantees the following: a ciphertext of x does not provide any information beyond $(f_1(x), \ldots, f_Q(x))$ even if the malicious authority generates a possibly malformed public key $\widetilde{\mathsf{pk}}$ where $(f_1, \ldots, f_Q) \leftarrow \mathsf{Ext}(1^\lambda, \widetilde{\mathsf{pk}})$ and Ext is a possibly inefficient extractor. As in PCE, malicious security can be weakened to semi-malicious, which can be further weakened to semi-honest. When security holds against a computationally unbounded authority, we say that it satisfies unconditional security. As in the case of PCE, security against semi-honest/semi-malicious/malicious authority and function-hiding imply the standard indistinguishability notion.

Pre-Constrained Input Obfuscation. A PCIO has three algorithms, namely $(\mathsf{Setup}, \mathcal{O}, \mathsf{Eval})$. Setup takes as input a security parameter and an input-set $\mathcal{X} := (x_1, \ldots, x_Q)$, and outputs a public key and evaluation key ek. $\mathcal{O}$ takes pk and a circuit C and outputs an obfuscated circuit $\widetilde{C}$. Eval takes ek, $\widetilde{C}$, and x', and outputs y. Correctness posits that if $x' \in \mathcal{X}$, $y = C(x')$.

We observe that PCIO is the dual of PCSFE. We can obtain PCIO from PCSFE if we set $f_i := U[x_i]$ where $U[x_i]$ is a universal circuit that takes a circuit C and outputs $C(x_i)$, $\mathsf{ek} := (\mathsf{sk}_{f_1}, \ldots, \mathsf{sk}_{f_Q})$, and encrypt C in PCSFE. PCIO should have input-set-hiding and virtual black-box security against malicious authority. The former and latter correspond to function-hiding and security against malicious authority, respectively. On the other side, we can also obtain PCSFE from PCIO via a universal circuit. Please see Section 5 for details.

Warmup: Constructions based on OT. As discussed above, Ananth et al. [AJJM22] provide constructions by using the punctured proof technique of Boneh et al. [BGG+14]. This technique is very well suited for constructing pre-constrained encryption since it naturally lends itself to constraining the master key, providing constructions for the restricted primitives of IB-PCE for general constraints and AB-PCE for point constraints. However, this technique is highly specific to LWE-based ABE constructions and appears very hard to generalize (even to punctured proofs in pairing-based ABE constructions, for instance).

From broader assumptions, it appears very challenging to guarantee confidentiality against a malicious authority. This task seems even more difficult when considering security against an unbounded authority. To tackle this difficulty, we take advantage of our simpler definition, which does not require key delegation. In this setting, we then leverage two-message secure two-party computation between an authority with input C (constraint) and a sender with input x (plaintext). The authority obtains $x \cdot C(x)$ (and the sender obtains nothing). If $C(x) = 0$, the authority cannot obtain information about x.

To implement this idea, we make use of two-message statistically sender-private oblivious transfer (SSP-OT) [HK12, BD20]. This primitive satisfies the following two requirements: (1) computational indistinguishability between $\mathsf{ot}_1(0)$ and $\mathsf{ot}_1(1)$ where $\mathsf{ot}_1(\beta)$ is the receiver's message with choice bit β, and (2) statistical indistinguishability between the sender message generated from $(\mu_0, \mu_1, \mathsf{ot}_1)$ and one generated from $(\mu_{\beta'}, \mu_{\beta'}, \mathsf{ot}_1)$ where $\beta' \leftarrow \mathsf{Ext}(\mathsf{ot}_1)$, Ext is a possibly inefficient extractor, and ot_1 is the receiver's message. We can instantiate this primitive with many standard assumptions [HK12, BD20] such as the DDH, QR, and LWE assumptions.

Our PCE scheme can be constructed as follows. The setup algorithm generates an SSP-OT receiver's message $\mathsf{ot}_{1,i}$ with choice bit $\beta_i := C[i]$ for all $i \in [\ell]$ where $|C| = \ell$ and C is a constraint. The public key is $\{\mathsf{ot}_{1,i}\}_{i\in[\ell]}$. The encryption algorithm generates a garbled circuit $\widetilde{P}$ and its labels $\{\mathsf{lb}_{i,b}\}_{i\in[\ell],b\in\{0,1\}}$ of a circuit $P[x]$ that takes as input a circuit C and outputs $x \cdot C(x)$. It also generates an SSP-OT sender's message $\mathsf{ot}_{2,i}$ of $(\mathsf{lb}_{i,0}, \mathsf{lb}_{i,1})$ for all $i \in [\ell]$. A ciphertext consists of $(\widetilde{P}, \{\mathsf{ot}_{2,i}\}_{i\in[\ell]})$. The authority can recover $\{\mathsf{lb}_{i,C[i]}\}_{i\in[\ell]}$ from $\{\mathsf{ot}_{2,i}\}_{i\in[\ell]}$ and $P[x](C) = x \cdot C(x)$ from $\widetilde{P}$ and the labels. The receiver security of SSP-OT guarantees constraint-hiding. The statistical sender security (against malicious receiver) and the garbled circuit security guarantee security against malicious authority since the (inefficient) extractor can extract C from $\{\mathsf{ot}_{1,i}\}_{i\in[\ell]}$ and we can simulate the SSP-OT sender's message and the garbled circuit using only $\{\mathsf{lb}_{i,C[i]}\}_{i\in[\ell]}$ and 0 if $C(x) = 0$.

We then extend this construction to PCSFE to support Q functions. The setup algorithm uses more SSP-OT instances. That is, it generates $Q \times \ell$ SSP-OT receiver's messages where Q is the number of functions and ℓ is the size of functions. The encryption algorithm garbles Q universal circuit $U_i[x]$ that takes as input f_i for $i \in [Q]$ and outputs $f_i(x)$. The rest of the construction follows a similar outline as the PCE scheme above. The public key size is linear in Q.

Although we use statistical sender privacy for security against malicious authority, the constructions do not achieve unconditional security since we use garbled circuits in the ciphertext. If we restrict the constraint/function class to NC^1, these constructions achieve unconditional security since an information-theoretic version of Yao's garbled circuit exists for NC^1 circuits [IK02]. Please see Sections 3.2 and C.1 for details.

Unconditionally secure constructions based on FHE. To obtain unconditional security in PCE and PCSFE, we need to use other tools that support all circuits since information-theoretically secure garbled circuits for all circuits do not exist so far. Our next idea is using circuit private fully homomorphic encryption (FHE) to implement the two-message secure two-party computation. The setup algorithm generates a key pair $(\mathsf{fhe.pk}, \mathsf{fhe.sk}) \leftarrow \mathsf{FHE.Gen}(1^\lambda)$ and a ciphertext $\mathsf{fhe.ct}_C$ of constraint C and outputs $(\mathsf{fhe.pk}, \mathsf{fhe.ct}_C)$ as a public key. The encryption algorithm applies the evaluation algorithm of FHE to $\mathsf{fhe.ct}$ and a circuit $P[x]$ above. The evaluated ciphertext should be statistically indistinguishable from $\mathsf{FHE.Enc}(\mathsf{fhe.pk}, x \cdot C(x))$ due to the circuit privacy of FHE. The authority can recover $x \cdot C(x)$ and nothing beyond that. If $C(x) = 0$, the information about x completely disappears from the evaluated ciphertext. Hence, it guarantees unconditional security. Constraint-hiding follows from the indistinguishability of FHE.

The big issue in this idea is that a malicious authority may generate a malformed public key (and ciphertext), and the circuit privacy of FHE is not guaranteed. Hence, we use *maliciously circuit private* FHE by Ostrovsky, Paskin-Cherniavsky, and Paskin-Cherniavsky [OPP14], which guarantees statistical circuit privacy even when the adversary generates a pair of malformed public key and ciphertext. This security notion perfectly fits our setting. Let $(\widetilde{\mathsf{fhe.pk}}, \widetilde{\mathsf{fhe.ct}})$ be a pair of *adversarially generated* public key and ciphertext. A simulator of maliciously circuit private FHE is given $(\widetilde{\mathsf{fhe.pk}}, \widetilde{\mathsf{fhe.ct}}, G(m))$ and can output a ciphertext which is *statistically* indistinguishable from $\mathsf{Eval}(\widetilde{\mathsf{fhe.pk}}, G, \widetilde{\mathsf{fhe.ct}})$ (i.e., a circuit G is applied to $\widetilde{\mathsf{fhe.ct}}$). Here, m is a plaintext extracted from $\widetilde{\mathsf{fhe.ct}}$. Hence, the unbounded authority obtains $x \cdot C(x)$ and nothing beyond if we set $m := C$ and $G := P[x]$. The simulated ciphertext has no information about x if $C(x) = 0$. It is easy to extend this construction to PCSFE. We generate $(\mathsf{FHE.Enc}(\mathsf{fhe.pk}, f_1), \ldots, \mathsf{FHE.Enc}(\mathsf{fhe.pk}, f_Q))$ as a public key and use $U[x]$ instead of $P[x]$ at the encryption phase. The rest of the construction is the same. The public key is linear in Q. Please see Sections 3.3 and C.2 for details.

Lower bounds for PCSFE. It is not hard to observe that achieving PCSFE with succinct ciphertexts is as hard as achieving indistinguishability obfuscation. Succinct ciphertexts mean the ciphertext size is sublinear in Q. If PCSFE has succinct ciphertexts, we can transform such a PCSFE into a single-key succinct standard FE, which implies indistinguishability obfuscation [BV18] by using known transformations [KNT21, BV18, LPST16a, LPST16b]. Although PCSFE does not have a delegation mechanism, its absence does not prevent the use of these transformations.

We then consider succinct public keys for PCSFE, namely we require that the public key size is sublinear in Q. First, we show that it is impossible to achieve maliciously secure PCSFE with succinct public keys via an incompressibility style argument inspired by the impossibility of simulation-based secure FE [AGVW13]. We sketch this argument next. Let u_i be a uniformly random string. Suppose that we generate $(\mathsf{pk}, \mathsf{sk}_1, \ldots, \mathsf{sk}_Q)$ from Q functions $(g[u_1], \ldots, g[u_Q])$ where $g[u_i]$ is a constant function that outputs u_i for any input and $|\mathsf{pk}| = O(Q^{1-\gamma})$ for some $0 < \gamma < 1$. If a

PCSFE scheme satisfies security against malicious authority, there exists a possibly inefficient extractor Ext that extracts $(g[u_1], \ldots, g[u_Q])$ from pk. That is, Ext can recover $(u_1, \ldots, u_Q)$ only from pk. This extraction is information theoretically impossible since $|\mathsf{pk}| = O(Q^{1-\gamma})$, but $\sum_{i=1}^{Q} |u_i| = O(Q)$. See Section 4.2 for the details. Hence, the best security we can achieve for succinct public key constructions is semi-malicious security against authority.

Succinct Semi-Malicious PCSFE. We now turn to the question of constructing PCSFE with succinct public key in the semi-malicious setting. To begin, we consider the simpler security requirement of semi-*honest* security against the authority – recall that in this notion, the setup algorithm is run honestly, but the adversary is allowed to see the random coins used for the execution.

Our starting point is the idea that to achieve succinct public key in a pre-constrained static FE scheme, we can leverage any FE scheme where the running times of setup and key generation algorithms are independent of the number of collusions supported. Hopefully, encryption and decryption can work as in the underlying FE scheme, yielding the desired functionality, and we can then try to adapt the scheme to obtain semi-malicious security.

A construction of bounded key ciphertext policy functional encryption by Agrawal et al. [AMVY21] enjoys the above feature and serves as a useful starting point. At a high level, their construction works as follows. They make use of a reusable dynamic MPC (RDMPC) protocol [AV19], an identity-based encryption (IBE) scheme and a garbled circuits scheme. An RDMPC consists of a single client and N servers where the client offloads an apriori bounded number of computations Q to the N servers in two phases: (i) an offline phase, in which the secret circuit C held by the client is encoded into N shares (via a circuit encoding algorithm), and one share is provided to each of the N servers, (ii) An online phase, in which the client runs an input encoding algorithm on each of its Q inputs and provides this to all the servers. Each server now performs some local computation on its circuit encoding and the given input encodings, after which, any subset S of servers of some minimum size (say n) can combine their partial outputs to obtain the final output of the computation.

To leverage an RDMPC to build FE, [AMVY21] do the following: the setup algorithm runs the setup of an IBE scheme and outputs a public and master secret key. The key generator, given an input f[4], computes its input encoding using the RDMPC scheme. It then samples a random set of servers Δ and provides an IBE secret key corresponding to this set of servers and the given input encoding. The encryptor computes garbled circuits for the RDMPC local computation circuit for each share of the circuit encoding, and encrypts the labels of these garbled circuits using IBE encryption. To decrypt, the user first performs IBE decryption to obtain the labels corresponding to the input encodings and chosen servers, then executes the garbled circuit to perform the RDMPC local computation and then performs the RDMPC final evaluation to recover the desired output.

For our setting of PCSFE, we first observe that it is beneficial to use the simpler primitive of hash encryption [DGHM18] in place of IBE for our purposes. A hash encryption scheme is similar to a witness encryption scheme and is specified as follows: there is a hash algorithm that hashes an input x to some short value h, an encryption algorithm, which given the hash, encrypts a message μ against h, a position $i \in [|x|]$ and a bit b, and a decrypt algorithm which, given the preimage x to h, recovers μ if and only if $x_i = b$. While hash encryption is known to imply IBE [DGHM18], it is far better for us as a building block since we desire security against a semi-malicious authority. In more detail, the construction of HE from LWE by Döttling et al. has a random matrix as its public key and does *not* have any master secret! Thus, it is immediate that this HE is secure against semi-honest authority in the standard model and against semi-malicious authority in the ROM (simply by using the RO to generate the public matrix). This is in stark contrast to an IBE, which also has an MSK and is much harder to secure against a semi-malicious authority.

The careful reader may object that using HE to build IBE does not really help with security against authority since the resultant IBE must nevertheless have some master key. Here, we use a second trick – instead of using IBE, we leverage the static setting of our FE to directly use HE to construct FE. In more detail, we collapse the setup and keygen of [AMVY21] into setup for our PCSFE where the functions are pre-specified. We use the RDMPC input encoding to encode the Q functions $f_1, \ldots, f_Q$, then hash the concatenation of these encodings using the hash algorithm of the HE scheme. Since the hash is compressing, the public key is succinct. This hash h is now provided to the encryptor, who computes the shares of the circuit encoding, garbles the local computation circuit and uses HE encryption to encode the labels of the garbled circuits. Decryption proceeds by recovering the garbled circuit labels using HE decryption, and executing the garbled circuits to get the RDMPC partial outputs, which are then combined using the RDMPC combine

[4]Although the construction of [AMVY21] is ciphertext-policy, it will be more useful for us to swap the role of the circuit and the input

procedure. The next challenge that is encountered is that the construction obtained via the above route does not satisfy function hiding. We then provide a generic way to lift a construction without function hiding to one with function hiding. Please see Section 4.4 for details.

In the above construction, the only randomness used by setup is in sampling the CRS or public key of the hash encryption scheme. Hence, we immediately get security against authority in the semi-honest setting. Moreover, by shifting to the ROM, we also obtain security against a semi-malicious authority. Can we get security against a semi-malicious authority even in the standard model?

Heuristic: PCSFE with Semi-Malicious Security. We do not have a complete answer to the above question. In the standard model, with randomness chosen adversarially, there are several attacks that the adversary can mount to break the security of a hash encryption scheme. While we are unable to provide a construction that can be proven secure under any clean assumptions, we adapt the LWE-based HE scheme by Döttling et al. [DGHM18] to incorporate several additional safeguards which allow us to weaken the requirement on the hash function which was modeled as a random oracle in the aforementioned construction.

At a high level, our construction adapts the ROM-based construction explained in the previous section to the standard model by (i) reducing the attack surface of the adversary to *linear attacks* by moving LWE samples to the exponent of a suitable group, (ii) Identifying possible linear attacks and constructing countermeasures against them. Nevertheless, we are far from a proof of security, which appears very challenging, even in the generic group model. Our goal in providing this heuristic is to engender efforts in the community to find an attack or a different construction with a proof. We refer the reader to Section 4.5 for details.

Pre-Constrained Group Signatures. Bartusek et al. [BGJP23] define the notion of set pre-constrained group signatures which can be implemented in an end-to-end secure messaging application. This primitive allows to encode a set of predefined illegal content into the public key of a group signature scheme. The main idea is that if a user signs a message that belongs to the predefined illegal set, then the user can be de-anonymized by the group manager. On the contrary, signers of messages outside this set remain anonymous even to the group manager.

We generalize the notion of pre-constrained group signatures (PCGS) to support general constraints beyond set membership. Our definitions for PCGS are game-based instead of ideal-functionality-based definitions of [BGJP23]. We improve the PCGS constructed by [BGJP23] by (i) supporting general constraints, (ii) achieving unconditional security, (iii) obtaining plausible post-quantum security. The construction by Bartusek et al. [BGJP23] designs an SPC group signature scheme from an SPC encryption scheme plus standard cryptographic tools, namely, a one-way function, a digital signature scheme, and a zero-knowledge non-interactive argument of knowledge. Our construction follows the same broad outline except that we use our general PCE and dual-mode NIZK instead of SPC encryption and standard NIZK argument of knowledge, respectively. Additional details need care to handle – for instance, we must use a dual-mode NIZK to achieve unconditional security against malicious authority. In the end, we obtain a PCGS for general constraints against unbounded authorities from the LWE assumption. We refer the reader to Section 6 for the details.

Other Related Work. Next we discuss additional notions related to pre-constrained cryptography.

Conditional disclosure of secrets. We mention the related primitive of conditional disclosure of secrets (CDS). While CDS bears some similarities in syntax to PCE, it is a fundamentally different primitive. In particular, PCE generalizes basic PKE by setting constraint C as the constant function that always outputs 1. It is not known how to achieve PKE only from CDS.

In more detail, in conditional disclosure of secrets [GIKM00], Alice and Bob have access to a *joint source of randomness* r and secret s, and compute $F_1(x,s,r)$ and $F_2(y,s,r)$ from inputs x and y, respectively. Here, x and y are public. They can send some secret s to Carol if $f(x,y) = 1$ using their joint randomness. If $f(x,y) = 0$, Carol cannot obtain any information about s (and r). In PCE, the two parties, i.e., parties running Setup and Enc, do *not* have common randomness. Setup has input a constraint C (unlike CDS, this must be hidden) and outputs pk and sk, Enc has input x and message m and uses pk to generate ct, and Dec uses sk to recover m if $U(C,x) = 1$ (where U is the universal circuit). Here, the output pk of the first party can be used an unbounded number of times for encryption of

different (x_i, m_i) whereas CDS is a one-time primitive. CDS can be seen as a weak symmetric key, one time attribute based encryption scheme and does not satisfy any of the scenarios targeted by our work (or by the work of Ananth et al. [AJJM22]).

Group signature with Message-Dependent Opening (GS-MDO). We also compare our pre-constrained group signatures to the notion of group signature with message-dependent opening (GS-MDO). As discussed in [BGJP23], in a GS-MDO scheme [OSEH13], the trust is divided between two entities : the group manager and the admitter. Both entities need to pool their private information in order to trace a user from a signature. We note that in this notion, if the group manager and the admitter collaborate in a malicious way, they can open any signature to reveal the user's identity. In our pre-constrained group signature scheme, even a malicious group manager cannot open any signature if the underlying message does not satisfy the constraint which was committed to during the setup phase.

Exceptional Access for Law Enforcement. A line of work [Sav18, GKVL21, GSW23] has provided approaches to allowing law enforcement agencies exceptional and controlled access to private user data. As noted by [AJJM22], the work of Green et al. [GKVL21] can be seen as an IB-PCE scheme. We also mention the work of [GP17, FPS+18b] which seeks to enforce accountability on secret laws. These works study questions in the domain of privacy versus accountability but use very different assumptions and techniques than ours.

2 Preliminaries

2.1 Lattice Background

Definition 2.1 (Discrete Gaussian distribution). For any real $s > 0$, $c \in \mathbb{R}^n$, and n-dimensional lattice $\Lambda \subset \mathbb{R}^n$, the discrete gaussian distribution over Λ with parameter s, centered around c, is defined as

$$\forall x \in \Lambda, D_{\Lambda,s,c} = \frac{\rho_{s,c}(x)}{\rho_{s,c}(\Lambda)}$$

where $\rho_{s,c}(x) = \exp(-\pi \cdot ||x-c||^2/s^2)$ and $\rho_{s,c}(\Lambda) = \sum_{x\in\Lambda} \rho_{s,c}(x)$. We use the shorthand $D_{\Lambda,s}$ when the distribution is centered at *zero*.

2.2 Statistical Sender-Private Two-Message Oblivious Transfer

Here we provide the definition of a two-message statistically sender-private oblivious transfer (SSP-OT) scheme, adapted from [BD18].
A two-message oblivious transfer scheme, for an input space $\mathcal{I}$, consists of three algorithms $(\mathsf{OTR}, \mathsf{OTS}, \mathsf{OTD})$ with the following syntax.

- $\mathsf{OTR}(1^\lambda, \beta) \to (\mathsf{ot}_1, \mathsf{st})$. This algorithm takes as input the security parameter λ and a choice bit $\beta \in \{0,1\}$ and outputs a message ot_1 and a secret state st.
- $\mathsf{OTS}(1^\lambda, (\mu_0, \mu_1), \mathsf{ot}_1) \to \mathsf{ot}_2$. This algorithm takes as input the security parameter λ, two inputs $\mu_0, \mu_1 \in \mathcal{I}$, and a message ot_1 and outputs a message ot_2.
- $\mathsf{OTD}(1^\lambda, \beta, \mathsf{st}, \mathsf{ot}_2) \to \mu$. This algorithm takes as input the security parameter λ, the bit $\beta \in \{0,1\}$, the secret state st, and the message ot_2 and outputs $\mu \in \mathcal{I}$.

Next, we define the properties satisfied by a statistical sender-private two-message oblivious transfer scheme.

Definition 2.2 (Correctness). A SSP-OT scheme is said to be correct if for any $\lambda \in \mathbb{N}$, and inputs $\mu_0, \mu_1 \in \mathcal{I}$ the following holds.

$$\Pr\left[\mu = \mu_\beta : \begin{array}{l} \beta \leftarrow \{0,1\}; (\mathsf{ot}_1, \mathsf{st}) \leftarrow \mathsf{OTR}(1^\lambda, \beta); \\ \mathsf{ot}_2 \leftarrow \mathsf{OTS}(1^\lambda, (\mu_0, \mu_1), \mathsf{ot}_1); \\ \mu = \mathsf{OTD}(1^\lambda, \beta, \mathsf{st}, \mathsf{ot}_2) \end{array}\right] = 1.$$

Definition 2.3 (Receiver Privacy). A SSP-OT scheme is said to satisfy receiver privacy if the following two distributions are computationally indistinguishable

$$\{\mathsf{ot}_1 \mid (\mathsf{ot}_1, \mathsf{st}) \leftarrow \mathsf{OTR}(1^\lambda, 0)\} \approx_c \{\mathsf{ot}_1 \mid (\mathsf{ot}_1, \mathsf{st}) \leftarrow \mathsf{OTR}(1^\lambda, 1)\}.$$

Definition 2.4 (Statistical Sender Privacy). A SSP-OT scheme is said to satisfy statistical sender-privacy if there exists an *admissible* unbounded extractor algorithm $\mathsf{OT.Ext}$ such that for any sequence of messages ot_1 output by an unbounded receiver and for any inputs $\mu_0, \mu_1 \in \mathcal{I}$, the following two distributions are statistically indistinguishable

$$\{\mathsf{ot}_2 \mid \mathsf{ot}_2 \leftarrow \mathsf{OTS}(1^\lambda, (\mu_0, \mu_1), \mathsf{ot}_1)\} \approx_s \{\mathsf{ot}_2 \mid \mathsf{ot}_2 \leftarrow \mathsf{OTS}(1^\lambda, (\mu_{\beta'}, \mu_{\beta'}), \mathsf{ot}_1)\}$$

where $\beta' = \mathsf{OT.Ext}(\mathsf{ot}_1)$. We say that $\mathsf{OT.Ext}$ is admissible if for any $\beta \in \{0,1\}$ and randomness r, we have $\beta = \beta'$ where (i) $\mathsf{ot}_1 \leftarrow \mathsf{OTR}(1^\lambda, \beta; r)$ and (ii) $\beta' = \mathsf{OT.Ext}(\mathsf{ot}_1)$.

SSP-OT can be based on a wide variety of assumptions – number-theoretic assumptions such as DDH [AIR01, NP01], QR and DCR [HK12], LWE [BD18, DGI+19, ADD+23], LPN and Nisan-Wigderson style derandomization [BF22].

2.3 Maliciously Circuit-Private FHE

A fully homomorphic encryption (FHE) scheme, for circuit class $\mathcal{C}_n$ of all efficiently computable circuits of input length $n = n(\lambda)$, consists of four algorithms $(\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Eval}, \mathsf{Dec})$ with the following syntax.

$\mathsf{KeyGen}(1^\lambda) \to (\mathsf{pk}, \mathsf{sk})$. The key generation algorithm takes as input the security parameter and outputs a public key pk and a secret key sk.

$\mathsf{Enc}(\mathsf{pk}, m) \to \mathsf{ct}$. The encryption algorithm takes as input the public key pk and a message $m \in \{0,1\}$, and outputs a ciphertext ct.

$\mathsf{Eval}(\mathsf{pk}, C, c_1, \ldots, c_n) \to \hat{\mathsf{ct}}$. The evaluation algorithm takes as input the public key pk, a circuit $C \in \mathcal{C}_n$ with input size n and ciphertexts $c_1, \ldots, c_n$, where $c_i \leftarrow \mathsf{Enc}(\mathsf{pk}, m_i)$ for $i \in [n]$, and outputs a ciphertext $\hat{\mathsf{ct}}$.

$\mathsf{Dec}(\mathsf{sk}, c) \to m$. The decryption algorithm takes as input the secret key sk and a ciphertext c and outputs a message m.

Definition 2.5 (Correctness). A FHE scheme for circuit class $\mathcal{C}_n$ is correct if, for any key-pair $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{KeyGen}(1^\lambda)$, any circuit $C \in \mathcal{C}_n$, any plaintexts $m \in \{0,1\}, m_1 \in \{0,1\}, \ldots, m_n \in \{0,1\}$, the following two condition holds

$$\Pr[m = \mathsf{Dec}(\mathsf{sk}, \mathsf{Enc}(\mathsf{pk}, m))] = 1$$

and

$$\Pr[C(m_1, \ldots, m_n) = \mathsf{Dec}(\mathsf{sk}, \mathsf{Eval}(\mathsf{pk}, C, \mathsf{Enc}(\mathsf{pk}, m_1), \ldots, \mathsf{Enc}(\mathsf{pk}, m_n)))] = 1.$$

Definition 2.6 (Semantic Security). A FHE scheme is said to satisfy semantic security if for any PPT adversary $\mathcal{A}$, there exists a negligible function $\mathsf{negl}(\cdot)$ such that for any two messages $m_0, m_1 \in \{0,1\}$, the following holds

$$\Pr\left[\beta' = \beta : \begin{array}{l} (\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{KeyGen}(1^\lambda); \\ \beta \leftarrow \{0,1\}; \mathsf{ct}_\beta \leftarrow \mathsf{Enc}(\mathsf{pk}, m_\beta); \\ \beta' \leftarrow \mathcal{A}(\mathsf{pk}, \mathsf{ct}_\beta) \end{array}\right] \leq \frac{1}{2} + \mathsf{negl}(\lambda).$$

Definition 2.7 (Malicious Circuit Privacy). A FHE scheme is said to satisfy malicious circuit privacy if there exists unbounded simulator Sim, a and an *admissible* deterministic extractor Ext, such that for all $\lambda \in \mathbb{N}$, all $C \in \mathcal{C}_n$ and any adversary $\mathcal{A}$, the following two distributions are statistically indistinguishable

$$\mathsf{Sim}(\mathsf{pk}^*, c_1^*, \ldots, c_n^*, C(m_1^*, \ldots, m_n^*)) \approx_s \mathsf{Eval}(\mathsf{pk}^*, C, c_1^*, \ldots, c_n^*)$$

where $(\mathsf{pk}^*, c_1^*, \ldots, c_n^*) \leftarrow \mathcal{A}(1^\lambda)$ and $(m_1^*, \ldots, m_n^*) = \mathsf{Ext}(\mathsf{pk}^*, c_1^*, \ldots, c_n^*)$. We say that Ext is admissible if for any $(m_1, \ldots, m_n) \in \{0,1\}^n$, keygen randomness r and encryption randomness $\bar{r}$, we have $(m_1, \ldots, m_n) = (m_1', \ldots, m_n')$ where (i) $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{KeyGen}(1^\lambda; r)$, (ii) $c_i \leftarrow \mathsf{Enc}(\mathsf{pk}, m_i; \bar{r})$ for all $i \in [n]$ and (iii) $m_i' = \mathsf{Ext}(\mathsf{pk}, c_1, \ldots, c_n)$.

Definition 2.8 (Compactness). A FHE scheme is compact if there exists a fixed polynomial bound $p(\cdot)$ such that for any circuit $C \in \mathcal{C}_n$ and ciphertexts $\{c_i\}_{i\in[n]}$ where $c_i \leftarrow \mathsf{Enc}(\mathsf{pk}, m_i)$, we have $|\hat{\mathsf{ct}}| \leq p(\lambda)$ where $\hat{\mathsf{ct}} \leftarrow \mathsf{Eval}(\mathsf{pk}, C, c_1, \ldots, c_n)$, i.e., the size of the evaluated ciphertext is independent of the size of the evaluated circuit.

Definition 2.9 (Linear efficiency). A FHE scheme satisfies linear efficiency if the ciphertext size is linear in the plaintext size. That is, $|\mathsf{ct}| = \mathrm{poly}(\lambda) \cdot |m|$ where $\mathsf{ct} \leftarrow \mathsf{Enc}(\mathsf{pk}, m)$.

Ostrovsky et al. [OPP14] showed that any compact FHE scheme can be converted to one that satisfies malicious circuit privacy using statistically sender private OT and Brakerski and Dottling [BD18] showed how to achieve statistically sender private OT from the Learning With Errors (LWE) assumption. Also, the resulting scheme retains the linear efficiency property of the underlying compact FHE scheme. Since compact FHE with linear efficiency can also be constructed from LWE [GSW13], we obtain the following:

Theorem 2.10 ([OPP14, BD18, GSW13]). There exists a malicious circuit-private compact fully homomorphic encryption scheme with linear efficiency from the polynomially hard learning with errors (LWE) assumption.

2.4 Reusable, Dynamic MPC Protocol

Here we provide the definition of a reusable, dynamic multi-party computation (RDMPC) protocol, adapting the syntax from [AMVY21], for circuit class $\mathcal{C}_{\mathsf{inp}}$ consisting of circuits with input length $\mathsf{inp} = \mathsf{inp}(\lambda)$. The protocol is further associated with polynomial functions $N = N(\lambda, Q)$, $n = n(\lambda, Q)$, and $t = t(\lambda, Q)$.

$\mathsf{CktEnc}(1^\lambda, 1^Q, 1^{\mathsf{inp}}, C) \to (\widehat{C}_1, \ldots, \widehat{C}_N)$**.** The circuit encoding algorithm takes as input the security parameter λ, the number of sessions Q, input length of circuit inp, and a circuit $C \in \mathcal{C}_{\mathsf{inp}}$. It then outputs an encoding $(\widehat{C}_1, \ldots, \widehat{C}_N)$ of the circuit C.

$\mathsf{InpEnc}(1^\lambda, 1^Q, 1^{\mathsf{inp}}, x) \to \widehat{x}$**.** The input encoding algorithm takes as input the security parameter λ, the number of sessions Q, input length of circuit inp, and an input $x \in \{0,1\}^{\mathsf{inp}}$. It then outputs an encoding $\widehat{x}$ of the input x.

$\mathsf{Local}(\widehat{C}_u, \widehat{x}) \to \widehat{y}_u$**.** The local computation algorithm takes as input the u-th encoding $\widehat{C}_u$ of C and an encoding $\widehat{x}$ of x and outputs $\widehat{y}_u$. We assume that this algorithm is deterministic.

$\mathsf{Decode}(\{\widehat{y}_u\}_{u\in S}, S) \to z$**.** The decoding algorithm takes as input a set of encodings $\{\widehat{y}_u\}_{u\in S}$ and a set $S \subseteq [N]$ and outputs z.

Definition 2.11 (Correctness). An RDMPC protocol with parameter (N, n, t) is correct if for all $\mathsf{inp} \in \mathbb{N}$, $x \in \{0,1\}^{\mathsf{inp}}$, $C \in \mathcal{C}_{\mathsf{inp}}$, and set $S \subset [N]$ of size n, we have

$$\Pr\left[\begin{array}{c}(\widehat{C}_1, \ldots, \widehat{C}_N) \leftarrow \mathsf{CktEnc}(1^\lambda, 1^Q, 1^{\mathsf{inp}}, C),\\ \widehat{x} \leftarrow \mathsf{InpEnc}(1^\lambda, 1^Q, 1^{\mathsf{inp}}, x),\\ \mathsf{Decode}\left(\left\{\mathsf{Local}(\widehat{C}_u, \widehat{x})\right\}_{u\in S}, S\right) = C(x)\end{array}\right] = 1$$

where probability is taken over the random coins of CktEnc, InpEnc and Decode.

Definition 2.12 (Security). For a RDMPC protocol for the circuit family $\mathcal{C}_{\mathsf{inp}}$ with parameter (N, n, t), a stateful PPT adversary $\mathcal{A}$, a simulator $\mathsf{Sim} = (\mathsf{Sim}_0, \mathsf{Sim}_1)$, and a coin $\beta \in \{0,1\}$ consider the following experiment $\mathsf{Expt}^{\mathsf{RDMPC}}_{\beta,\mathcal{A}}(1^\lambda)$:

1. **Setup phase:** On input 1^λ, $\mathcal{A}$ submits the query bound 1^Q and input length 1^{inp} of the circuits to the challenger. Note that this defines the total number of parties $N = N(\lambda, Q)$, number of parties $n = n(\lambda, Q)$ participating in any session, threshold $t = t(\lambda, Q)$. The adversary $\mathcal{A}$ also chooses $S_{\mathsf{crr}} \subset [N]$ of size at most t and sets $\Delta^{(1)}, \ldots, \Delta^{(Q)} \subseteq [N]$ such that $|\Delta^{(i)}| = n$ and submits it to the challenger.

2. **Query phase:** During the game, $\mathcal{A}$ is allowed to make a total of Q input encoding queries. First, it makes $R_1 \leq Q$ adaptive input encoding queries. Namely, when $\mathcal{A}$ makes the i-th input encoding query $x^{(i)}$ with $i \leq Q$, the challenger runs $\widehat{x}^{(i)} \leftarrow \mathsf{InpEnc}(1^\lambda, 1^Q, 1^{\mathsf{inp}}, x^{(i)})$ and returns $\widehat{x}^{(i)}$ to $\mathcal{A}$.

3. **Challenge phase:** During the game, $\mathcal{A}$ is allowed to make single circuit encoding query. When $\mathcal{A}$ submits a circuit $C \in \mathcal{C}_{\mathsf{inp}}$, the challenger proceeds as follows.

   - **Real World.** If $\beta = 0$, the challenger runs $(\widehat{C}_1, \ldots, \widehat{C}_N) \leftarrow \mathsf{CktEnc}(1^\lambda, 1^Q, 1^{\mathsf{inp}}, C)$ and returns $\left(\left\{\widehat{C}_j\right\}_{j\in S_{\mathsf{crr}}}, \left\{\mathsf{Local}(\widehat{C}_j, \widehat{x}^{(i)})\right\}_{i\in[R_1], j\in\Delta^{(i)}}\right)$ to $\mathcal{A}$.
   - **Ideal World.** If $\beta = 1$, we define $\mathcal{V}$ as $\mathcal{V} = \{C(x^{(i)}), x^{(i)}\}_{i\in[R_1]}$. Then, the simulator is run as $\left(\mathsf{st}, \left\{\widehat{C}_j\right\}_{j\in S_{\mathsf{crr}}}, \left\{\widehat{y}_j^{(i)}\right\}_{i\in[R_1], j\in\Delta^{(i)}}\right) \leftarrow \mathsf{Sim}_0(1^{|C|}, S_{\mathsf{crr}}, \mathcal{V})$ and the output is returned to $\mathcal{A}$. Here, st is the internal state of the simulator.

4. **Query phase:** $\mathcal{A}$ then makes $R_2 \leq Q - R_1$ input encoding queries. When $\mathcal{A}$ submits an input $x^{(i)} \in \{0,1\}^{\mathsf{inp}}$, the challenger proceeds as follows.

   - **Real World.** If $\beta = 0$, the challenger runs $\widehat{x}^{(i)} \leftarrow \mathsf{InpEnc}(1^\lambda, 1^Q, 1^{\mathsf{inp}}, x^{(i)})$ and returns $\left(\widehat{x}^{(i)}, \left\{\widehat{C}_j(\widehat{x}^{(i)})\right\}_{j\in\Delta^{(i)}}\right)$ to $\mathcal{A}$.
   - **Ideal World.** If $\beta = 1$, we run the simulator as $\left(\widehat{x}^{(i)}, \left\{\widehat{y}_j^{(i)}\right\}_{j\in\Delta^{(i)}}\right) \leftarrow \mathsf{Sim}_1(\mathsf{st}, \Delta^{(i)}, C(x^{(i)}), x^{(i)})$ and returns the output to $\mathcal{A}$.

5. **Output phase:** $\mathcal{A}$ outputs a guess bit β' as the output of the experiment.

We say that an RDMPC protocol is secure if for every adversary $\mathcal{A}$, there exists a PPT simulator Sim such that

$$\mathsf{Adv}_{\mathcal{A}}^{\mathsf{RDMPC}}(\lambda) = \left|\Pr\left[\mathsf{Expt}_{0,\mathcal{A}}^{\mathsf{RDMPC}}(1^\lambda) = 1\right] - \Pr\left[\mathsf{Expt}_{1,\mathcal{A}}^{\mathsf{RDMPC}}(1^\lambda) = 1\right]\right| \leq \mathsf{negl}(\lambda).$$

Theorem 2.13 (Adapted from [AV19]). Assuming the existence of one-way functions, there exists an RDMPC protocol with parameter $N = \Theta(Q^2\lambda)$, $t = \Theta(Q\lambda)$, and $n = \Theta(t)$ for $\mathcal{C}_{\mathsf{inp}}$ with any $\mathsf{inp} = \mathrm{poly}(\lambda)$.

2.5 Hash Encryption

Here we provide the definition of a hash encryption (HE) scheme from [DGHM18].
A HE scheme consists of four algorithms $(\mathsf{Gen}, \mathsf{Hash}, \mathsf{Enc}, \mathsf{Dec})$ with the following syntax.

$\mathsf{Gen}(1^\lambda, m) \to \mathsf{key}$. The generation algorithm takes as input the security parameter, input parameter m and outputs a key key,

$\mathsf{Hash}(\mathsf{key}, x) \to h$. The hashing algorithm takes as input a key key, an input $x \in \{0,1\}^m$ and outputs a hash value h of λ bits.

$\mathsf{Enc}(\mathsf{key}, (h, i, c), \mu) \to \mathsf{ct}$. The encryption algorithm takes as input a key key, a hash value h, an index $i \in [m]$, a bit $c \in \{0,1\}$, and a message $\mu \in \{0,1\}^*$ and outputs a ciphertext ct.

$\mathsf{Dec}(\mathsf{key}, x, \mathsf{ct}) \to \mu'$. The decryption algorithm takes as input a key key, an input $x \in \{0,1\}^m$, and a ciphertext ct and outputs $\mu' \in \{0,1\}^* \cup \{\perp\}$.

Definition 2.14 (Correctness). A HE scheme is said to be correct if for any input $x \in \{0,1\}^m$ and index $i \in [m]$, the following holds

$$\Pr[\mathsf{Dec}(\mathsf{key}, x, \mathsf{Enc}(\mathsf{key}, (\mathsf{Hash}(\mathsf{key}, x), i, x[i]), \mu)) = \mu] \geq 1 - \mathsf{negl}(\lambda)$$

where $x[i]$ denotes the i-th bit of x and the randomness is taken over $\mathsf{key} \leftarrow \mathsf{Gen}(1^\lambda, m)$.

Definition 2.15 (Semi-Honest Security). For a HE scheme and an adversary $\mathcal{A}$, we consider the following experiment $\mathsf{Expt}^{\mathsf{HE}}_{\beta,\mathcal{A}}(1^\lambda)$.

1. $\mathcal{A}$ outputs an input $x \in \{0,1\}^m$.
2. The challenger generates $\mathsf{key} \leftarrow \mathsf{Gen}(1^\lambda, m)$ using some randomness R and sends key to $\mathcal{A}$.
3. $\mathcal{A}$ outputs an index $i \in [m]$, $c \in \{0,1\}$, such that $x_i \neq c$ and two messages μ_0, μ_1.
4. The challenger samples $\beta \leftarrow \{0,1\}$ and returns ct_β to $\mathcal{A}$ where $\mathsf{ct}_\beta \leftarrow \mathsf{Enc}(\mathsf{key}, \mathsf{Hash}(\mathsf{key}, x), i, c), \mu_\beta)$.
5. $\mathcal{A}$ outputs a guess bit β' as the output of the experiment.

We define the advantage $\mathsf{Adv}^{\mathsf{HE}}_{\mathcal{A}}(\lambda)$ of $\mathcal{A}$ in the above game as

$$\mathsf{Adv}^{\mathsf{HE}}_{\mathcal{A}}(\lambda) := \left|\Pr\left[\mathsf{Expt}^{\mathsf{HE}}_{0,\mathcal{A}}(1^\lambda) = 1\right] - \Pr\left[\mathsf{Expt}^{\mathsf{HE}}_{1,\mathcal{A}}(1^\lambda) = 1\right]\right|.$$

We say that a HE scheme is selectively secure if for any PPT adversary $\mathcal{A}$, $\mathsf{Adv}^{\mathsf{HE}}_{\mathcal{A}}(\lambda) \leq \mathsf{negl}(\lambda)$.

Definition 2.16 (Succinctness). We say that a HE scheme is *succinct* if $|\mathsf{key}| = \mathrm{poly}(\lambda)$, where $\mathsf{key} \leftarrow \mathsf{Gen}(1^\lambda, m)$, i.e., the size of key is independent of the size of input $x \in \{0,1\}^m$.

Definition 2.17 (Semi-Malicious security). We say that a HE scheme satisfies *semi-malicious* security if it is secure in the above sense even if the random coins R used by the challenger to generate key in Step 2 are provided by the adversary in Step 1.

Theorem 2.18 ([DGHM18]). There exists a HE scheme satisfying semi-honest security assuming learning with errors (LWE).

Remark 2.19 (Multi-challenge Security). We note that a security game where an adversary can adaptively send polynomially many challenge queries $\{i, \mu_0^{(i)}, \mu_1^{(i)}\}$, for a single challenge input string x, is implied by the single-challenge query security game as defined above via a simple hybrid argument. In the argument we can consider polynomially many single-challenge security sub-hybrids, one for each query.

2.6 Garbling Scheme

Here we provide the definition of a garbling scheme for circuit class $\mathcal{C} = \{C : \{0,1\}^{\ell_{\mathsf{in}}} \to \{0,1\}^{\ell_{\mathsf{out}}}\}$. A garbling scheme for circuit class $\mathcal{C}$ consists of a pair of algorithms $(\mathsf{Garble}, \mathsf{Eval})$ with the following syntax.

$\mathsf{Garble}(1^\lambda, C) \to (\tilde{C}, \mathsf{lb})$. The garbling algorithm takes as input the security parameter λ and a circuit $C \in \mathcal{C}$, and outputs a garbled circuit $\tilde{C}$ and a set of labels $\mathsf{lb} = \{\mathsf{lb}_{i,b}\}_{i\in[\ell_{\mathsf{in}}],b\in\{0,1\}}$.

$\mathsf{Eval}(\tilde{C}, \mathsf{lb}_x) \to y$. The evaluation algorithm takes as input the garbled circuit $\tilde{C}$ and labels corresponding to an input $x \in \{0,1\}^{\ell_{\mathsf{in}}}$, $\mathsf{lb}_x = \{\mathsf{lb}_{i,x_i}\}_{i\in[\ell_{\mathsf{in}}]}$ where x_i denotes the i-th bit of x, and it outputs $y \in \{0,1\}^{\ell_{\mathsf{out}}}$.

A garbling scheme satisfies the following properties.

Definition 2.20 (Correctness). A garbling scheme is said to be correct if for any circuit $C \in \mathcal{C}$ and any input $x \in \{0,1\}^{\ell_{\mathsf{in}}}$, the following holds

$$\Pr\left[\, y = C(x) : (\tilde{C}, \mathsf{lb}) \leftarrow \mathsf{Garble}(1^\lambda, C); y \leftarrow \mathsf{Eval}(\tilde{C}, \mathsf{lb}_x)\right] = 1.$$

Definition 2.21 (Security). A garbling scheme is secure if there exists a PPT simulator SIM such that for any circuit $C \in \mathcal{C}$ and any input $x \in \{0,1\}^{\ell_{\mathsf{in}}}$, the following holds

$$\{(\tilde{C}, \mathsf{lb}_x) \mid (\tilde{C}, \mathsf{lb}) \leftarrow \mathsf{Garble}(1^\lambda, C)\} \approx_c \{(\bar{C}, \bar{\mathsf{lb}}) \mid (\bar{C}, \bar{\mathsf{lb}}) \leftarrow \mathsf{SIM}(1^\lambda, C(x))\}$$

where $\mathsf{lb} = \{\mathsf{lb}_{i,b}\}_{i\in[\ell_{\mathsf{in}}],b\in\{0,1\}}$ and $\mathsf{lb}_x = \{\mathsf{lb}_{i,x_i}\}_{i\in[\ell_{\mathsf{in}}]}$.

Remark 2.22 (Multi-challenge Security). Similar to Remark 2.19, we note that the above security definition implies multi-challenge security of a garbled circuit scheme, where the adversary can adaptively make polynomial number of garbling queries to the challenger,i.e., adversary can query (C_1, x_1) and receive $(\tilde{C}_1, \mathsf{lb}_{x_1})$, then query (C_2, x_2), and so forth. All queries are answered by honestly using Garble algorithm in the real world whereas they are all simulated in the ideal world.

We provide some additional preliminaries in Appendix A.

3 Pre-Constrained Encryption

3.1 Definition

A pre-constrained encryption (PCE) scheme for a circuit family $\mathcal{C}_L = \{C : \{0,1\}^L \to \{0,1\}\}$, consisting of circuits with input length $L = L(\lambda)$, and message space $\mathcal{M} = \{0,1\}^L$ consists of three algorithms $(\mathsf{Setup}, \mathsf{Enc}, \mathsf{Dec})$ with the following syntax.

$\mathsf{Setup}(1^\lambda, C) \to (\mathsf{pk}, \mathsf{sk})$. The setup algorithm takes as input the security parameter, a circuit $C \in \mathcal{C}_L$ and outputs a public key pk and a secret key sk.

$\mathsf{Enc}(\mathsf{pk}, x) \to \mathsf{ct}$. The encryption algorithm takes as input the public key pk, a message $x \in \mathcal{M}$ and outputs a ciphertext ct.

$\mathsf{Dec}(\mathsf{sk}, \mathsf{ct}) \to y$. The decryption algorithm takes as input the secret key sk and a ciphertext ct and outputs a string $y \in \mathcal{M} \cup \{\perp\}$.

Next, we define the properties of a pre-constrained encryption scheme.

Definition 3.1 (Correctness). A PCE scheme is said to be correct if for any $C \in \mathcal{C}_L$, $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Setup}(1^\lambda, C)$ and $x \in \mathcal{M}$ such that $C(x) = 1$, the following holds

$$\Pr[\mathsf{Dec}(\mathsf{sk}, \mathsf{Enc}(\mathsf{pk}, x)) = x] \geq 1 - \mathsf{negl}(\lambda).$$

If correctness holds with probability 1, the scheme is said to satisfy *perfect* correctness.

Definition 3.2 (Constraint-Hiding). A PCE scheme is said to satisfy constraint-hiding security if for any PPT adversary $\mathcal{A}$ the following holds

$$\Pr\left[\begin{array}{l|ll} \mathcal{A}(\mathsf{pk}_\beta) = \beta : & (C_0, C_1) \leftarrow \mathcal{A}; \\ & \beta \leftarrow \{0,1\}; (\mathsf{pk}_\beta, \mathsf{sk}) \leftarrow \mathsf{Setup}(1^\lambda, C_\beta) \end{array}\right] \leq \frac{1}{2} + \mathsf{negl}(\lambda)$$

where $\mathcal{A}$ is admissible if $|C_0| = |C_1|$ and $C_0, C_1 \in \mathcal{C}_L$.

Definition 3.3 (Security against Semi-Honest Authority). A PCE scheme is said to satisfy security against the semi-honest authority if for any PPT and admissible adversary $\mathcal{A}$, the following holds

$$\Pr\left[\begin{array}{l|ll} \mathcal{A}(r, \mathsf{pk}, \mathsf{sk}, \mathsf{ct}_\beta) = \beta : & C \leftarrow \mathcal{A}; (\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Setup}(1^\lambda, C; r) \\ & (x_0, x_1) \leftarrow \mathcal{A}(r, \mathsf{pk}, \mathsf{sk}); \\ & \beta \leftarrow \{0,1\}; \mathsf{ct}_\beta \leftarrow \mathsf{Enc}(\mathsf{pk}, x_\beta) \end{array}\right] \leq \frac{1}{2} + \mathsf{negl}(\lambda)$$

where $\mathcal{A}$ is admissible if and $x_0, x_1 \in \mathcal{M}$ and $C(x_b) = 0$ for $b \in \{0,1\}$.

Definition 3.4 (Unconditional Security against Semi-Honest Authority). We say that a PCE scheme satisfies unconditional security against a semi-honest authority if the above holds for *any* (unbounded) admissible adversary $\mathcal{A}$.

Definition 3.5 (Security against Semi-Malicious Authority). We say that a PCE scheme satisfies (computational/unconditional) security against a semi-malicious authority exactly as (Definition 3.3/3.4) except that in the beginning of the game $\mathcal{A}$ outputs the challenge circuit $C \in \mathcal{C}_L$ *as well as* randomness $r \leftarrow \{0,1\}^{\mathrm{poly}(\lambda)}$.

Definition 3.6 (Security against Malicious Authority). A PCE scheme is said to satisfy security against malicious authority if there exists an *admissible* (possibly inefficient) extractor Ext such that for any non-uniform (stateful) PPT and admissible adversary $\mathcal{A}$, the following holds

$$\Pr\left[\begin{array}{ll} \mathcal{A}(\mathsf{ct}_\beta) = \beta : & (\mathsf{pk}, x_0, x_1) \leftarrow \mathcal{A}; \\ & \beta \leftarrow \{0,1\}; \mathsf{ct}_\beta \leftarrow \mathsf{Enc}(\mathsf{pk}, x_\beta) \end{array}\right] \leq \frac{1}{2} + \mathsf{negl}(\lambda)$$

where $\mathcal{A}$ is admissible if (i) $C \in \mathcal{C}_L$ where $C \leftarrow \mathrm{Ext}(1^\lambda, \mathsf{pk})$, and (ii) $C(x_\beta) = 0$ for $\beta \in \{0,1\}$. We say that Ext is admissible if for every $C \in \mathcal{C}_L$ and randomness r, we have that $C = C'$, where (i) $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Setup}(1^\lambda, C; r)$ and (ii) $C' \leftarrow \mathrm{Ext}(1^\lambda, \mathsf{pk})$.

Definition 3.7 (Unconditional Security against Malicious Authority). We say that a PCE scheme satisfies unconditional security against a malicious authority if the above holds for *any* (unbounded) admissible adversary $\mathcal{A}$.

Definition 3.8 (Security against Outsiders). A PCE scheme is said to satisfy security against outsiders if for any PPT adversary $\mathcal{A}$, any $x_0, x_1 \in \mathcal{M}$, any $C \in \mathcal{C}_L$, the following holds

$$\Pr\left[\begin{array}{ll} \beta' = \beta : & (\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Setup}(1^\lambda, C); \\ & \beta \leftarrow \{0,1\}; \mathsf{ct}_\beta \leftarrow \mathsf{Enc}(\mathsf{pk}, x_\beta); \\ & \beta' \leftarrow \mathcal{A}(\mathsf{pk}, \mathsf{ct}_\beta) \end{array}\right] \leq \frac{1}{2} + \mathsf{negl}(\lambda).$$

We note that if a PCE scheme satisfies constraint-hiding and security against a semi-honest authority, then it also satisfies security against outsiders.

Lemma 3.9. If PCE is constraint-hiding and secure against a semi-honest authority, PCE is also secure against outsiders.

Proof Sketch. Recall that in the security against outsiders we prove that $\mathsf{Enc}(\mathsf{pk}, x_0) \approx_c \mathsf{Enc}(\mathsf{pk}, x_1)$ where $\mathsf{pk} \leftarrow \mathsf{Setup}(1^\lambda, C)$ for any circuit C and messages $x_0, x_1 \in \mathcal{M}$. To prove this, we define the first hybrid game as follows. We change the circuit C to C_0 where $C_0(y) = 0$ for any $y \in \{0,1\}^L$. By the constraint-hiding property of the PCE scheme, the first hybrid game is indistinguishable from the original game where x_0 is encrypted. Next, we define the second hybrid game as follows. We encrypt x_1 instead of x_0. By the security against the semi-honest authority, it holds $\mathsf{Enc}(\mathsf{pk}, x_0) \approx_c \mathsf{Enc}(\mathsf{pk}, x_1)$ and the first and second games are indistinguishable since $C_0(x_0) = 0 = C_0(x_1)$. The second hybrid game is indistinguishable from the original game where x_1 is encrypted due to the constraint-hiding property. Thus, we obtain the lemma. □

Special case PCE. We also consider a special case of PCE, for circuit class $\mathcal{C}_L = \{C : \{0,1\}^L \to \{0,1\}\}$, attribute space $\{0,1\}^L$ and message space $\mathcal{M}$.

Definition 3.10. In a special case of PCE, we split the plaintext into two parts: an attribute $x \in \{0,1\}^L$ and a message $m \in \mathcal{M}$. For correctness, we want that $\mathsf{Dec}(\mathsf{sk}, \mathsf{Enc}(\mathsf{pk}, (x, m))) = m$ if $C(x) = 1$. The security requirements are similar to those defined in Section 3, except that if the scheme does not hide the attribute then we refer to the security against authority/outsiders as message-hiding security against authority/outsider.

3.2 PCE from SSP-OT and Garbled Circuits

In this section we construct a PCE scheme satisfying computational security against a malicious authority for the circuit class $\mathcal{C}_L = \{C : \{0,1\}^L \to \{0,1\}\}$ and a message space $\mathcal{M} = \{0,1\}^L$. For a circuit $C \in \mathcal{C}_L$, we denote the circuit C using a binary string of length ℓ.

Building blocks. We use the following ingredients for our construction.

1. A garbling scheme $\mathsf{GC} = (\mathsf{GC.Garble}, \mathsf{GC.Eval})$ for universal circuit $U[x]$, with $x \in \mathcal{M}$ hardwired, that takes as input a circuit from circuit class $\mathcal{C}_L$. This can be instantiated from Yao's scheme [Yao82], which can be based on any one way function.
2. A two-message SSP-OT scheme $\mathsf{OT} = (\mathsf{OTR}, \mathsf{OTS}, \mathsf{OTD})$ with input space as the space of labels of the above garbled circuit scheme. This can be instantiated from a wide variety of assumptions as discussed in Section 2.2.

Construction. We describe our construction below.

$\mathsf{Setup}(1^\lambda, C) \to (\mathsf{pk}, \mathsf{sk})$. The setup algorithm does the following.

- Parse C as an ℓ bit string and let $C[i]$ denote the i-th bit of C.
- For $i = 1, \cdots, \ell$, compute $(\mathsf{ot}_{1,i}, \mathsf{st}_i) \leftarrow \mathsf{OTR}(1^\lambda, C[i])$.
- Output $\mathsf{pk} = \{\mathsf{ot}_{1,i}\}_{i\in[\ell]}$ and $\mathsf{sk} = \{\mathsf{st}_i, C[i]\}_{i\in[\ell]}$.

$\mathsf{Enc}(\mathsf{pk}, x) \to \mathsf{ct}$. The encryption algorithm does the following.

- Define the circuit $U[x]$, with x hardwired, as follows :
  On input a circuit C, $U[x](C) = x$ if $C(x) = 1$ and $\perp$ if $C(x) = 0$.[5]
- Compute $(\tilde{U}, \{\mathsf{lb}_{i,b}\}_{i\in[\ell],b\in\{0,1\}}) \leftarrow \mathsf{GC.Garble}(1^\lambda, U[x])$.
- Parse $\mathsf{pk} = \{\mathsf{ot}_{1,i}\}_{i\in[\ell]}$
- Compute $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, \mathsf{lb}_{i,0}, \mathsf{lb}_{i,1}, \mathsf{ot}_{1,i})$ for all $i \in [\ell]$.
- Output $\mathsf{ct} = (\tilde{U}, \{\mathsf{ot}_{2,i}\}_{i\in[\ell]})$.

$\mathsf{Dec}(\mathsf{sk}, \mathsf{ct}) \to y$. The decryption algorithm does the following.

- Parse $\mathsf{sk} = \{\mathsf{st}_i, C[i]\}_{i\in[\ell]}$ and $\mathsf{ct} = (\tilde{U}, \{\mathsf{ot}_{2,i}\}_{i\in[\ell]})$.
- For $i = 1, \cdots, \ell$, compute $\mathsf{lb}'_i = \mathsf{OTD}(1^\lambda, C[i], \mathsf{st}_i, \mathsf{ot}_{2,i})$.
- Compute $y \leftarrow \mathsf{GC.Eval}(\tilde{U}, \{\mathsf{lb}'_i\}_{i\in[\ell]})$.
- Output y.

Theorem 3.11. (Correctness.) Suppose the OT scheme and the GC scheme satisfy correctness as defined in Definition 2.2 and Definition 2.20, respectively. Then the PCE construction satisfies *perfect* correctness as defined in Definition 3.1.

Proof. We note that for any $\mathsf{ct} \leftarrow \mathsf{Enc}(\mathsf{pk}, x)$, we have $\mathsf{ct} = (\tilde{U}, \{\mathsf{ot}_{2,i}\}_{i\in[\ell]})$, where $(\tilde{U}, \{\mathsf{lb}_{i,b}\}) \leftarrow \mathsf{GC.Garble}(1^\lambda, U[x])$ and $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, \mathsf{lb}_{i,0}, \mathsf{lb}_{i,1}, \mathsf{ot}_{1,i})$ for $i \in [\ell]$. By the correctness of the OT scheme we have for all $i \in [\ell]$, $\mathsf{OTD}(1^\lambda, C[i], \mathsf{st}_i, \mathsf{ot}_{2,i}) = \mathsf{lb}_{i,C[i]}$ with probability 1. Also, from the correctness of the GC scheme it follows that $\mathsf{GC.Eval}(\tilde{U}, \{\mathsf{lb}_{i,C[i]}\}_{i\in[\ell]}) = U[x](C)$ with probability 1.
Now, if $C(x) = 1$, we get $U[x](C) = x$ with probability 1. Hence the scheme is perfectly correct. □

Constraint-hiding. Constraint-hiding directly follows from the receiver security of the underlying OT scheme since the public key consists of OT messages encoding the bits of the receiver. We prove it formally in Theorem B.1.

Security against malicious authority. This follows from the statistical sender security of the underlying OT scheme and the simulation security of GC scheme.

Theorem 3.12. Suppose the OT scheme satisfies statistical sender-privacy (Definition 2.4) and the GC scheme satisfies simulation security (Definition 2.21). Then the construction of the PCE scheme satisfies security against a malicious authority (Definition 3.6).

Proof. Recall that to prove security against a malicious authority, we want

$$\mathsf{Enc}(\mathsf{pk}, x_0) \approx_c \mathsf{Enc}(\mathsf{pk}, x_1)$$

where $C(x_0) = C(x_1) = 0$, for the circuit C associated with the, possibly malformed, public key pk.
Here we let the extractor Ext of PCE to be the extractor $\mathsf{OT}.\mathsf{Ext}$ of the underlying statistically sender private OT scheme. The admissibility of the Ext follows from the admissibility of $\mathsf{OT}.\mathsf{Ext}$. The proof proceeds via the following sequence of hybrid games between the challenger and a non-uniform PPT adversary $\mathcal{A}$.

[5] Here, $\perp$ is a *fixed* special string. We set $\perp = 0^\ell$.

Hyb_0. This is the real world with $\beta = 0$, i.e., the challenge ciphertext is computed using the input x_0. We write the complete game here to set up the notations and easy reference in later hybrids.

1. $\mathcal{A}$ outputs a public key pk and two inputs $x_0, x_1 \in \{0,1\}^L$ where $|x_0| = |x_1|$.
2. The challenger defines the circuit $U[x_0]$ as in the construction and computes $(\tilde{U}, \{\mathsf{lb}_{i,b}\}_{i\in[\ell],b\in\{0,1\}}) \leftarrow \mathsf{GC.Garble}(1^\lambda, U[x_0])$. It parses $\mathsf{pk} = \{\mathsf{ot}_{1,i}\}_{i\in[\ell]}$ and computes $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, \mathsf{lb}_{i,0}, \mathsf{lb}_{i,1}, \mathsf{ot}_{1,i})$ for all $i \in [\ell]$. It sets $\mathsf{ct} = (\tilde{U}, \{\mathsf{ot}_{2,i}\})$ and returns ct to $\mathcal{A}$.
3. $\mathcal{A}$ outputs a guess bit β'.

Hyb_1. This hybrid is same as the previous hybrid except that the challenger computes $\mathsf{ot}_{2,i}$ differently, i.e., $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, \mathsf{lb}_{i,b}, \mathsf{lb}_{i,b}, \mathsf{ot}_{1,i})$ for all $i \in [\ell]$, where $b = \mathsf{OT.Ext}(\mathsf{ot}_{1,i})$ and $\mathsf{OT.Ext}$ is the extractor in the statistical sender private security of the OT scheme.

Hyb_2. This hybrid is same as the previous hybrid except that the challenger computes the garbled circuit and the labels differently using $\mathsf{GC.Sim}$, i.e., $(\tilde{U}, \{\mathsf{lb}_i\}_{i\in[\ell]}) \leftarrow \mathsf{GC.Sim}(1^\lambda, U[x_0](C))$ where $C = C[1]\dots C[\ell]$; $C[i] = \mathsf{OT.Ext}(\mathsf{ot}_{1,i})$ for $i \in [\ell]$.

Hyb_3. This hybrid is same as the previous hybrid except that the challenger runs the $\mathsf{GC.Sim}$ on $U[x_1](C)$, i.e., $(\tilde{U}, \{\mathsf{lb}_i\}_{i\in[\ell]}) \leftarrow \mathsf{GC.Sim}(1^\lambda, U[x_1](C))$.

Hyb_4. This hybrid is same as the previous hybrid except that the challenger computes the garbled circuit and the labels honestly for the circuit $U[x_1]$, i.e., $(\tilde{U}, \{\mathsf{lb}_{i,b}\}_{i\in[\ell],b\in\{0,1\}}) \leftarrow \mathsf{GC.Garble}(1^\lambda, U[x_1])$.

Hyb_5. This hybrid is same as the previous hybrid except that the challenger computes $\mathsf{ot}_{2,i}$ honestly, i.e., $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(\mathsf{lb}_{i,0}, \mathsf{lb}_{i,1}, \mathsf{ot}_{1,i})$. This is the real world with $\beta = 1$.

Indistinguishability of hybrids. We prove the indistinguishability of the above hybrids in Appendix B.1 □

Security against outsiders. This follows from Lemma 3.9.

We observe that the PCE construction can achieve unconditional security against a malicious authority if we restrict the circuit class to NC^1.

Lemma 3.13. Using an information theoretic version of Yao's garbled circuit [IK02], efficient for NC^1, and a statistically sender private OT scheme, we can achieve unconditional security against a malicious authority for PCE scheme for circuits in NC^1.

3.3 PCE with Unconditional Security from FHE

In this section we provide our construction of a PCE scheme, for circuit family $\mathcal{C}_L = \{C : \{0,1\}^L \to \{0,1\}\}$ and a message space $\mathcal{M} = \{0,1\}^L$, satisfying unconditional security against a malicious authority. For a circuit $C \in \mathcal{C}_L$, we denote the circuit C using a binary string of length ℓ.

Building block. We use a FHE scheme $\mathsf{FHE} = \mathsf{FHE.}(\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Eval}, \mathsf{Dec})$ satisfying malicious circuit privacy. This can be instantiated from LWE (Theorem 2.10).

Construction. We describe our PCE construction below.

$\mathsf{Setup}(1^\lambda, C) \to (\mathsf{pk}, \mathsf{sk})$. The setup algorithm does the following.

- Generate $(\mathsf{FHE.pk}, \mathsf{FHE.sk}) \leftarrow \mathsf{FHE.Gen}(1^\lambda)$.
- Parse C as an ℓ bit string and compute $\mathsf{FHE.ct}_C \leftarrow \mathsf{FHE.Enc}(\mathsf{FHE.pk}, C)$.

– Output $\mathsf{pk} = (\mathsf{FHE.pk}, \mathsf{FHE.ct}_C)$ and $\mathsf{sk} = \mathsf{FHE.sk}$.

$\mathsf{Enc}(\mathsf{pk}, x) \to \mathsf{ct}$. The encryption algorithm does the following.

– Parse $\mathsf{pk} = (\mathsf{FHE.pk}, \mathsf{FHE.ct}_C)$.
– Let $G[x] : \{0,1\}^\ell \to \{0,1\}^L$ be a circuit with x hardwired defined as:
On input a circuit C, $G[x](C) = x$ if $C(x) = 1$, else it outputs $\perp$[6].
– Compute $\mathsf{FHE.ct} \leftarrow \mathsf{FHE.Eval}(\mathsf{FHE.pk}, G[x], \mathsf{FHE.ct}_C)$
– Output $\mathsf{ct} = \mathsf{FHE.ct}$.

$\mathsf{Dec}(\mathsf{sk}, \mathsf{ct}) \to y$. The decryption algorithm does the following.

– Parse $\mathsf{sk} = \mathsf{FHE.sk}$ and $\mathsf{ct} = \mathsf{FHE.ct}$.
– Return $\mathsf{FHE.Dec}(\mathsf{FHE.sk}, \mathsf{FHE.ct})$.

Theorem 3.14. (Correctness.) Suppose the FHE scheme is correct. Then the PCE construction satisfies *perfect* correctness as defined in Definition 3.1.

Proof. Note that for any ciphertext $\mathsf{ct} \leftarrow \mathsf{Enc}(\mathsf{pk}, x)$, we have $\mathsf{ct} = \mathsf{FHE.ct}$, where $\mathsf{FHE.ct} \leftarrow \mathsf{FHE.Eval}(\mathsf{FHE.pk}, G[x], \mathsf{FHE.ct}_C)$. Also, for any key-pair $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Setup}(1^\lambda, C)$, we have $\mathsf{pk} = (\mathsf{FHE.pk}, \mathsf{FHE.ct}_C)$ and $\mathsf{sk} = \mathsf{FHE.sk}$, where $(\mathsf{FHE.pk}, \mathsf{FHE.sk}) \leftarrow \mathsf{FHE.Gen}(1^\lambda)$ and $\mathsf{FHE.ct}_C \leftarrow \mathsf{FHE.Enc}(\mathsf{FHE.pk}, C)$. So, from the correctness of the Eval algorithm of FHE scheme, we have

$$\mathsf{FHE.Dec}(\mathsf{FHE.sk}, \mathsf{FHE.ct}) = G[x](C)$$

with probability 1. Now, if $C(x) = 1$, then $\mathsf{Dec}(\mathsf{sk}, \mathsf{ct}) = \mathsf{FHE.Dec}(\mathsf{FHE.sk}, \mathsf{FHE.ct}) = x$ with probability 1. Hence the construction satisfies correctness. □

Constraint-hiding. This follows immediately from the semantic security of underlying FHE. We prove it formally using the following theorem.

Theorem 3.15. Assume that the FHE scheme satisfies semantic security (Definition 2.6). Then the construction of the PCE scheme satisfies constraint-hiding (Definition 3.2).

Proof. Recall that to show constraint-hiding, we want

$$\{\mathsf{pk} \mid (\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Setup}(1^\lambda, C_0)\} \approx_c \{\mathsf{pk} \mid (\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Setup}(1^\lambda, C_1)\}$$

for any circuits $C_0, C_1 \in \mathcal{C}_L$. We show that if there exists a PPT adversary $\mathcal{A}$ who can distinguish between the above two distributions with non-negligible advantage ϵ, then there exists a PPT adversary $\mathcal{B}$ against the semantic security of the FHE scheme with the same advantage ϵ. The reduction is as follows.

1. $\mathcal{B}$ first runs $\mathcal{A}$. $\mathcal{A}$ outputs the challenge circuits C_0, C_1 such that $C_0, C_1 \in \mathcal{C}_L$.
2. $\mathcal{B}$ parses C_0 and C_1 as an ℓ bit string and forwards it to the FHE challenger as challenge inputs. The FHE challenger generates $(\mathsf{FHE.pk}, \mathsf{FHE.sk}) \leftarrow \mathsf{FHE.Gen}(1^\lambda)$, samples $\beta \leftarrow \{0,1\}$, computes $\mathsf{FHE.ct}_\beta \leftarrow \mathsf{FHE.Enc}(\mathsf{FHE.pk}, C_\beta)$, and returns $\mathsf{FHE.pk}$ and $\mathsf{FHE.ct}_\beta$ to $\mathcal{B}$.
3. $\mathcal{B}$ sets $\mathsf{pk} = (\mathsf{FHE.pk}, \mathsf{FHE.ct}_\beta)$ and forwards it to $\mathcal{A}$.
4. In the end $\mathcal{A}$ outputs a bit β'. $\mathcal{B}$ forwards β' to the FHE challenger.

We observe that if the FHE challenger samples $\beta = 0$, then $\mathcal{B}$ simulated the distribution $D_0 = \{\mathsf{pk} | \mathsf{pk} \leftarrow \mathsf{Setup}(1^\lambda, C_0)\}$, else $D_1 = \{\mathsf{pk} | \mathsf{pk} \leftarrow \mathsf{Setup}(1^\lambda, C_1)\}$ with $\mathcal{A}$. Hence, advantage of $\mathcal{B}$ = $|\Pr(\beta' = 1|\beta = 0) - \Pr(\beta' = 1|\beta = 1)|$ = $|\Pr(\beta' = 1|D_0) - \Pr(\beta' = 1|D_1)|$ = ϵ (by assumption). □

[6]We note that $\perp$ is a *fixed* special string. We set $\perp = 0^\ell$, which lies in the output space of the circuit $G[x]$.

Unconditional security against malicious authority. This follows from the malicious circuit privacy property of the underlying FHE. Recall that maliciously circuit-private FHE [OPP14] ensures that even for possibly malformed $\mathsf{pk}^*, \mathsf{ct}^*$, there exists m^* such that for all G, $\mathsf{Sim}(G(m^*))$ is statistically indistinguishable from $\mathsf{FHE.Eval}(\mathsf{pk}^*, G, \mathsf{ct}^*)$.

Thus, if the challenge circuit C^* satisfies $C^*(x_\beta) = 0$ for $\beta \in \{0,1\}$, and if ct^* is the (possibly malicious) encryption of C^*, then the challenge ciphertext $\mathsf{FHE.Eval}(\mathsf{pk}^*, G[x_\beta], \mathsf{ct}^*)$ is statistically indistinguishable from $\mathsf{Sim}(G[x_\beta](C)) = \mathsf{Sim}(\perp)$ which is clearly independent of x_β and hence the challenge bit β. We prove this formally below.

Theorem 3.16. Assume that the FHE scheme satisfies unconditional malicious circuit privacy (Definition 2.7). Then the construction of the PCE scheme satisfies unconditional security against a malicious authority (Definition 3.7).

Proof. Recall that to show unconditional security against a malicious authority, we want

$$\mathsf{Enc}(\mathsf{pk}, x_0) \approx_s \mathsf{Enc}(\mathsf{pk}, x_1)$$

where $C(x_0) = C(x_1) = 0$, for the circuit C associated with the, possibly malformed, public key pk.
Here we let the extractor Ext of PCE to be the extractor of the underlying malicious circuit private FHE. The admissibility of the Ext follows from the admissibility of FHE extractor. The proof proceeds via the following sequence of hybrid games between the challenger and an unbounded adversary $\mathcal{A}$.

Hyb_0. This is the real world with $\beta = 0$, i.e., the challenge ciphertext is computed using the input x_0. We write the complete game here to set up the notations and easy reference in later hybrids.

1. The adversary outputs a public key pk and two inputs $x_0, x_1 \in \{0,1\}^L$ where $|x_0| = |x_1|$.
2. The challenger parses $\mathsf{pk} = (\mathsf{FHE.pk}, \mathsf{FHE.ct}_C)$, defines $G[x_0]$ as in the construction, and computes $\mathsf{FHE.ct} \leftarrow \mathsf{FHE.Eval}(\mathsf{FHE.pk}, G[x_0], \mathsf{FHE.ct}_C)$. It sets $\mathsf{ct} = \mathsf{FHE.ct}$, and returns ct to the adversary.
3. The adversary outputs a guess bit β'.

Hyb_1. This hybrid is same as the previous hybrid except that the challenger computes FHE.ct as

$$\mathsf{FHE.ct} \leftarrow \mathsf{FHE.Sim}(\mathsf{FHE.pk}, \mathsf{FHE.ct}_C, G[x_0](C))$$

where $C = \mathsf{FHE.Ext}(\mathsf{FHE.pk}, \mathsf{FHE.ct}_C)$.

Hyb_2. This hybrid is same as the previous hybrid except that the challenger computes FHE.ct as

$$\mathsf{FHE.ct} \leftarrow \mathsf{FHE.Sim}(\mathsf{FHE.pk}, \mathsf{FHE.ct}_C, G[x_1](C)).$$

Hyb_3. This hybrid is same as the previous hybrid except that the challenger computes FHE.ct as

$$\mathsf{FHE.ct} \leftarrow \mathsf{FHE.Eval}(\mathsf{FHE.pk}, G[x_1], \mathsf{FHE.ct}_C)$$

where $G[x_1]$ is as defined in the construction.
This is the real world with $\beta = 1$.

Indistinguishability of hybrids. We now show that the consecutive hybrids are indistinguishable.

Claim 3.17. Assume that FHE satisfies malicious circuit privacy, then $\mathsf{Hyb}_0 \approx_s \mathsf{Hyb}_1$.

Proof. We show that if there exists an unbounded adversary $\mathcal{A}$ who can distinguish between Hyb_0 and Hyb_1 with non-negligible advantage ϵ, then there exists an unbounded adversary $\mathcal{B}$ against the malicious circuit privacy security of FHE scheme with the same advantage ϵ. The reduction is as follows.

1. $\mathcal{B}$ first runs $\mathcal{A}$. $\mathcal{A}$ outputs a public key pk and two inputs x_0, x_1.

2. $\mathcal{B}$ parses $\mathsf{pk} = (\mathsf{FHE.pk}, \mathsf{FHE.ct}_C)$ and defines the circuit $G[x_0]$ as in the Enc algorithm. It sends $\mathsf{FHE.pk}$, $\mathsf{FHE.ct}_C$ and $G[x_0]$ as the challenge query to the FHE challenger. The FHE challenger computes $C = \mathsf{FHE.Ext}(\mathsf{FHE.pk}, \mathsf{FHE.ct}_C)$ and returns $\mathsf{FHE.ct}_{\beta^*}$ to $\mathcal{B}$, where $\mathsf{FHE.ct}_0 \leftarrow \mathsf{FHE.Eval}(\mathsf{FHE.pk}, G[x_0], \mathsf{FHE.ct}_C)$ and $\mathsf{FHE.ct}_1 \leftarrow \mathsf{FHE.Sim}(\mathsf{FHE.pk}, \mathsf{FHE.ct}_C, G[x_0](C))$.

3. $\mathcal{B}$ sets and forwards $\mathsf{ct} = \mathsf{FHE.ct}_{\beta^*}$ to $\mathcal{A}$.

4. In the end, $\mathcal{A}$ outputs a bit β'. $\mathcal{B}$ sends β' to the FHE challenger.

We observe that if the FHE challenger samples $\beta^* = 0$, then $\mathcal{B}$ simulated Hyb_0, else Hyb_1 with $\mathcal{A}$. Hence, advantage of $\mathcal{B} = |\Pr(\beta' = 1|\beta^* = 0) - \Pr(\beta' = 1|\beta^* = 1)| = |\Pr(\beta' = 1|\mathsf{Hyb}_0) - \Pr(\beta' = 1|\mathsf{Hyb}_1)| = \epsilon$ (by assumption). □

We note that $\mathsf{Hyb}_1 = \mathsf{Hyb}_2$. By the admissibility of $\mathcal{A}$, we have $C(x_0) = C(x_1) = 0$ and thus $G[x_0](C) = G[x_1](C) = \perp$.
Also, we note that $\mathsf{Hyb}_2 \approx_s \mathsf{Hyb}_3$. The indistinguishability follows from the malicious circuit privacy of underlying FHE scheme and the proof is similar to that of indistinguishability of Hyb_0 and Hyb_1, hence omitted. □

Security against outsiders. This follows from Lemma 3.9.

PCE for Database with Unconditional Security from SSP-OT. We also construct a PCE scheme that supports the database checking constraint, for a database $\mathcal{D} \subseteq \{0,1\}^\ell$ consisting of a single element, $\mathcal{D} = \{y\}$ and message space $\mathcal{M} = \{0,1\}^{\mathsf{len}}$, satisfying unconditional security against a malicious authority using a two-message SSP-OT scheme. For the database functionality, we need to consider special case PCE as described in Definition 3.10, where a plaintext consists of two parts: attribute part x and message part m. We provide this construction in Appendix B.2.

Pre-Constrained Group Signatures from PCE. We observe that our PCE schemes can be used to achieve pre-constrained group signatures for general constraints with simpler game based definitions as compared to the simulation style definitions of [BGJP23] for database constraint. We give our definitions, constructions and security proof in Appendix E.

4 Pre-Constrained Static Functional Encryption

We introduce pre-constrained *static* functional encryption (PCSFE) in this section. This notion is a relaxation of pre-constrained FE introduced by Ananth et al. [AJJM22]. The big difference is that our PCSFE does not have a delegation mechanism. Although we can generate functional decryption keys, all functions are fixed at the setup phase.

4.1 Definition

A pre-constrained static functional encryption scheme (PCSFE) for a function family $\mathcal{F} = \{f : \mathcal{X} \to \mathcal{Y}\}$ with input space $\mathcal{X}$ and output space $\mathcal{Y}$ consists of three algorithms $(\mathsf{Setup}, \mathsf{Enc}, \mathsf{Dec})$ defined as follows.

$\mathsf{Setup}(1^\lambda, \{f_1, \ldots, f_Q\}) \to (\mathsf{pk}, \mathsf{sk}_{f_1}, \ldots, \mathsf{sk}_{f_Q})$. The setup algorithm takes as input the security parameter λ, and functions $f_1, \ldots, f_Q \in \mathcal{F}$, and returns the public key pk and secret keys $\mathsf{sk}_{f_1}, \ldots, \mathsf{sk}_{f_Q}$.

$\mathsf{Enc}(\mathsf{pk}, x) \to \mathsf{ct}$. The encryption algorithm takes as input the public key pk and an input $x \in \mathcal{X}$, and outputs a ciphertext ct.

$\mathsf{Dec}(\mathsf{sk}, \mathsf{ct}) \to y$. The decryption algorithm takes as input a secret key sk and a ciphertext ct, and outputs $y \in \mathcal{Y}$.

Definition 4.1 (Correctness). A PCSFE scheme is said to be correct if for any $f \in \mathcal{F}$ and $x \in \mathcal{X}$, the following holds

$$\Pr\Big[\mathsf{Dec}(\mathsf{sk}_{f_i}, \mathsf{Enc}(\mathsf{pk}, x)) = f_i(x)\Big] \geq 1 - \mathsf{negl}(\lambda)$$

where $i \in [Q]$ and $(\mathsf{pk}, \mathsf{sk}_{f_1}, \ldots, \mathsf{sk}_{f_Q}) \leftarrow \mathsf{Setup}(1^\lambda, \{f_1, \ldots, f_Q\})$.

If correctness holds with probability 1, the scheme is said to be *perfectly* correct.

Definition 4.2 (Function-Hiding). A PCSFE scheme is said to satisfy function-hiding security if for any PPT adversary $\mathcal{A}$, the following holds

$$\Pr\left[\begin{array}{ll} & \{(f_1^0, f_1^1), \ldots, (f_Q^0, f_Q^1)\} \leftarrow \mathcal{A}; \\ \mathcal{A}(\mathsf{pk}_\beta) = \beta : & \beta \leftarrow \{0,1\}; \\ & (\mathsf{pk}_\beta, \mathsf{sk}_{f_1}^\beta, \ldots, \mathsf{sk}_{f_Q}^\beta) \leftarrow \mathsf{Setup}(1^\lambda, \{f_1^\beta, \ldots, f_Q^\beta\}); \end{array}\right] \leq \frac{1}{2} + \mathsf{negl}(\lambda)$$

where $\mathcal{A}$ is admissible if $|f_i^0| = |f_i^1|$ and $f_i^0, f_i^1 \in \mathcal{F}$ for all $i \in [Q]$.

Definition 4.3 (SIM Security Against Semi-Malicious Authority). For a PCSFE scheme, an adversary $\mathcal{A}$ and a PPT simulator SIM we define the experiment for security against semi malicious authority $\mathsf{Expt}_{\beta,\mathcal{A}}^{\mathsf{SMS}}(1^\lambda)$ as follows.

1. $\mathcal{A}$ outputs functions $\{f_1, \ldots, f_Q\}$ and randomness $r \in \{0,1\}^\lambda$.
2. On input $1^\lambda, \{f_1, \ldots, f_Q\}$ and randomness r, the challenger generates $(\mathsf{pk}, \mathsf{sk}_{f_1}, \ldots, \mathsf{sk}_{f_Q}) \leftarrow \mathsf{Setup}(1^\lambda, \{f_1, \ldots, f_Q\}; r)$. It sends the public key pk to $\mathcal{A}$.
3. $\mathcal{A}$ outputs x. the challenger samples $\beta \leftarrow \{0,1\}$. If $\beta = 0$, it computes $\mathsf{ct}_0 \leftarrow \mathsf{Enc}(\mathsf{pk}, x)$ else if $\beta = 1$, it computes $\mathsf{ct}_1 \leftarrow \mathsf{SIM}(\mathsf{pk}, 1^{|x|}, \{f_1(x), \ldots, f_Q(x)\})$. It sends ct_β to $\mathcal{A}$.
4. $\mathcal{A}$ outputs a guess bit β' as the output of the experiment.

We define the advantage $\mathsf{Adv}_{\mathcal{A}}^{\mathsf{SMS}}(\lambda)$ of $\mathcal{A}$ in the above game as

$$\mathsf{Adv}_{\mathcal{A}}^{\mathsf{SMS}}(\lambda) := \left|\Pr\left[\mathsf{Expt}_{0,\mathcal{A}}^{\mathsf{SMS}}(1^\lambda) = 1\right] - \Pr\left[\mathsf{Expt}_{1,\mathcal{A}}^{\mathsf{SMS}}(1^\lambda) = 1\right]\right|.$$

We say that a PCSFE scheme satisfies security against a semi malicious authority if there exists a PPT simulator SIM such that for every PPT adversary $\mathcal{A}$, we have $\mathsf{Adv}_{\mathcal{A}}^{\mathsf{SMS}}(\lambda) \leq \mathsf{negl}(\lambda)$.

Definition 4.4 (Unconditional SIM Security against Semi-Malicious Authority). We say that a PCSFE scheme satisfies unconditional simulation security against a semi malicious authority if there exists a PPT simulator SIM such that for *any* (unbounded) adversary $\mathcal{A}$, the advantage of $\mathsf{Adv}_{\mathcal{A}}^{\mathsf{SMS}}(\lambda)$ of $\mathcal{A}$ (as defined in Definition 4.3) is negligible in the security parameter.

Definition 4.3 is slightly strong compared with the simulation-based security of standard FE [GVW12] in the sense that Sim takes only $(f_1(x), \ldots, f_Q(x))$. In the standard simulation-based security [GVW12], Sim can take $(f_1(x), \ldots, f_Q(x))$, $(f_1, \ldots, f_Q)$, and $(\mathsf{sk}_{f_1}, \ldots, \mathsf{sk}_{f_Q})$. We consider this variant as a relaxed definition.

Definition 4.5 (Relaxed-SIM Security against a Semi-Malicious Authority). We define relaxed-SIM security against a semi-malicious authority exactly as above except that the simulator SIM takes as input $(\mathsf{pk}, 1^{|x|}, \mathcal{V})$ where $\mathcal{V} = \left\{f_i(x), f_i, \mathsf{sk}_{f_i}\right\}_{i \in [Q]}$.

This relaxed definition is also meaningful since Sim does not use information about x beyond $\{f_i(x)\}_{i \in [Q]}$ (and $|x|$).

Definition 4.6 (SIM Security against Malicious Authority). For a PCSFE scheme, an adversary $\mathcal{A}$ and a PPT simulator Sim we define the experiment for security against malicious authority $\mathsf{Expt}_{\beta,\mathcal{A}}^{\mathsf{MS}}(1^\lambda)$ as follows.

1. $\mathcal{A}$ outputs a public key pk and an input x.
2. The challenger samples a random bit $\beta \leftarrow \{0,1\}$. If $\beta = 0$, it computes $\mathsf{ct}_0 \leftarrow \mathsf{Enc}(\mathsf{pk}, x)$ else if $\beta = 1$, it computes $\mathsf{ct}_1 \leftarrow \mathsf{Sim}(\mathsf{pk}, 1^{|x|}, f_1(x), \ldots, f_Q(x))$, where $(f_1, \ldots, f_Q) \leftarrow \mathsf{Ext}(1^\lambda, \mathsf{pk})$. It sends ct_β to $\mathcal{A}$. Here Ext is an extractor algorithm.

3. $\mathcal{A}$ outputs a guess bit β' as the output of the experiment.

We define the advantage $\mathsf{Adv}^{\mathsf{MS}}_{\mathcal{A}}(\lambda)$ of $\mathcal{A}$ in the above game as

$$\mathsf{Adv}^{\mathsf{MS}}_{\mathcal{A}}(\lambda) := \left|\Pr\left[\mathsf{Expt}^{\mathsf{MS}}_{0,\mathcal{A}}(1^\lambda)=1\right] - \Pr\left[\mathsf{Expt}^{\mathsf{MS}}_{1,\mathcal{A}}(1^\lambda)=1\right]\right|.$$

We say that a PCSFE scheme satisfies security against a malicious authority if there exists a PPT simulator Sim and an *admissible* (possibly inefficient) extractor Ext such that for every PPT adversary $\mathcal{A}$, we have $\mathsf{Adv}^{\mathsf{MS}}_{\mathcal{A}}(\lambda) \leq \mathsf{negl}(\lambda)$. We say that Ext is admissible if for every $(f_1,\ldots,f_Q)$ and randomness r, we have that $(f_1,\ldots,f_Q) = (f'_1,\ldots,f'_Q)$, where (i) $(\mathsf{pk},\mathsf{sk}_{f_1},\ldots,\mathsf{sk}_{f_Q}) \leftarrow \mathsf{Setup}(1^\lambda,\{f_1,\ldots,f_Q\};r)$ and (ii) $(f'_1,\ldots,f'_Q) \leftarrow \mathsf{Ext}(1^\lambda,\mathsf{pk})$.

Definition 4.7 (Unconditional SIM Security against Malicious Authority). We say that a PCSFE scheme satisfies unconditional simulation security against a malicious authority if there exists a PPT simulator Sim such that for *any* (unbounded) adversary $\mathcal{A}$, the advantage of $\mathsf{Adv}^{\mathsf{MS}}_{\mathcal{A}}(\lambda)$ of $\mathcal{A}$ (as defined in Definition 4.6) is negligible in the security parameter.

Definition 4.8 (Security against Outsiders). A PCSFE scheme for function family $\mathcal{F}$ is said to satisfy security against outsiders if for any PPT adversary $\mathcal{A}$, any $x_0,x_1 \in \mathcal{X}$, $\{f_1,\ldots,f_Q\} \in \mathcal{F}$, the following holds

$$\Pr\left[\begin{array}{ll} \mathcal{A}(\mathsf{pk},\mathsf{ct}_\beta)=\beta: & (\mathsf{pk},\mathsf{sk}_{f_1},\ldots,\mathsf{sk}_{f_Q}) \leftarrow \mathsf{Setup}(1^\lambda,\{f_1,\ldots,f_Q\}); \\ & \beta \leftarrow \{0,1\}; \mathsf{ct}_\beta \leftarrow \mathsf{Enc}(\mathsf{pk},x_\beta) \end{array}\right] \leq \frac{1}{2} + \mathsf{negl}(\lambda).$$

Lemma 4.9. If a PCSFE scheme satisfies function-hiding and is secure against a (semi-malicious/malicious) authority, then it is also secure against outsiders.

Proof. Recall that in the security against outsiders we prove that $\mathsf{Enc}(\mathsf{pk},x_0) \approx_c \mathsf{Enc}(\mathsf{pk},x_1)$ where $\mathsf{pk} \leftarrow \mathsf{Setup}(1^\lambda,f_1,\ldots,f_Q)$ for any functions $(f_1,\ldots,f_Q)$ and inputs $x_0,x_1 \in \{0,1\}^L$. To prove this, we define the first hybrid game as follows. We change functions f_i to f_0 for all i where $f_0(x)=0$ for any $x \in \{0,1\}^L$. By function-hiding property of the PCSFE scheme, the first hybrid game is indistinguishable from the original game where x_0 is encrypted. Next we define the second hybrid game as follows. We encrypt x_1 instead of x_0. By the simulation security against the malicious authority, it holds that $\mathsf{Enc}(\mathsf{pk},x_0) \approx_c \mathsf{Sim}(\mathsf{pk},1^{|x_0|},\{f_0(x_0),\ldots,f_0(x_0)\})$. Since $|x_0|=|x_1|$ and $f_0(x_0)=f_0(x_1)=0$, we have $\mathsf{Sim}(\mathsf{pk},1^{|x_0|},\{f_0(x_0),\ldots,f_0(x_0)\}) = \mathsf{Sim}(\mathsf{pk},1^{|x_1|},\{f_0(x_1),\ldots,f_0(x_1)\})$. Again using the simulation security against the authority, we have $\mathsf{Sim}(\mathsf{pk},1^{|x_1|},\{f_0(x_1),\ldots,f_0(x_1)\}) \approx_c \mathsf{Enc}(\mathsf{pk},x_1)$. The second hybrid game is indistinguishable from the original game where x_1 is encrypted due to the function-hiding property. Thus, we obtain the lemma. □

Definition 4.10 (Succinct Public Key). We say that a PCSFE scheme has succinct public keys when the size of the public key is sublinear in Q, that is $|\mathsf{pk}| = O(Q^{1-\gamma})$ for some $0<\gamma<1$.

On non-trivial PCSFE. We observe that the notion of PCSFE is trivially achievable if both of the following properties hold.

1. The public key size $|\mathsf{pk}|$ is linear in Q (i.e., non-succinct public keys).
2. The public key pk reveals the information about $(f_1,\ldots,f_Q)$ (i.e., non-function-hiding).

We can easily achieve non-function-hiding PCSFE that does not have succinct public keys. The public key consists of $(f_1,\ldots,f_Q)$, and the encryptor computes and sends $(f_1(x),\ldots,f_Q(x))$ as the ciphertext. The notion is primarily meaningful when the public key hides the functions, or (ideally) is sublinear in Q.

4.2 Lower Bounds for Succinct PCSFE

We present lower bounds of PCSFE in this section. These lower bounds hold even for a weaker IND style security definition provided below.

Definition 4.11 (IND Security against Malicious Authority). A PCSFE scheme is said to satisfy indistinguishability against a malicious authority if for any PPT and admissible adversary $\mathcal{A}$, the following holds

$$\Pr\left[\begin{array}{ll} \mathcal{A}(\mathsf{ct}_\beta) = \beta : & \mathsf{pk}, (x_0, x_1) \leftarrow \mathcal{A}; \\ & \beta \leftarrow \{0,1\}; \mathsf{ct}_\beta \leftarrow \mathsf{Enc}(\mathsf{pk}, x_\beta) \end{array}\right] \leq \frac{1}{2} + \mathsf{negl}(\lambda)$$

$\mathcal{A}$ is admissible if (i) $f_1, \ldots, f_Q \in \mathcal{F}$ where $(\{f_1, \ldots, f_Q\}) \leftarrow \mathsf{Ext}(1^\lambda, \mathsf{pk})$, where Ext is an extractor algorithm and (ii) $f_i(x_0) = f_i(x_1)$ for all $i \in [Q]$.

We observe that for an indistinguishability-based security definition, if $|\mathsf{ct}|$ is sublinear in Q (we call it weakly CT-collusion-succinct in this paper), PCSFE implies IO by known results (via [LPST16b]). In addition, this holds even for security against *semi-honest authority* since the transformation from single-key weakly succinct PKFE to IO needs the indistinguishability-based security for standard FE. Thus, we can say achieving PCSFE with succinct ciphertexts is as hard as achieving IO. We formalise it using the following lemma

Lemma 4.12. If there exists a PCSFE scheme that supports all polynomial size circuits, satisfies indistinguishability against semi-honest authority and the succinct ciphertexts property (i.e., $|\mathsf{ct}|$ is $O(Q^{1-\gamma})$ for some $0 < \gamma < 1$), there exists IO for all polynomial size circuits.

Proof. To prove the lemma, it suffices to show that a PCSFE scheme with succinct ciphertext satisfying IND security implies IO. We observe that

1. We can construct single-key weakly CT-succinct PKFE from Q-key weakly CT-collusion-succinct PCSFE via the transformation by [KNT21, Section 4] (almost the same as [BV18, Section 4.2]). This transformation works since weakly selective security for standard FE (both target plaintexts and functions are fixed at the beginning of the game) is sufficient for our purpose. Note that the syntax of single-key PKFE by Bitansky and Vaikuntanathan [BV18] is the same as single-key PCSFE. Although the transformation by Kitagawa et al. or Bitansky and Vaikuntanathan is for weakly succinct schemes (i.e., the *encryption circuit* is weakly succinct), it works for weakly CT-succinct schemes if the goal is weakly CT-succinct.)

2. We can construct single-key weakly succinct PKFE from single-key weakly CT-succinct and PKFE and the LWE assumption via the transformation by [LPST16a]. This implies IO [BV18]. Or we can use output compressing randomized encoding in the CRS model in [LPST16b]. We can construct weakly sublinear compact randomized encoding scheme for Turing machines in the CRS model from single-key weakly CT-succinct and PKFE (note that [LPST16b] calls weakly CT-succinct as weakly sublinear compact).

□

We also show that it is impossible to achieve PCSFE with succinct public key which is secure against a malicious authority, using a natural incompressibility style argument inspired from [AGVW13]. Moreover, in our setting, the impossibility holds even for the indistinguishability-based definition of PCSFE. We prove it using the following theorem.

Theorem 4.13. There does not exist a PCSFE scheme that supports all polynomial size circuits, satisfies indistinguishability against malicious authority, and whose public key size is $O(Q^{1-\gamma})$ for some $0 < \gamma < 1$ where Q is the number of the functions used in the setup algorithm.

Proof. We consider the following constant function $C[\rho]$ where a uniformly random string $\rho \leftarrow \{0,1\}^\lambda$ is hardwired. The function $C[\rho]$ takes $x \in \{0,1\}^{\lambda/2}$ as as input, and outputs ρ. Let $\mathcal{F}_c := \{C[\rho] : \{0,1\}^{\lambda/2} \to \{0,1\}^\lambda\}$. Assume that there exists a PCSFE scheme that is indistinguishable against malicious authority and has succinct public keys for all polynomial size circuits. Then, there also exists such a PCSFE scheme that supports the function family $\mathcal{F}_c$ above. This implies that there exists an adversary $\mathcal{A}$ that outputs a public key pk and two inputs $(x_0, x_1) \in (\{0,1\}^{\lambda/2})^2$. There

also exists a (possibly inefficient) extractor Ext that takes pk and outputs $(C[\rho_1], \ldots, C[\rho_Q])$. Note that it holds that $C[\rho_i](x_0) = C[\rho_i](x_1)$ for all $i \in [Q]$. The extractor can also compute $\rho_i = C[\rho_i](x')$ for all $i \in [Q]$ where x' is an arbitrary input. Here, $|\mathsf{pk}| = O(Q^{1-\gamma})$ holds due the succinct public key property. That is, $\mathcal{A}$ compressed a uniformly random $Q \times \lambda$ bit string into an $O(Q^{1-\gamma})$ bit string since the extractor can recover a uniformly random $Q \times \lambda$ bit string from pk, which is an $O(Q^{1-\gamma})$ bit string. $O(Q^{1-\gamma})$ is asymptotically smaller than $Q \times \lambda$. This compression is information theoretically impossible since the Kolmogorov complexity of a uniformly random $Q \times \lambda$ string is at least $Q \times \lambda$ by the definition. Thus, the proof is concluded. □

The high level idea is as follows. We consider uniformly random strings for functions f_i. That is, for any input x, $f_i[\rho_i](x)$ outputs ρ_i where $\rho_i \in \{0,1\}^\lambda$ is uniformly random. The compressor is the malicious authority and decompressor is the challenger in the malicious security game. In more detail:

1. The malicious authority $\mathcal{A}$ generates $(\mathsf{pk}, x) \leftarrow \mathcal{A}(f_1, ..., f_Q)$ from Q functions where $|\mathsf{pk}|$ is sublinear in Q. So the size of (pk, x) is $O(Q^{1-\gamma}) + \lambda$.
2. The challenger can reconstruct $f_1(x), ..., f_Q(x)$ from pk and x by running the extractor. Note that such an extractor exists since we assume malicious security[7]. In more detail, the challenger obtains $(f_1, ..., .f_Q) \leftarrow \mathsf{Ext}(\mathsf{pk})$ and $f_1(x), ..., f_Q(x)$.
3. The length of $f_1(x), ..., f_Q(x)$ is $\lambda \times Q$, which is asymptotically larger than $O(Q^{1-\gamma}) + \lambda$.
4. This is impossible since $f_1(x), ..., f_Q(x)$ are uniformly random strings. In more detail, the adversary compressed random strings linear in Q into the short pk sublinear in Q, which is impossible.

4.3 PCSFE with Security against Malicious Authority

For security against a malicious authority, we achieve the following.

1. We can construct a PCSFE scheme for general constraints from a two-message SSP-OT scheme and GC scheme. The construction is similar to that in Section 3.2. Please see Appendix C.1 for the details.
2. We can construct a PCSFE scheme for general constraints from a circuit-private FHE scheme, achieving *unconditional* security against a malicious authority. The construction is similar to that in Section 3.3. Please see Appendix C.2 for the details.

4.4 PCSFE with Succinct PK and Semi-Malicious Security in ROM

We must consider semi-malicious security to achieve succinct public keys due to the impossibility result in Section 4.2. In this section we provide a two-step construction for PCSFE with succinct public keys, as follows.

- First we provide a PCSFE construction, with succinct public key, that does not achieve function-hiding (and consequently security against outsiders).
- Next, we show how to compile the above PCSFE into a function-hiding PCSFE.

4.4.1 Construction without Function-Hiding.

Building blocks. We use the following ingredients for our construction.

1. A reusable, dynamic MPC protocol $\mathsf{RDMPC} = (\mathsf{CktEnc}, \mathsf{InpEnc}, \mathsf{Local}, \mathsf{Decode})$ with $N = \Theta(R^2\lambda)$, $t = \Theta(R\lambda)$, and and $n = \Theta(t)$. This can be instantiated from one way functions (Theorem 2.13). We use $\widehat{\ell}$ to denote the size of the output of InpEnc algorithm. We also denote the size of the circuit $\mathsf{Local}(\widehat{C}_j, \cdot)$ by $\widehat{s}(\lambda, |C|)$, where $\widehat{C}_j$ is the output of CktEnc on input a circuit C.

[7]The extractor may be inefficient but this does not affect the argument, which is information theoretic.

2. A garbled circuit scheme $\mathsf{GC} = (\mathsf{GC.Garble}, \mathsf{GC.Eval})$ for circuit class $\mathcal{C}_{\widehat{\ell}}$, where $\mathcal{C}_{\widehat{\ell}}$ is the set of all polynomial size circuits with input length $\widehat{\ell}$. [8]

3. A hash encryption scheme $\mathsf{HE} = (\mathsf{HE.Gen}, \mathsf{HE.Hash}, \mathsf{HE.Enc}, \mathsf{HE.Dec})$ for hash domain $\{-1, 0, 1\}^{\overline{\ell}}$ satisfying succinctness and semi-malicious security. This can be instantiated using LWE in ROM (Appendix C.5).

Construction. We describe the construction of the succinct PCSFE scheme below.

$\mathsf{Setup}(1^\lambda, \{f_1, \ldots, f_Q\})$. The setup algorithm does the following.

1. For each f_i, where $i \in [Q]$, do the following.
   (a) Parse f_i as an ℓ bit string.
   (b) Compute $\hat{f}_i \leftarrow \mathsf{InpEnc}(1^\lambda, 1^\ell, f_i)$ and let $|\hat{f}_i| = \widehat{\ell}$.
2. For each $i \in [Q]$, sample random set $\Delta^{(i)} \subset [N]$ such that $|\Delta^{(i)}| = n$.
   If $\left|\bigcup_{i,i' \in [Q], i \neq i'} \left(\Delta^{(i)} \cap \Delta^{(i')}\right)\right| > t$, abort.
3. For $i \in [Q]$, let $S_i = \{(i, j, k, \hat{f}_i[k])\}_{j \in \Delta^{(i)}, k \in [\widehat{\ell}]}$, where $\hat{f}_i[k]$ is the k-th bit of $\hat{f}_i$. Set $S := \|_{i \in [Q]} S_i$.
4. Let str be a binary string of length with $\overline{\ell} = Q \cdot N \cdot \widehat{\ell}$, indexed as (i, j, k), for $i \in [Q], j \in [N]$ and $k \in [\widehat{\ell}]$, where
$$\mathsf{str}_{i,j,k} = \begin{cases} 1, & \text{if } (i, j, k, 1) \in S \\ 0, & \text{if } (i, j, k, 0) \in S \\ -1, & \text{if } j \notin \cup_{i \in [Q]} \Delta^{(i)} \end{cases}$$
5. Generate $\mathsf{key} \leftarrow \mathsf{HE.Gen}(1^\lambda, \overline{\ell})$ and compute $h_{\mathsf{str}} \leftarrow \mathsf{HE.Hash}(\mathsf{key}, \mathsf{str})$.
6. Output $\mathsf{pk} = (\mathsf{key}, h_{\mathsf{str}})$ and for $i \in [Q]$,
$$\mathsf{sk}_{f_i} = \left(i, \Delta^{(i)}, \hat{f}_i\right). \tag{1}$$

$\mathsf{Enc}(\mathsf{pk}, x, 1^Q)$. On input the public key $\mathsf{pk} = (\mathsf{key}, h_{\mathsf{str}})$, an input x and the query bound $1 \leq Q \leq 2^\lambda$ in unary form, do the following.

1. Define circuit C_x, which on input a function f, outputs $f(x)$.
2. Compute $(\widehat{C}_{i,1}, \ldots, \widehat{C}_{i,N}) \leftarrow \mathsf{CktEnc}(1^\lambda, 1^\lambda, 1^\ell, C_x)$ for $i \in [Q]$.
3. Define the circuit $\mathsf{L}_{i,j}(\cdot) := \mathsf{Local}(\widehat{C}_{i,j}, \cdot)$ with input length $\widehat{\ell}$.
   For all $i \in [Q]$ and $j \in [N]$, do the following.
   (a) Run the garbling algorithm
$$\left(\left\{\mathsf{lab}_{i,j,k,b}\right\}_{k \in [\widehat{\ell}], b \in \{0,1\}}\right) \leftarrow \mathsf{GC.Garble}(1^\lambda, \mathsf{L}_{i,j}).$$
   (b) For all $k \in [\widehat{\ell}]$ and $b \in \{0, 1\}$, compute [9]
$$\mathsf{HE.ct}_{i,j,k,b} \leftarrow \mathsf{HE.Enc}(\mathsf{key}, (h_{\mathsf{str}}, (i, j, k), b), \mathsf{lab}_{i,j,k,b}).$$

[8] For the ease of notations, we will use the syntax of the garbling scheme by [AMVY21], where the output of the Garble algorithm and the input to the Eval algorithm consists of only labels. This can be shown equivalent to the standard syntax (Section 2.6), by including the garbled circuit into a label.

[9] Note that we do not generate any ciphertext for $b = -1$, as it is not required for the functionality of our scheme. We want to recover the labels $\mathsf{lab}_{i,j,k,b}$ when $\mathsf{str}_{i,j,k} = b$ using the secret key as generated in the setup phase and we set $\mathsf{str}_{i,j,k} = -1$ only when $j \notin \cup_{i \in [Q]} \Delta^{(i)}$, i.e., for those $j \in [N]$ which is not a part of any of the secret keys. So, we do not need $\mathsf{HE.ct}$ corresponding to $b = -1$.

4. Output

$$\mathsf{ct} = \left\{\mathsf{HE.ct}_{i,j,k,b}\right\}_{i\in[Q],j\in[N],k\in[\widehat{\ell}],b\in\{0,1\}}. \tag{2}$$

$\mathsf{Dec}(\mathsf{ct},\mathsf{sk},1^Q)$. On input a secret key sk, a ciphertext ct, and the query bound $1 \le Q \le 2^\lambda$ in unary form, do the following.

1. Parse the secret key as Eq. (1) and the ciphertext as Eq. (2).
2. For all $j \in \Delta^{(i)}$, do the following.
   (a) Compute $\mathsf{lab}'_{i,j,k} := \mathsf{HE.Dec}(\mathsf{key}, (h_{\mathsf{str}}, i, j, k, \hat{f}_i[k]), \mathsf{HE.ct}_{i,j,k,\hat{f}_i[k]})$, for all $k \in [\widehat{\ell}]$, where $\hat{f}_i[k]$ is the k-th bit of $\hat{f}_i$.
   (b) Compute $\widehat{y}'_{i,j} := \mathsf{GC.Eval}(\{\mathsf{lab}'_{i,j,k}\}_{k\in[\widehat{\ell}]})$.
3. Compute and output $z_i = \mathsf{Decode}(\{\widehat{y}'_{i,j}\}_{j\in\Delta^{(i)}}, \Delta^{(i)})$.

Succinct public keys. We note that in the above construction $\mathsf{pk} = (\mathsf{key}, h_{\mathsf{str}})$, where $\mathsf{key} \leftarrow \mathsf{HE.Gen}(1^\lambda, \overline{\ell})$ and $h_{\mathsf{str}} \leftarrow \mathsf{HE.Hash}(\mathsf{key}, \mathsf{str})$. For the succinct public keys we instantiate the underlying HE scheme as described in Appendix C.5, where $|\mathsf{key}| = \lambda$ and $|h_{\mathsf{str}}| = \lambda$. So it follows that $|\mathsf{pk}| = 2\lambda$, which is independent of the number of functions(Q) in the setup phase. Hence the above construction achieves full succinctness.

Correctness. We prove that the above construction of PCSFE is correct via the following theorem.

Theorem 4.14. Suppose the HE scheme, GC and RDMPC scheme satisfies correctness as defined in Definition 2.14, 2.20 and 2.11 respectively. Then the above construction of PCSFE is correct.

Proof. By the correctness of HE scheme, we have

$$\mathsf{lab}_{i,j,k,\hat{f}[k]} = \mathsf{HE.Dec}(\mathsf{key}, (\mathsf{Hash}(\mathsf{key},\mathsf{str}), i, j, k, \hat{f}[k]), \mathsf{HE.ct}_{i,j,k,\hat{f}[k]})$$

with all but negligible probability, since $\mathsf{str}_{i,j,k} = \hat{f}[k]$ by the definition of str. So, $\mathsf{lab}'_{i,j,k} = \mathsf{lab}_{i,j,k,\hat{f}[k]}$ holds for all $\mathsf{lab}'_{i,j,k}$ recovered in Step 2a of the decryption algorithm. Next, by the correctness of GC scheme, we have

$$\mathsf{L}_{i,j}(\hat{f}_i) = \mathsf{GC.Eval}(\widetilde{L}_{i,j}, \{\mathsf{lab}_{i,j,k,\hat{f}[k]}\}_{k\in[\widehat{\ell}]}).$$

So, $\widehat{y}'_{i,j} = \mathsf{L}_{i,j}(\hat{f}_i) = \mathsf{Local}(\widehat{C}_{i,j}, \hat{f}_i)$ for all $\widehat{y}'_{i,j}$ recovered in Step 2b of the decryption algorithm. Finally, since $|\Delta^{(i)}| = n$ we have

$$C_x(f_i) = \mathsf{Decode}(\{\mathsf{Local}(\widehat{C}_{i,j}, \hat{f}_i)\}_{j\in\Delta^{(i)}}, \Delta^{(i)})$$

by the correctness of the RDMPC scheme. So, $z_i = C_x(f_i) = f_i(x)$ for z_i recovered in Step 3 of the decryption algorithm. □

Relaxed-SIM security against a semi-malicious authority. We prove the security of our succinct PCSFE scheme using the following theorem.

Theorem 4.15. Assume that HE scheme is secure against a semi malicious setup , GC is a secure garbled circuit scheme, and RDMPC is secure as per Definition 2.15, 2.21, and 2.12 respectively. Then the above construction of succinct PCSFE scheme satisfies security against a semi malicious authority Definition 4.5.

We prove the theorem in Appendix C.3.

4.4.2 Construction with Function-Hiding.

In this section we provide a compiler to convert a succinct and semi-malicious PCSFE without function-hiding and security against outsiders into a PCSFE with function-hiding and security against outsiders using a maliciously circuit-private FHE with linear efficiency. We also show that the resulting scheme retains the succinctness and security against a semi-malicious authority property of the underlying PCSFE scheme.

Building Blocks. We use the following ingredients for our construction.

1. A PCSFE scheme $\mathsf{SFE} = (\mathsf{SFE.Setup}, \mathsf{SFE.Enc}, \mathsf{SFE.Dec})$ satisfying (SIM/relaxed-SIM) security against a semi-malicious authority with succinct public keys.
2. A compact maliciously circuit-private FHE scheme with linear efficiency $\mathsf{FHE} = (\mathsf{FHE.KeyGen}, \mathsf{FHE.Enc}, \mathsf{FHE.Dec})$.

Construction. Below we provide the description of our compiler.

$\mathsf{Setup}(1^\lambda, f_1, \ldots, f_Q)$. The setup algorithm does the following.

1. Generate $(\mathsf{FHE.pk}, \mathsf{FHE.sk}) \leftarrow \mathsf{FHE.KeyGen}(1^\lambda)$ and $(\mathsf{SFE.pk}, \mathsf{SFE.sk}_{f_1}, \ldots, \mathsf{SFE.sk}_{f_Q}) \leftarrow \mathsf{SFE.Setup}(1^\lambda, f_1, \ldots, f_Q)$.
2. Compute $\mathsf{FHE.ct} \leftarrow \mathsf{FHE.Enc}(\mathsf{FHE.pk}, \mathsf{SFE.pk})$.
3. Output $\mathsf{pk} = (\mathsf{FHE.pk}, \mathsf{FHE.ct})$ and $\mathsf{sk}_{f_i} = (\mathsf{FHE.sk}, \mathsf{SFE.sk}_{f_i})$ for $i \in [Q]$.

$\mathsf{Enc}(\mathsf{pk}, x)$. The encryption algorithm does the following.

1. Parse $\mathsf{pk} = (\mathsf{FHE.pk}, \mathsf{FHE.ct})$.
2. Choose encryption randomness r for $\mathsf{SFE.Enc}$.
3. Let $G[x, r]$ be a circuit that on input key evaluates $\mathsf{SFE.Enc}(\mathsf{key}, x; r)$ (this notation means encryption randomness is r).
4. Compute $\mathsf{FHE.}\widehat{\mathsf{ct}} \leftarrow \mathsf{FHE.Eval}(\mathsf{FHE.pk}, G[x, r], \mathsf{FHE.ct})$.
5. Output $\mathsf{ct} = \mathsf{FHE.}\widehat{\mathsf{ct}}$.

$\mathsf{Dec}(\mathsf{sk}, \mathsf{ct})$. The decryption algorithm does the following.

1. Parse $\mathsf{sk} = (\mathsf{FHE.sk}, \mathsf{SFE.sk}_{f_i})$ for some $i \in [Q]$ and $\mathsf{ct} = \mathsf{FHE.}\widehat{\mathsf{ct}}$.
2. Compute $y \leftarrow \mathsf{FHE.Dec}(\mathsf{FHE.sk}, \mathsf{FHE.}\widehat{\mathsf{ct}})$.
3. Output $\mathsf{SFE.Dec}(\mathsf{SFE.sk}_{f_i}, y)$.

Succinct public keys. First we note that $|\mathsf{SFE.pk}| = O(Q^{1-\gamma})$ since SFE has succinct public keys. Also, it is easy to see that $|\mathsf{FHE.pk}|$ is independent of Q and $|\mathsf{FHE.ct}| = \mathrm{poly}(\lambda) \cdot |\mathsf{SFE.pk}| = \mathrm{poly}(\lambda) \cdot O(Q^{1-\gamma})$ due to the linear efficiency of FHE. Hence, we have $|\mathsf{pk}| = O(Q^{1-\gamma})$.

Next, we show that the above compiler satisfies function hiding. We refer the readers to Appendix C.4 for the correctness and other security properties of the scheme.

Function-hiding. This follows from the semantic security of underlying FHE scheme. We prove it using the following theorem.

Theorem 4.16. Assume that the FHE scheme satisfies semantic security (Definition 2.6). Then the above construction of the PCSFE scheme satisfies function-hiding (Definition 4.2).

Proof. Recall that to show constraint hiding, we want

$$\{\mathsf{pk}|\mathsf{pk} \leftarrow \mathsf{Setup}(1^\lambda, f_1^0, \dots f_Q^0)\} \approx_c \{\mathsf{pk}|\mathsf{pk} \leftarrow \mathsf{Setup}(1^\lambda, f_1^1, \dots f_Q^1)\}$$

where $f_i^b \in \mathcal{F}$ for $i \in [Q]$ and $b \in \{0,1\}$. We show that if there exists a PPT adversary $\mathcal{A}$ who can distinguish between the above two distributions with non-negligible advantage ϵ, then there exists a PPT adversary $\mathcal{B}$ against the semantic security of the FHE scheme with the same advantage ϵ. The reduction is as follows.

1. $\mathcal{B}$ first runs $\mathcal{A}$. $\mathcal{A}$ outputs the challenge functions $\{(f_1^0, f_1^1), \dots, (f_Q^0, f_Q^1)\}$.
2. $\mathcal{B}$ computes $(\mathsf{SFE.pk}^b, \mathsf{SFE.sk}_{f_1}^b, \dots, \mathsf{SFE.sk}_{f_Q}^b) \leftarrow \mathsf{SFE.Setup}(1^\lambda, f_1^b, \dots, f_Q^b)$ for $b \in \{0,1\}$.
3. $\mathcal{B}$ sends $(\mathsf{SFE.pk}^0, \mathsf{SFE.pk}^1)$ to the FHE challenger. The challenger generates $(\mathsf{FHE.pk}, \mathsf{FHE.sk}) \leftarrow \mathsf{FHE.KeyGen}(1^\lambda)$, samples a bit $\beta \leftarrow \{0,1\}$, computes $\mathsf{FHE.ct} \leftarrow \mathsf{FHE.Enc}(\mathsf{FHE.pk}, \mathsf{SFE.pk}^\beta)$, and returns $\mathsf{FHE.pk}, \mathsf{FHE.ct}$ to $\mathcal{B}$.
4. $\mathcal{B}$ sets $\mathsf{pk} = (\mathsf{FHE.pk}, \mathsf{FHE.ct})$ and forwards it to $\mathcal{A}$.
5. In the end $\mathcal{A}$ outputs a bit β'. $\mathcal{B}$ forwards β' to the FHE challenger.

We observe that if the FHE challenger samples $\beta = 0$, then $\mathcal{B}$ simulated the distribution $D_0 = \{\mathsf{pk}|\mathsf{pk} \leftarrow \mathsf{Setup}(1^\lambda, f_1^0, \dots f_Q^0)\}$, else $D_1 = \{\mathsf{pk}|\mathsf{pk} \leftarrow \mathsf{Setup}(1^\lambda, f_1^1, \dots f_Q^1)\}$ with $\mathcal{A}$. Hence, advantage of $\mathcal{B} = |\Pr(\beta' = 1|\beta = 0) - \Pr(\beta' = 1|\beta = 1)| = |\Pr(\beta' = 1|D_0) - \Pr(\beta' = 1|D_1)| = \epsilon$ (by assumption). □

4.5 Heuristic: PCSFE with Succinct PK and Semi-Malicious Security

In this section, we provide a heuristic construction for succinct PCSFE with semi-malicious security in the standard model. While the random oracle is itself a heuristic, the hope here is to rely on a weaker heuristic than ROM. For this, it suffices to construct a succinct hash encryption scheme with semi-malicious security as described above.

Recap of DGHM [DGHM18] Hash Encryption from LWE.

$\mathsf{Gen}(1^\lambda, m)$: Sample $\mathbf{A} \in \mathbb{Z}_p^{m \times \lambda}$.

$\mathsf{Hash}(\mathsf{key}, x \in \{0,1\}^m)$: Interpret $\mathsf{key} = \mathbf{A}$. Output $x^\top \mathbf{A}$.

$\mathsf{Enc}(\mathsf{key}, (h, i \in [m], \beta \in \{0,1\}), \mu)$: Do the following:

1. Sample $s \leftarrow \mathbb{Z}_p^\lambda$ and $e \leftarrow D_\alpha^m$.
2. Compute $c_1 = \mathbf{A}_{-i}s + e_{-i}$ and $c_2 = (h - \beta\, a_i)s + e_i + \lfloor p/2 \rceil \cdot m$.
3. Output $\mathsf{ct} = (c_1, c_2)$

$\mathsf{Dec}(\mathsf{key}, x, \mathsf{ct})$:. The decryption algorithm does the following.

1. Parse $\mathsf{ct} = (c_1, c_2)$.
2. Compute $c_2 - x_{-i}^\top c_1$. If this is closer to $p/2$ output 1, else output 0.

Above the parameter α for the discrete Gaussian distribution only needs to be small enough so that the error incurred in the decryption equation permits correct decryption. Dottling et al. [DGHM18] choose $\alpha = o(\frac{1}{\sqrt{\lambda}\log(\lambda)})$. We refer the reader to [DGHM18] for additional details.

Semi-Malicious Security. To understand the intuition for our construction, let us consider possible adversarial strategies in the construction of hash encryption above, where the matrix $\mathbf{A}$ can be arbitrarily chosen. The view of the adversary, when the i^{th} bit of the witness does not match the bit value input to the encrypt algorithm, comprises

$$c_1 = \mathbf{A}_{-i}s + e_{-i}, \quad c_2 = \langle a_i,\ s\rangle + e_i + \lfloor p/2 \rceil \mu$$

where $\mathbf{A}$ is adversarially chosen. We must establish that μ remains hidden in such a scenario.

Clearly, if $\mathbf{A}$ were a random matrix, the privacy of μ would hold by the hardness of LWE. However, if $\mathbf{A}$ is adversarial, then it is easy to break security. For instance, one strategy for the adversary, discussed in [AJJM21] (henceforth AJJM) is to simply choose $\mathbf{A}$ with a trapdoor, which trivially allows to recover the message. Another possibility is that a_i could simply be chosen as the zero vector, or it could be the sum of two other vectors, say a_1 and a_2 in $\mathbf{A}_{-i}$. In the former case, the message is revealed trivially, while in the latter case, the mask $\langle a_i, s\rangle$ in c_m can be trivially computed (upto noise) given c_1. Other attack strategies may also be possible.

The approach of AJJM. A similar challenge was encountered by AJJM [AJJM21] in the context of an attribute-based secure function evaluation scheme. Their goals and final construction are quite different from ours and the details are not relevant here. The key idea they use (which in turn builds upon [BD18]) to achieve security against a semi-malicious receiver is: (i) choose $\mathbf{A}$ to have some special structure, which should allow to "isolate" the entropy of the LWE secret used during encryption, (ii) using an extractor during encryption allows to extract this entropy and use it to mask the message.

In more detail, instead of choosing $\mathbf{A}$ randomly, they choose it as:

$$\mathbf{A} = \left(\mathbf{A}',\ \mathbf{A}' \cdot \mathbf{S} + \mathbf{E}\right) \in \mathbb{Z}_p^{m \times \lambda}$$

Thus, $\mathbf{A}$ is a structured matrix with $\lambda/2$ short vectors. This ensures that the mapping from s to $\mathbf{A}s + e$ is lossy in s, which allows to "save" $\lambda/2$ bits of entropy from s. Then, the encryptor can extract the entropy of s using a strong extractor, which is used to mask the message. To decrypt, the decryptor is given a "hint", which is a trapdoor to a matrix – this allows to recover the randomness s which in turn can be fed into the extractor to recover the mask used to hide the message.

However, the above approach, applied to our setting, quickly gets stuck – if we use an extractor on s to derive the message mask, then it becomes necessary to recover s in order to decrypt. However, in a hash encryption scheme, the decryptor does not have a secret key, or any privileged information that would allow it to recover s – indeed, the ability to recover s would harm security.

In the absence of an extractor, the above technique appears insufficient, and to protect against an arbitrary $\mathbf{A}$ seems very challenging.

Our Approach. We do not have a complete solution to this problem. However, below we suggest some mechanisms to reduce the power of the adversary. This in turn allows us to weaken the requirement that the hash function must behave like a random oracle.

1. As a first counter-measure, we mimic the technique of AJJM to choose $\mathbf{A}$ as a structured matrix with $\lambda/2$ short vectors. This ensures that the mapping from s to $\mathbf{A}s + e$ is lossy in s, letting us "save" $\lambda/2$ bits of entropy from s which we can then hope to use for masking the message.

2. As a second counter-measure, we restrict the space of attacks that an adversary can perform to *linear* attacks by moving the LWE ciphertexts to the exponent of a group where discrete log is hard. By matching the LWE modulus with the group order, such a "composition" is possible and it is not difficult to show that decryption for the HE scheme still works. A similar approach was used by Agrawal and Yamada in the context of broadcast encryption [AY20].

3. Next, we randomize the ciphertext using a random scalar r in the exponent. By modifying the decryption procedure suitably, it can be ensured that decryption still works, and having fresh randomness which cannot be cancelled provides extra heuristic security as discussed below.

4. Next, to protect against the linear attacks described earlier, we generate $\mathbf{A}'$ using a hash function which ensures that:
   (a) The rows of $\mathbf{A}'$ have large entries.
   (b) The rows of $\mathbf{A}'$ should not have "small" linear dependencies between them, i.e. it should be hard to find a vector y with small (polynomial) coefficients such that $\mathbf{A}'\, y = 0 \mod p$.

   Note that the above two requirements are satisfied if $\mathbf{A}'$ is sampled randomly but that is exactly what we are trying not to assume! We conjecture that these are the only possible linear attacks on the scheme, and that by protecting against these, the only remaining attacks are non-linear, which in turn are ruled out by the generic group.

Construction. We now provide our heuristic construction.

$\mathsf{Gen}(1^\lambda, m) \to \mathsf{key}$. The setup algorithm does the following.

- Sample $r_1, r_2 \leftarrow \{0,1\}^\lambda$.
- Choose a group $\mathbb{G}$ of order p on which discrete log is hard. Let g be its generator.
- Output $\mathsf{key} = (r_1, r_2, g)$.

$\mathsf{Hash}(\mathsf{key}, x) \to h$. The hash algorithm does the following.

- Parse $\mathsf{key} = (r_1, r_2, g)$.
- Compute $\mathbf{A}' := H(\mathsf{PRG}(r_1)) \in \mathbb{Z}_p^{m \times \lambda/2}$.
- Using randomness from $H(\mathsf{PRG}(r_2))$, sample $\mathbf{S} \in D_\sigma^{\lambda/2 \times \lambda/2}$ and $\mathbf{E} \in D_\sigma^{m \times \lambda/2}$.
- Let $\mathbf{A} = \left(\mathbf{A}',\ \mathbf{A}' \cdot \mathbf{S} + \mathbf{E}\right) \in \mathbb{Z}_p^{m \times \lambda}$
- Output $h = x^T \mathbf{A}$.

$\mathsf{Enc}(\mathsf{key}, (h, i, c), \mu)$. The encryption algorithm does the following.

- Parse $\mathsf{key} = (r_1, r_2, g)$.
- Compute $\mathbf{A}' := H(\mathsf{PRG}(r_1)) \in \mathbb{Z}_p^{m \times \lambda/2}$.
- Using randomness from $H(\mathsf{PRG}(r_2))$, sample $\mathbf{S} \in D_\sigma^{\lambda/2 \times \lambda/2}$ and $\mathbf{E} \in D_\sigma^{m \times \lambda/2}$.
- Let $\mathbf{A} = \left(\mathbf{A}',\ \mathbf{A}' \cdot \mathbf{S} + \mathbf{E}\right) \in \mathbb{Z}_p^{m \times \lambda}$.
- Sample $s \leftarrow D_{\mathbb{Z}^\lambda, \alpha}$, $e \leftarrow D_{\mathbb{Z}^m, \alpha}$, and compute

$$c_1' := \mathbf{A}_{-i} s + e_{-i}$$

$$c_2' := (h - c \cdot a_i) s + e_i + \left\lfloor \frac{p}{2} \right\rfloor \cdot \mu$$

  where $\mathbf{A}_{-i}$ denotes the matrix obtained after dropping the i-th row of $\mathbf{A}$ and a_i denotes the i-th row of $\mathbf{A}$. Similarly, e_{-i} denotes the vector obtained after removing the i-th component of e and e_i denotes the i-th component of e.
- Sample random scalar $r \in \mathbb{Z}_p$ and let $c_0 = g^r,\ c_1 = g^{r \cdot c_1'},\ c_2 = g^{r \cdot c_2'}$.
- Output $\mathsf{ct} = (c_0, c_1, c_2)$.

$\mathsf{Dec}(\mathsf{key}, x, \mathsf{ct}) \to \{0,1\}$. The decryption algorithm does the following:

1. Compute $d = g^{r \cdot (c_2 - x_{-i}^T c_1)}$.
2. Given $c_0 = g^r$, check if $(c_2 - x_{-i}^T c_1)$ is small via brute force discrete log. In more detail, check whether $c_0^j = d$ for $j \in [B]$, where B is a parameter described below.
3. If yes, output 0, else output 1.

Correctness. For correctness, observe that since $c_1 := \mathbf{A}_{-i}s + e_{-i}$ and $c_2 := (h - c \cdot a_i)s + e_i + \lfloor \frac{p}{2} \rceil \cdot \mu$ and $h = x^T\mathbf{A}$, we have that $c_2 - x_{-i}^T c_1 = \lfloor \frac{p}{2} \rceil \cdot \mu + e_i - x_{-i}e_{-i}$. We set the parameters α and the bound B such that $e_i - x_{-i}e_{-i}$ is bounded by B – for this it suffices to set $\alpha = o(\frac{1}{\sqrt{\lambda}\log\lambda})$ and $B = m \cdot \alpha$. Thus, B is polynomially bounded and we can check whether $c_0^j = d$ for $j \in [B]$ via brute force discrete logarithm.

Security. We leverage the fact that only generic non-linear attacks are possible due to the message being in the exponent of a group. The randomizer r which is additionally used in the construction "spreads" $\langle a_i, s\rangle$ to the whole domain – in particular, if $\langle a_i, s\rangle$ is in a bounded range then $r \cdot \langle a_i, s\rangle$ still takes values over the entire range. Note that r cannot be cancelled by the adversary. Finally, we expect the hash function to output a matrix whose entries are large and whose rows do not have small linear dependencies. This requirement is weaker than the requirement that the hash function produce a random output, and may be satisfied more easily by hash functions used in practice. Basing a construction in the standard model on any meaningful assumption is a fascinating open problem.

5 Pre-Constrained Input Obfuscation

In this section we introduce the notion of pre-constrained input obfuscation (PCIO) in this section. This notion is a relaxation of virtual black-box obfuscation or indistinguishability obfuscation. A notable difference is that we can compute outputs of obfuscated circuits only for pre-determined polynomially many inputs.

A pre-constrained input obfuscation for circuit family $\mathcal{C} = \{C : \mathcal{X} \to \mathcal{Y}\}$ with input space $\mathcal{X}$ and output space $\mathcal{Y}$ consists of three algorithms $(\mathsf{Setup}, \mathcal{O}, \mathsf{Eval})$ defined as follows.

$\mathsf{Setup}(1^\lambda, \mathcal{S}) \to (\mathsf{pk}, \mathsf{ek})$. The setup algorithm takes as input a set of input $\mathcal{S}$ where $\mathcal{S} \subset \mathcal{X}$ and $|\mathcal{S}| = \mathrm{poly}(\lambda)$, and outputs a public key pk and evaluation key ek.

$\mathcal{O}(\mathsf{pk}, C) \to \widetilde{C}$. The obfuscation algorithm takes as input the public key pk and a circuit C, and outputs an obfuscated circuit $\widetilde{C}$.

$\mathsf{Eval}(\mathsf{ek}, \widetilde{C}, x) \to y$. The evaluation algorithm takes as input the evaluation key ek, obfuscated circuit $\widetilde{C}$, and an input $x \in \mathcal{X}$, and outputs an $y \in \mathcal{Y}$.

Here, pk is reusable for multiple C.

Definition 5.1 (Correctness). A PCIO scheme is correct if for any $\mathcal{S} \subset \mathcal{X}$, $|\mathcal{S}| = \mathrm{poly}(\lambda)$, and $(\mathsf{pk}, \mathsf{ek}) \leftarrow \mathsf{Setup}(1^\lambda, \mathcal{S})$, the following holds

1. $\forall x \in \mathcal{S}, \Pr[\mathsf{Eval}(\mathsf{ek}, \mathcal{O}(\mathsf{pk}, C), x) = C(x)] \geq 1 - \mathsf{negl}(\lambda)$.
2. $\forall x \in \mathcal{X} \setminus \mathcal{S}, \Pr[\mathsf{Eval}(\mathsf{ek}, \mathcal{O}(\mathsf{pk}, C), x) = \bot] \geq 1 - \mathsf{negl}(\lambda)$.

Definition 5.2 (Input-Set-Hiding). A PCIO scheme satisfies input-set-hiding security if for any PPT adversary $\mathcal{A}$, the following holds

$$\Pr\left[\begin{array}{ll} \mathcal{A}(\mathsf{pk}_\beta) = \beta : & (\mathcal{S}_0, \mathcal{S}_1) \leftarrow \mathcal{A}; \\ & \beta \leftarrow \{0,1\}; (\mathsf{pk}_\beta, \mathsf{ek}_\beta) \leftarrow \mathsf{Setup}(1^\lambda, \mathcal{S}_\beta) \end{array}\right] \leq \frac{1}{2} + \mathsf{negl}(\lambda)$$

where $\mathcal{A}$ is admissible if $\mathcal{S}_b \subset \mathcal{X}$ and $\mathcal{S}_b = \mathrm{poly}(\lambda)$ for $b \in \{0,1\}$.

Definition 5.3 (Virtual Black-Box Security against Malicious Authority). For a PPT adversary $\mathcal{A}$ and a PPT simulator Sim, consider the following experiment $\mathsf{Exp}^{\mathsf{PC\text{-}VBB}}_{\beta,\mathcal{A},\mathsf{Sim}}(1^\lambda)$.

1. $\mathcal{A}$ outputs the public key pk and circuit C.

2. On input (pk, C), the challenger samples a random bit β uniformly and generates $\widetilde{C} \leftarrow \mathcal{O}(\mathsf{pk}, C)$ if $\beta = 0$. Otherwise, the challenger generates $\widetilde{C} \leftarrow \mathsf{Sim}(1^\lambda, 1^{|C|}, \mathsf{pk}, C(x_1), \ldots, C(x_q))$ where $(x_1, \ldots, x_q) \leftarrow \mathrm{Ext}(1^\lambda, \mathsf{pk})$ and Ext is an (possibly inefficient) extractor. It returns $\widetilde{C}$ to $\mathcal{A}$.

3. $\mathcal{A}$ outputs a guess bit β' as the output of the experiment.

We say that a PCIO scheme satisfies the virtual black-box security against malicious authority if there exists an (possibly inefficient) extractor Ext such that for any PPT adversary $\mathcal{A}$,

$$\left|\Pr\left[\mathsf{Exp}^{\mathsf{PC\text{-}VBB}}_{0,\mathcal{A},\mathsf{Sim}}(1^\lambda) = 1\right] - \Pr\left[\mathsf{Exp}^{\mathsf{PC\text{-}VBB}}_{1,\mathcal{A},\mathsf{Sim}}(1^\lambda) = 1\right]\right| = \mathsf{negl}(\lambda).$$

Definition 5.4 (Indistinguishability against Malicious Authority). A PCIO scheme satisfies input-set-hiding security if there exists an admissible (possibly inefficient) extractor Ext such that for any PPT and admissible adversary $\mathcal{A}$, the following holds

$$\Pr\left[\begin{array}{ll} \mathcal{A}(\mathsf{ct}_\beta) = \beta : & \mathsf{pk}, (C_0, C_1) \leftarrow \mathcal{A}; \\ & \beta \leftarrow \{0,1\}; \mathsf{ct}_\beta \leftarrow \mathsf{Enc}(\mathsf{pk}, C_\beta) \end{array}\right] \leq \frac{1}{2} + \mathsf{negl}(\lambda)$$

where $\mathcal{A}$ is admissible if (i) $\mathcal{S} \subset \mathcal{X}$ and $|\mathcal{S}| = \mathrm{poly}(\lambda)$ where $\mathcal{S} \leftarrow \mathrm{Ext}(1^\lambda, \mathsf{pk})$, and (ii) $C_b \in \mathcal{C}$ for $b \in \{0,1\}$, $|C_0| = |C_1|$, and $\forall x \in \mathcal{S}, C_0(x) = C_1(x)$. We say that Ext is admissible if for every $\mathcal{S} = (x_1, \ldots, x_Q)$ and randomness r, we have that $(x_1, \ldots, x_Q) = (x'_1, \ldots, x'_Q)$, where (i) $(\mathsf{pk}, \mathsf{ek}_{x_1}, \ldots, \mathsf{ek}_{x_Q}) \leftarrow \mathsf{Setup}(1^\lambda, \{x_1, \ldots, x_Q\}; r)$ and (ii) $\mathcal{S}' = (x'_1, \ldots, x'_Q) \leftarrow \mathrm{Ext}(1^\lambda, \mathsf{pk})$.
In addition, if $\mathcal{A}$ is unbounded, we say that PCIO is unconditional IND secure.

Definition 5.5 (Succinct Public Key). We say that a PCIO scheme has succinct public keys when the size of the public key is sublinear in $\mathcal{S}$, that is $|\mathsf{pk}| = O(|\mathcal{S}|^{1-\gamma})$ for some $0 < \gamma < 1$ where $(\mathsf{pk}, \mathsf{ek}) \leftarrow \mathsf{Setup}(1^\lambda, \mathcal{S})$.

Constructions and Implications to PCSFE. We observe that a PCSFE scheme implies PCIO scheme. The implication also holds the other way round. We prove both the implications in Appendix D.

6 Pre-Constrained Group Signatures

In this section, we provide our definition and construction for pre-constrained group signatures. Our definition largely follows the definition of [BGJP23] except that we generalize the constraint of database membership to arbitrary constraints, and favour simpler game based definitions for security against authorities as compared to the simulation style definitions of [BGJP23][10]. We provide the details in Appendix E.

[10]Their definition also includes the step of authorizing the database while ours does not. We note that such an authorization can be performed via a separate protocol (using zero-knowledge or multiparty computation protocols).

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A Additional Preliminaries

A.1 Dual-Mode Non-Interactive Zero-Knowledge Proof Systems

Let L be a language in NP, and let R be the associated binary relation such that for every $x \in \{0,1\}^*$, $x \in L$ if and only if there exists a witness w such that $(x,w) \in R$. A dual-mode non-interactive proof system for R consists of the following probabilistic polynomial-time algorithms.

$\mathsf{Hsetup}(1^\lambda) \to \mathsf{crs}$. This algorithm takes as input the security parameter λ and outputs a hiding common reference string crs.

$\mathsf{Bsetup}(1^\lambda) \to (\mathsf{crs}, \mathsf{td}_{\mathsf{ext}})$. This algorithm takes as input the security parameter λ and outputs a binding common reference string crs and an extraction trapdoor $\mathsf{td}_{\mathsf{ext}}$.

$\mathsf{Prove}(\mathsf{crs}, x, w) \to \pi$. This algorithm takes as input a crs, a statement x, and a witness w, and outputs a proof π.

$\mathsf{Verify}(\mathsf{crs}, x, \pi) \to 0/1$. The verification algorithm takes as input a crs, a statement x, and a proof π, and outputs either 1 (accept) or 0 (reject).

We require the dual mode NIZK proof system to satisfy the following properties.

Definition A.1 (Perfect completeness in both modes). A dual-mode NIZK proof system for the language L and the associated binary relation R is said to satisfy perfect completeness if for any $\lambda \in \mathbb{N}$, any statement $x \in L$ and corresponding witness w, the following holds

$$\Pr[\mathsf{Verify}(\mathsf{crs}, x, \mathsf{Prove}(\mathsf{crs}, x, w)) = 1] = 1$$

where $\mathsf{crs} \leftarrow \mathsf{Hsetup}(1^\lambda)$ or $(\mathsf{crs}, \mathsf{td}_{\mathsf{ext}}) \leftarrow \mathsf{Bsetup}(1^\lambda)$.

Definition A.2 (CRS indistinguishability). A dual-mode NIZK proof system for the language L and the associated binary relation R is said to satisfy CRS indistinguishability if there exists a negligible function $\mathsf{negl}(\cdot)$ such that for any adversary $\mathcal{A}$, the following two distribution ensembles are computationally indistinguishable

$$\{\mathsf{crs} \mid \mathsf{crs} \leftarrow \mathsf{Hsetup}(1^\lambda)\} \approx_c \{\mathsf{crs} \mid (\mathsf{crs}, \mathsf{td}_{\mathsf{ext}}) \leftarrow \mathsf{Bsetup}(1^\lambda)\}.$$

Definition A.3 (Statistical zero-knowledge in hiding mode). A dual-mode NIZK proof system for the language L and the associated binary relation R is said to satisfy statistical zero-knowledge if there exists a probabilistic polynomial-time simulator Sim such that for any $(x, w) \in R$, the following two distribution ensembles are statistically indistinguishable for any $\lambda \in \mathbb{N}$

$$\{(\mathsf{crs}, \pi) : (\mathsf{crs}, \pi) \leftarrow \mathsf{Sim}(1^\lambda, x)\} \approx_s \{(\mathsf{crs}, \mathsf{Prove}(\mathsf{crs}, x, w)) : \mathsf{crs} \leftarrow \mathsf{Hsetup}(1^\lambda)\}.$$

Definition A.4 (Common random string). If crs output by Hsetup (resp. Bsetup) is uniformly random, we call it common random string in the hiding (resp. binding) mode.

Definition A.5 (Knowledge extraction in the binding mode). A dual-mode NIZK proof system for the language L and the associated binary relation R is said to satisfy knowledge extraction if there exists an extractor Ext, such that for any PPT adversary $\mathcal{A}$, the following holds

$$\Pr\left[\begin{array}{l|l} \mathsf{Verify}(\mathsf{crs}, x, \pi) = 0 \vee (x, w) \in R : & (\mathsf{crs}, \mathsf{td}_{\mathsf{ext}}) \leftarrow \mathsf{Bsetup}(1^\lambda); \\ & (x, \pi) \leftarrow \mathcal{A}(\mathsf{crs}); \\ & w \leftarrow \mathsf{Ext}(\mathsf{td}_{\mathsf{ext}}, x, \pi) \end{array}\right] \geq 1 - \mathsf{negl}(\lambda).$$

Remark A.6. In the standard definition of dual-mode NIZK, we do not require that the CRS is a uniformly random string in the hiding mode. However, the construction based on the LWE assumption [CCH+19, PS19] has this property.

The dual-mode NIZK by Canetti et al. [CCH+19] satisfies the following adaptive soundness instead of Definition A.5 (in the binding mode).

Definition A.7 (Adaptive Soundness). A dual-mode NIZK proof system for the language L and the associated binary relation R is said to satisfy adaptive soundness if for any PPT adversary $\mathcal{A}$, the following holds

$$\Pr\left[\begin{array}{l|l} \mathsf{Verify}(\mathsf{crs}, x, \pi) = 1 \wedge x \notin L : & (\mathsf{crs}, \mathsf{td}_{\mathsf{ext}}) \leftarrow \mathsf{Bsetup}(1^\lambda); \\ & (x, \pi) \leftarrow \mathcal{A}(\mathsf{crs}) \end{array}\right] \leq \mathsf{negl}(\lambda).$$

We can upgrade adaptive soundness to knowledge extraction by the following theorem.

Theorem A.8 ([KNYY19]). If a NIZK proof system is adaptively sound, and there exists PKE, we can convert the NIZK proof system into a NIZK that has the knowledge extraction property.

It is easy to see that this conversion preserves the dual-mode property and the uniformly random CRS property in the hiding mode if we use lossy PKE [PVW08, BHY09, KN08] with uniformly random lossy public keys [CCH+19] for the conversion. This is because the conversion adds a public key to the NIZK CRS, appends a ciphertext of a witness to a NIZK proof for proving the knowledge and the ciphertext is an encryption of the witness.

Theorem A.9 ([Reg09]). The Regev PKE is lossy encryption with uniformly random lossy public keys under the LWE assumption.

Theorem A.10 ([CCH+19, PS19]). There exists a dual-mode non-interactive zero-knowledge proof system for any NP language, based on the plain LWE problem with (small) polynomial approximation factors, satisfying statistical zero-knowledge with common random string in the hiding mode and adaptive soundness in the binding mode.

We can obtain the following corollary from the theorems above.

Corollary A.11. There exists a dual-mode non-interactive zero-knowledge proof system for any NP language, based on the plain LWE problem with (small) polynomial approximation factors, satisfying statistical zero-knowledge with common random string in the hiding mode and knowledge extraction in the binding mode.

A.2 One-Way Relation

Next, we provide the definition of one-way relation.

A one-way relation for a relation $\mathcal{R}$ consists of two algorithms $(\mathsf{Gen}, \mathsf{Sample})$ with the following syntax.

$\mathsf{Gen}(1^\lambda) \to \mathsf{pp}$. The generation algorithm takes as input the security parameter and outputs the public parameter pp.

$\mathsf{Sample}(\mathsf{pp}) \to (x, w)$. The sample algorithm outputs an instance witness pair (x, w).

A one-way relation satisfies the following properties.

Definition A.12 (Correctness). A one-way relation for a relation $\mathcal{R}$ is said to be correct if for any $\mathsf{pp} \leftarrow \mathsf{Gen}(1^\lambda)$, the following holds

$$\Pr[(\mathsf{pp}, x, w) \in \mathcal{R} \mid (x, w) \leftarrow \mathsf{Sample}(\mathsf{pp})] \geq 1 - \mathsf{negl}(\lambda).$$

Definition A.13 (Security). A one-way relation for a relation $\mathcal{R}$ is said to be secure if for any PPT adversary $\mathcal{A}$, the following holds

$$\Pr\left[(\mathsf{pp}, x, w') \in \mathcal{R} : \begin{array}{l} \mathsf{pp} \leftarrow \mathsf{Gen}(1^\lambda); \\ (x, w) \leftarrow \mathsf{Sample}(\mathsf{pp}); \\ w' \leftarrow \mathcal{A}(\mathsf{pp}, x) \end{array} \right] \leq \mathsf{negl}(\lambda).$$

A.3 Digital Signatures

A digital signature scheme for a message space $\mathcal{M}$ consists of three algorithms $(\mathsf{Gen}, \mathsf{Sign}, \mathsf{Verify})$ with the following syntax.

$\mathsf{Gen}(1^\lambda) \to (\mathsf{vk}, \mathsf{sk})$. The key generation algorithm takes as input the security parameter λ and outputs a verification key vk and a signing key sk.

$\mathsf{Sign}(\mathsf{sk}, m) \to \sigma$. The signing algorithm takes as input the signing key sk and a message $m \in \mathcal{M}$, and outputs a signature σ.

$\mathsf{Verify}(\mathsf{vk}, m, \sigma) \to 0/1$. The verification algorithm takes as input a verification key vk, a message $m \in \mathcal{M}$ and a signature σ, and outputs either 1 (accept) or 0 (reject).

A digital signature scheme satisfies the following properties.

Definition A.14 (Correctness). A digital signature scheme is sad to be correct if for any $(\mathsf{vk}, \mathsf{sk}) \leftarrow \mathsf{Gen}(1^\lambda)$, and any $m \in \mathcal{M}$, the following holds

$$\Pr[\mathsf{Verify}(\mathsf{vk}, m, \mathsf{Sign}(\mathsf{sk}, m)) = 1] = 1.$$

Definition A.15 (EUF-CMA Security). A digital signature scheme is existentially unforgeable under chosen message attacks if for any PPT adversary $\mathcal{A}$, the following holds

$$\Pr\left[m \notin Q \wedge \mathsf{Verify}(\mathsf{vk}, m, \sigma) = 1 : \begin{array}{l} (\mathsf{vk}, \mathsf{sk}) \leftarrow \mathsf{Gen}(1^\lambda); \\ (m, \sigma) \leftarrow \mathcal{A}^{\mathsf{Sign}(\mathsf{sk}, \cdot)}(\mathsf{vk}) \end{array} \right] \leq \mathsf{negl}(\lambda)$$

where $Q \subset \mathcal{M}$ is the set of messages for which $\mathcal{A}$ makes the signing queries to $\mathsf{Sign}(\mathsf{sk}, \cdot)$.

B Missing details from Section 3

B.1 Proofs from Section 3.2

In this section, we prove the security properties of our PCE scheme constructed from two-message SSP-OT and garbling scheme in Section 3.2.

Proof for Constraint-hiding.

Theorem B.1. Suppose the OT scheme satisfies receiver privacy (Definition 2.3). Then the construction of the PCE scheme satisfies constraint-hiding (Definition 3.2).

Proof. Recall that to show constraint-hiding, we want

$$\{\mathsf{pk} \mid (\mathsf{pk},\mathsf{sk}) \leftarrow \mathsf{Setup}(1^\lambda, C_0)\} \approx_c \{\mathsf{pk} \mid (\mathsf{pk},\mathsf{sk}) \leftarrow \mathsf{Setup}(1^\lambda, C_1)\}$$

for any circuits $C_0, C_1 \in \mathcal{C}_L$. The proof proceeds via the following sequence of hybrid games between the challenger and a PPT adversary $\mathcal{A}$.

Hyb_0. This is the real world with bit $\beta = 0$, i.e., the challenge public key is computed using the circuit C_0. We write the complete game to set up the notations and easy reference in the later hybrids.

1. $\mathcal{A}$ outputs two circuits $C_0, C_1 \in \mathcal{C}_L$ where $|C_0| = |C_1|$.
2. The challenger computes $(\mathsf{ot}_{1,i}, \mathsf{st}_i) \leftarrow \mathsf{OTR}(1^\lambda, C_0[i])$ for all $i \in [\ell]$ where $C[i]$ denotes the i-th bit of C. It returns $\mathsf{pk} = \{\mathsf{ot}_{1,i}\}$ to $\mathcal{A}$.
3. In the end $\mathcal{A}$ outputs a bit β'.

$\mathsf{Hyb}_{k;1\le k\le \ell}$. This hybrid is same as the previous hybrid except that the challenger computes $(\mathsf{ot}_{1,i}, \mathsf{st}_i) \leftarrow \mathsf{OTR}(1^\lambda, C_1[i])$ for $1 \le i \le k$ and $(\mathsf{ot}_{1,i}, \mathsf{st}_i) \leftarrow \mathsf{OTR}(1^\lambda, C_0[i])$ for $k+1 \le i \le \ell$, where $C_b[i]$ denotes the i-th bit of the circuit C_b for $b \in \{0,1\}$.
Note that Hyb_ℓ is the real world with bit $\beta = 1$, i.e., the challenge public key is computed using the circuit C_1.

We note that it is sufficient to argue $\mathsf{Hyb}_{k-1} \approx_c \mathsf{Hyb}_k$, $k \in [\ell]$, to complete the proof. We show that if there exists a PPT adversary $\mathcal{A}$ who can distinguish between the two hybrids with non-negligible advantage ϵ, then there exists a PPT adversary $\mathcal{B}$ against the receiver privacy of the OT scheme with the same advantage ϵ. The reduction is as follows.

1. $\mathcal{B}$ first runs $\mathcal{A}$. $\mathcal{A}$ outputs the challenge circuits C_0, C_1 such that $C_0, C_1 \in \mathcal{C}_L$.
2. $\mathcal{B}$ parses C_0 and C_1 as ℓ bit strings and forwards $C_0[k]$ and $C_1[k]$ to the OT challenger, where $C_b[k]$ denotes the k-th bit of the circuit C_b for $b \in \{0,1\}$. The OT challenger samples a bit $\beta \leftarrow \{0,1\}$ and computes $(\mathsf{ot}_{1,k}, \mathsf{st}_{1,k}) \leftarrow \mathsf{OTR}(1^\lambda, C_\beta[k])$ and returns $\mathsf{ot}_{k,1}$ to $\mathcal{B}$.
3. $\mathcal{B}$ computes $(\mathsf{ot}_{1,i}, \mathsf{st}_{1,i}) \leftarrow \mathsf{OTR}(1^\lambda, C_1[i])$ for $1 \le i \le k-1$ and $(\mathsf{ot}_{1,i}, \mathsf{st}_i) \leftarrow \mathsf{OTR}(1^\lambda, C_0[i])$ for $k+1 \le i \le \ell$, sets $\mathsf{pk} = \{\mathsf{ot}_{1,i}\}_{i\in[\ell]}$ and forwards it to $\mathcal{A}$.
4. In the end $\mathcal{A}$ outputs a bit β'. $\mathcal{B}$ forwards β' to the OT challenger.

We observe that if the OT challenger samples $\beta = 0$, then $\mathcal{B}$ simulated the distribution Hyb_{k-1} , else Hyb_k with $\mathcal{A}$. Hence, advantage of $\mathcal{B} = |\Pr(\beta' = 1|\beta = 0) - \Pr(\beta' = 1|\beta = 1)| = |\Pr(\beta' = 1|\mathsf{Hyb}_{k-1}) - \Pr(\beta' = 1|\mathsf{Hyb}_k)| = \epsilon$ (by assumption). □

Security against malicious authority. Recall that to prove security against a malicious authority (Theorem 3.12), we want to show that the consecutive hybrids are indistinguishable.

Hyb_0. This is the real world with $\beta = 0$, i.e., the challenge ciphertext is computed using the input x_0. We write the complete game here to set up the notations and easy reference in later hybrids.

1. $\mathcal{A}$ outputs a public key pk and two inputs $x_0, x_1 \in \{0,1\}^L$ where $|x_0| = |x_1|$.
2. The challenger defines the circuit $U[x_0]$ as in the construction and computes $(\tilde{U}, \{\mathsf{lb}_{i,b}\}_{i\in[\ell],b\in\{0,1\}}) \leftarrow \mathsf{GC.Garble}(1^\lambda, U[x_0])$. It parses $\mathsf{pk} = \{\mathsf{ot}_{1,i}\}_{i\in[\ell]}$ and computes $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, \mathsf{lb}_{i,0}, \mathsf{lb}_{i,1}, \mathsf{ot}_{1,i})$ for all $i \in [\ell]$. It sets $\mathsf{ct} = (\tilde{U}, \{\mathsf{ot}_{2,i}\})$ and returns ct to $\mathcal{A}$.
3. $\mathcal{A}$ outputs a guess bit β'.

Hyb_1. This hybrid is same as the previous hybrid except that the challenger computes $\mathsf{ot}_{2,i}$ differently, i.e., $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, \mathsf{lb}_{i,b}, \mathsf{lb}_{i,b}, \mathsf{ot}_{1,i})$ for all $i \in [\ell]$, where $b \leftarrow \mathsf{OT.Ext}(\mathsf{ot}_{1,i})$ and $\mathsf{OT.Ext}$ is the extractor in the statistical sender private security of the OT scheme.

Hyb_2. This hybrid is same as the previous hybrid except that the challenger computes the garbled circuit and the labels differently using $\mathsf{GC.Sim}$, i.e., $(\tilde{U}, \{\mathsf{lb}_i\}_{i\in[\ell]}) \leftarrow \mathsf{GC.Sim}(1^\lambda, U[x_0](C))$ where $C = C[1]\dots C[\ell]; C[i] \leftarrow \mathsf{OT.Ext}(\mathsf{ot}_{1,i})$ for $i \in [\ell]$.

Hyb_3. This hybrid is same as the previous hybrid except that the challenger runs the $\mathsf{GC.Sim}$ on $U[x_1](C)$, i.e., $(\tilde{U}, \{\mathsf{lb}_i\}_{i\in[\ell]}) \leftarrow \mathsf{GC.Sim}(1^\lambda, U[x_1](C))$.

Hyb_4. This hybrid is same as the previous hybrid except that the challenger computes the garbled circuit and the labels honestly for the circuit $U[x_1]$, i.e., $(\tilde{U}, \{\mathsf{lb}_{i,b}\}_{i\in[\ell],b\in\{0,1\}}) \leftarrow \mathsf{GC.Garble}(1^\lambda, U[x_1])$.

Hyb_5. This hybrid is same as the previous hybrid except that the challenger computes $\mathsf{ot}_{2,i}$ honestly, i.e., $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(\mathsf{lb}_{i,0}, \mathsf{lb}_{i,1}, \mathsf{ot}_{1,i})$. This is the real world with $\beta = 1$.

Indistinguishability of hybrids. We now prove that the consecutive hybrids are indistinguishable.

Claim B.2. Assume that OT satisfies statistical sender privacy, then $\mathsf{Hyb}_0 \approx_s \mathsf{Hyb}_1$.

Proof. To prove the claim we consider sub hybrids $\mathsf{Hyb}_{0.k}$ for $k = 0$ to ℓ, where $\mathsf{Hyb}_{0.k}$ is same as Hyb_0 except that $\mathsf{ot}_{2,i}$ is generated differently for all $i \le k$, i.e., $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, \mathsf{lb}_{i,b}, \mathsf{lb}_{i,b}, \mathsf{ot}_{1,i})$, where $b \leftarrow \mathsf{OT.Ext}(\mathsf{ot}_{1,i})$ for $1 \le i \le k$ and $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, \mathsf{lb}_{i,0}, \mathsf{lb}_{i,1}, \mathsf{ot}_{1,i})$ for $k+1 \le i \le \ell$. We note that $\mathsf{Hyb}_0 = \mathsf{Hyb}_{0.0}$ and $\mathsf{Hyb}_{0.\ell} = \mathsf{Hyb}_1$.
To prove the above claim it suffices to show that $\mathsf{Hyb}_{0.k-1} \approx \mathsf{Hyb}_{0.k}$ for $k \in [\ell]$. We show that if there exists an unbounded adversary $\mathcal{A}$ who can distinguish between $\mathsf{Hyb}_{0.k-1}$ and $\mathsf{Hyb}_{0.k}$ with non-negligible advantage ϵ, then there exists an unbounded adversary $\mathcal{B}$ against the statistical sender privacy security of OT scheme with the same advantage ϵ. The reduction is as follows.

1. $\mathcal{B}$ first runs $\mathcal{A}$. $\mathcal{A}$ outputs a public key pk and two inputs x_0, x_1.
2. $\mathcal{B}$ parses $\mathsf{pk} = \{\mathsf{ot}_{1,i}\}_{i\in[\ell]}$, defines the circuit $U[x_0]$ as in the construction and computes $(\tilde{U}, \{\mathsf{lb}_{i,b}\}) \leftarrow \mathsf{GC.Garble}(1^\lambda, U[x_0])$ for all $i \in [\ell]$ and $b \in \{0,1\}$.
3. $\mathcal{B}$ sends $(\mathsf{lb}_{k,0}, \mathsf{lb}_{k,1}), \mathsf{ot}_{1,k}$ to the OT challenger. The challenger samples a bit $\beta \leftarrow \{0,1\}$ and computes $\mathsf{ot}_{2,k} \leftarrow \mathsf{OTS}(1^\lambda, \mathsf{lb}_{k,0}, \mathsf{lb}_{k,1}, \mathsf{ot}_{1,k})$ if $\beta = 0$, else if $\beta = 1$, it computes $\mathsf{ot}_{2,k} \leftarrow \mathsf{OTS}(1^\lambda, \mathsf{lb}_{k,b}, \mathsf{lb}_{k,b}, \mathsf{ot}_{1,k})$ where $b \leftarrow \mathsf{OT.Ext}(\mathsf{ot}_{1,k})$. It returns $\mathsf{ot}_{2,k}$ to $\mathcal{B}$.
4. $\mathcal{B}$ computes $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, \mathsf{lb}_{i,b}, \mathsf{lb}_{i,b}, \mathsf{ot}_{1,i})$, where $b \leftarrow \mathsf{OT.Ext}(\mathsf{ot}_{1,i})$ for $1 \le i \le k-1$ and $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, \mathsf{lb}_{i,0}, \mathsf{lb}_{i,1}, \mathsf{ot}_{1,i})$ for $k+1 \le i \le \ell$. It sets $\mathsf{ct} = (\tilde{U}, \{\mathsf{ot}_{2,i}\}_{i\in[\ell]})$ and returns ct to $\mathcal{A}$.
5. $\mathcal{A}$ outputs a guess bit β'. $\mathcal{B}$ forwards β' to the OT challenger.

We observe that if the OT challenger samples $\beta = 0$, then $\mathcal{B}$ simulated the distribution $\mathsf{Hyb}_{0.k-1}$, else $\mathsf{Hyb}_{0.k}$ with $\mathcal{A}$. Hence, advantage of $\mathcal{B} = |\Pr(\beta' = 1|\beta = 0) - \Pr(\beta' = 1|\beta = 1)| = |\Pr(\beta' = 1|\mathsf{Hyb}_{0.k-1}) - \Pr(\beta' = 1|\mathsf{Hyb}_{0.k})| = \epsilon$ (by assumption). □

Claim B.3. Assume that GC satisfies simulation security, then $\mathsf{Hyb}_1 \approx_c \mathsf{Hyb}_2$.

We show that if there exists a PPT adversary $\mathcal{A}$ who can distinguish between Hyb_1 and Hyb_2 with non-negligible advantage ϵ, then there exists an adversary $\mathcal{B}$ against the security of GC scheme with the same advantage ϵ. The reduction is as follows.

1. $\mathcal{B}$ first runs $\mathcal{A}$. $\mathcal{A}$ outputs a public key pk and two inputs x_0, x_1.
2. $\mathcal{B}$ parses $\mathsf{pk} = \{\mathsf{ot}_{1,i}\}_{i\in[\ell]}$ and computes $C = C[1]\dots C[\ell]$ where $C[i] \leftarrow \mathsf{OT.Ext}(\mathsf{ot}_{1,i})$ for $i \in [\ell]$.
3. $\mathcal{B}$ defines the circuit $U[x_0]$ as in the construction and sends $U[x_0]$ and C to the GC challenger as the challenge circuit and challenge input. The challenger chooses a bit $\beta \leftarrow \{0,1\}$ and does the following:
   - If $\beta = 0$, it computes $(\tilde{U}, \{\mathsf{lb}_{i,b}\}_{i\in[\ell],b\in\{0,1\}}) \leftarrow \mathsf{GC.Garble}(1^\lambda, U[x_0])$. It sets $U' = \tilde{U}$ and $\mathsf{lb}_i = \{\mathsf{lb}_{i,C[i]}\}$ for $i \in [\ell]$.
   - If $\beta = 1$, it computes $(\tilde{U}, \{\mathsf{lb}_i\}_{i\in[\ell]}) \leftarrow \mathsf{GC.Sim}(1^\lambda, U[x_0](C))$. It sets $U' = \tilde{U}$.

   The GC challenger returns $(U', \{\mathsf{lb}_i\}_{i\in[\ell]})$ to $\mathcal{B}$.
4. $\mathcal{B}$ computes $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(\mathsf{lb}_i, \mathsf{lb}_i, \mathsf{ot}_{1,i})$ for all $i \in [\ell]$. It returns $\mathsf{ct} = (U', \{\mathsf{ot}_{2,i}\}_{i\in[\ell]})$.
5. In the end, $\mathcal{A}$ outputs a bit β'. $\mathcal{B}$ sends β' to the GC challenger.

We observe that if the GC challenger samples $\beta = 0$, then $\mathcal{B}$ simulated Hyb_1, else Hyb_2 with $\mathcal{A}$. Hence, advantage of $\mathcal{B} = |\Pr(\beta' = 1|\beta = 0) - \Pr(\beta' = 1|\beta = 1)| = |\Pr(\beta' = 1|\mathsf{Hyb}_1) - \Pr(\beta' = 1|\mathsf{Hyb}_2)| = \epsilon$ (by assumption). We note that $\mathsf{Hyb}_2 = \mathsf{Hyb}_3$, since by the admissibility of $\mathcal{A}$, we have $C(x_0) = C(x_1) = 0$ and thus $U[x_0](C) = U[x_1](C) = \perp$. Also the rest of the hybrids, Hyb_4 and Hyb_5 are simply unwinding the previous hybrids and their proofs of indistinguishability are same as their corresponding counterparts in the first set of hybrids and hence, omitted.

B.2 PCE for Database with Unconditional Security from SSP-OT

In this section we construct a PCE scheme that supports the database checking constraint, for a database $\mathcal{D} \subseteq \{0,1\}^\ell$ consisting of a single element, $\mathcal{D} = \{y\}$ and message space $\mathcal{M} = \{0,1\}^{\mathsf{len}}$, satisfying unconditional security against a malicious authority using a two-message SSP-OT scheme. For the database functionality, we need to consider special case PCE as described in Definition 3.10, where a plaintext consists of two parts: attribute part x and message part m. If $x \in \mathcal{D}$, the receiver can recover m. Although we can consider both attribute-hiding and message-hiding against malicious authority, our construction satisfies only message-hiding against malicious authority. That is, m is hidden, but x is revealed.

Building blocks. We use a two-message statistically sender-private oblivious transfer scheme $\mathsf{OT} = (\mathsf{OTR}, \mathsf{OTS}, \mathsf{OTD})$ with input space $\{0,1\}^{\mathsf{len}+\mathsf{len}'}$. This can be instantiated from a wide variety of assumptions as discussed in Section 2.2.

Construction. We describe our PCE construction below.

$\mathsf{Setup}(1^\lambda, y) \to (\mathsf{pk}, \mathsf{sk})$. The setup algorithm does the following.

- Parse $y = y[1], \dots, y[\ell]$ as an ℓ bit string.
- For $i = 1, \cdots, \ell$, compute $(\mathsf{ot}_{1,i}, \mathsf{st}_i) \leftarrow \mathsf{OTR}(1^\lambda, y[i])$.
- Output $\mathsf{pk} = \{\mathsf{ot}_{1,i}\}_{i\in[\ell]}$ and $\mathsf{sk} = \{y[i], \mathsf{st}_i\}_{i\in[\ell]}$.

$\mathsf{Enc}(\mathsf{pk}, x, m) \to \mathsf{ct}$. The encryption algorithm does the following.

- Choose $\ell - 1$ uniformly random strings $s_1, \ldots, s_{\ell-1}$, where $s_i \in \{0,1\}^{\mathsf{len}+\mathsf{len}'}$ for all $i \in [\ell-1]$.
- Compute $s_\ell := \bigoplus_{i=1}^{\ell-1} s_i \oplus (m \| 0^{\mathsf{len}'}) \in \{0,1\}^{\mathsf{len}+\mathsf{len}'}$.
- Parse $\mathsf{pk} = \{\mathsf{ot}_{1,i}\}_{i\in[\ell]}$. For each $i \in [\ell]$ do the following:
  - Choose uniformly random $r_i \leftarrow \{0,1\}^{\mathsf{len}+\mathsf{len}'}$.
  - If $x[i] = 0$, compute $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, (s_i, r_i), \mathsf{ot}_{1,i})$.
  - Else if $x[i] = 1$, compute $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, (r_i, s_i), \mathsf{ot}_{1,i})$.
- Output $\mathsf{ct} = \{\mathsf{ot}_{2,i}\}_{i\in[\ell]}$.

$\mathsf{Dec}(\mathsf{sk}, \mathsf{ct}) \to \{m/\perp\}$. The decryption algorithm does the following.

- Parse $\mathsf{sk} = \{y[i], \mathsf{st}_i\}_{i\in[\ell]}$ and $\mathsf{ct} = \{\mathsf{ot}_{2,i}\}_{i\in[\ell]}$.
- Compute $s'_i = \mathsf{OTD}(1^\lambda, y[i], \mathsf{st}_i, \mathsf{ot}_{2,i})$ for $i \in [\ell]$.
- Compute $s' := \bigoplus_{i=1}^{\ell} s'_i$.
- Parse $s' = m' \| \mathsf{pad}$ where $m' \in \{0,1\}^{\mathsf{len}}$ and $\mathsf{pad} \in \{0,1\}^{\mathsf{len}'}$.
- If $\mathsf{pad} = 0^{\mathsf{len}'}$, output m'. Otherwise, output $\perp$.

A receiver can know that the recovered value is a correct message by the padding mechanism. It is easy to extend the above construction to one handing multiple elements: if we have n elements in a database, we prepare $\ell \times n$ instances of OT.

Correctness. We now show that the PCE construction satisfies correctness via the following theorem.

Theorem B.4. Suppose that the OT scheme is correct (Definition 2.2). Then the PCE construction satisfies *perfect* correctness as defined in Definition 3.1.

Proof. We note that for any $\mathsf{ct} \leftarrow \mathsf{Enc}(\mathsf{pk}, x, m)$, we have that $\mathsf{ct} = \{\mathsf{ot}_{2,i}\}_{i\in[\ell]}$, where m is split into ℓ shares $s_1, \ldots, s_\ell$ and $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, (s_i, r_i), \mathsf{ot}_{1,i})$ if $x[i] = 0$, else $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, (r_i, s_i), \mathsf{ot}_{1,i})$. Now if $x = y$, then the following holds for $i = 1 \ldots \ell$

- if $x[i] = 0 = y[i]$, $\mathsf{OTD}(1^\lambda, y[i], \mathsf{st}_i, \mathsf{ot}_{2,i}) = \mathsf{OTD}(1^\lambda, y[i], \mathsf{st}_i, \mathsf{OTS}(1^\lambda, (s_i, r_i), \mathsf{ot}_{1,i})) = s_i$,
- if $x[i] = 1 = y[i]$, $\mathsf{OTD}(1^\lambda, y[i], \mathsf{st}_i, \mathsf{ot}_{2,i}) = \mathsf{OTD}(1^\lambda, y[i], \mathsf{st}_i, \mathsf{OTS}(1^\lambda, (r_i, s_i), \mathsf{ot}_{1,i})) = s_i$,

with probability 1 by the correctness of OT scheme. So, we have $s_1 \oplus \ldots \oplus s_\ell = m \| 0^{\mathsf{len}'}$ with probability 1. The decryption outputs m since we have $\mathsf{pad} = 0^{\mathsf{len}'}$. Hence, the correctness follows. □

Constraint-hiding. Constraint-hiding directly follows from the receiver security of the underlying OT scheme.

Theorem B.5. Suppose the oblivious transfer OT scheme satisfies receiver privacy (Definition 2.3). Then the construction of the PCE scheme satisfies constraint-hiding (Definition 3.2).

Proof. The proof is the same as the proof of Theorem B.1, hence omitted. □

Unconditional message-hiding against malicious authority. This follows from the statistical sender security of the underlying OT scheme.

Theorem B.6. Suppose the oblivious transfer OT scheme satisfies statistical sender-privacy (Definition 2.4). Then the construction of the PCE scheme satisfies unconditional message-hiding against a malicious authority (Definition 3.7).

Proof. Recall that to prove unconditional message-hiding security against a malicious authority, we want

$$\mathsf{Enc}(\mathsf{pk}, x, m_0) \approx_s \mathsf{Enc}(\mathsf{pk}, x, m_1)$$

where $x \neq y$, for the database string y associated with the, possibly malformed, public key pk.
Here we let the extractor Ext of PCE to be the extractor $\mathsf{OT.Ext}$ of the underlying statistically sender private OT scheme. The admissibility of the Ext follows from the admissibility of $\mathsf{OT.Ext}$. The proof proceeds via the following sequence of hybrid games between the challenger and an unbounded adversary $\mathcal{A}$.

Hyb_0. This is the real world with $\beta = 0$, i.e., the challenge ciphertext is computed using the input m_0. We write the complete game here to set up the notations and easy reference in later hybrids.

1. $\mathcal{A}$ outputs a public key pk, an input x and a message pair $m_0, m_1 \in \{0,1\}^{\mathsf{len}}$.
2. The challenger parses $\mathsf{pk} = \{\mathsf{ot}_{1,i}\}_{i\in[\ell]}$. It chooses $\ell - 1$ uniformly random strings $s_1, \ldots, s_{\ell-1}$, where $s_i \leftarrow \{0,1\}^{\mathsf{len}+\mathsf{len}'}$ and computes $s_\ell := \bigoplus_{i=1}^{\ell-1} s_i \oplus (m_0 \| 0^{\mathsf{len}'})$. For each $i \in [\ell]$, it computes $\mathsf{ot}_{2,i}$ as in the construction. It returns $\mathsf{ct} = \{\mathsf{ot}_{2,i}\}_{i\in[\ell]}$ to $\mathcal{A}$.
3. $\mathcal{A}$ outputs a guess bit β'.

Hyb_1. This hybrid is same as the previous hybrid except that the challenger computes $\mathsf{ot}_{2,i}$, for all $i \in [\ell]$ differently, i.e., $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, s_i, s_i)$ if $y'[i] = x[i]$ and $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, r_i, r_i)$ if $y'[i] \neq x[i]$, where $y'[i] = \mathsf{OT.Ext}(\mathsf{ot}_{1,i})$.

Hyb_2. This hybrid is same as the previous hybrid except that the challenger computes s_j differently for $j \in [\ell]$, where j is the first index such that $y'[j] \neq x[j]$ and $y'[j] = \mathsf{OT.Ext}(\mathsf{ot}_{1,j})$. The challenger sets

$$s_j := \bigoplus_{i=1, i\neq j}^{\ell} s_i \oplus (m_1 \| 0^{\mathsf{len}'}).$$

Hyb_3. This hybrid is same as the previous hybrid except that the challenger computes $\mathsf{ot}_{2,i}$, for all $i \in [\ell]$ honestly as in the construction, i.e., if $x[i] = 0$, then $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, (s_i, r_i), \mathsf{ot}_{1,i})$, else $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, (r_i, s_i), \mathsf{ot}_{1,i})$, where $r_i \leftarrow \{0,1\}^{\mathsf{len}+\mathsf{len}'}$.
This is the real world with bit $\beta = 1$.

Indistinguishability of hybrids. We now prove that the consecutive hybrids are indistinguishable.

Claim B.7. Assume that OT satisfies statistical sender privacy, then $\mathsf{Hyb}_0 \approx_s \mathsf{Hyb}_1$.

Proof. To prove the above claim we consider sub hybrids $\mathsf{Hyb}_{0.k}$ for $k = 0$ to ℓ, where $\mathsf{Hyb}_{0.k}$ is same as Hyb_0 except that $\mathsf{ot}_{2,i}$ is generated differently for all $i \leq k$. We note that $\mathsf{Hyb}_0 = \mathsf{Hyb}_{0.0}$ and $\mathsf{Hyb}_{0.\ell} = \mathsf{Hyb}_1$.
To prove the above claim it suffices to show that $\mathsf{Hyb}_{0.k-1} \approx \mathsf{Hyb}_{0.k}$ for $k \in [\ell]$. We show that if there exists an unbounded adversary $\mathcal{A}$ who can distinguish between $\mathsf{Hyb}_{0.k-1}$ and $\mathsf{Hyb}_{0.k}$ with non-negligible advantage ϵ, then there exists an unbounded adversary $\mathcal{B}$ against the statistical sender privacy security of OT scheme with the same advantage ϵ. The reduction is as follows.

1. $\mathcal{B}$ first runs $\mathcal{A}$. $\mathcal{A}$ outputs a public key pk, an input x and a message pair $m_0, m_1 \in \{0,1\}^{\mathsf{len}}$.
2. $\mathcal{B}$ parses $\mathsf{pk} = \{\mathsf{ot}_{1,i}\}_{i\in[\ell]}$ and computes $y'_i = \mathsf{OT.Ext}(\mathsf{ot}_{1,i})$ for all $i \in [\ell]$.

3. It chooses $\ell - 1$ uniformly random strings $s_1, \ldots, s_{\ell-1}$ and computes $s_\ell := \bigoplus_{i=1}^{\ell-1} s_i \oplus m_0$. It also samples $r_i \leftarrow \{0,1\}^{\mathsf{len}+\mathsf{len}'}$ uniformly for $i \in [\ell]$.

4. For each $i \leq k-1$ if $x[i] = y'[i]$ then $\mathcal{B}$ computes $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, s_i, s_i)$ else it computes $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, r_i, r_i)$. For $i \geq k+1$, it computes $\mathsf{ot}_{2,i}$ as in the construction.

5. For $i = k$, if $x[k] = 0$, $\mathcal{B}$ sets $\mu_0 = s_k$ and $\mu_1 = r_k$, else it sets $\mu_0 = r_k$ and $\mu_1 = s_k$. It sends (μ_0, μ_1) and $\mathsf{ot}_{1,\ell}$ to the OT challenger. The challenger samples a bit $\beta \leftarrow \{0,1\}$ and computes $\mathsf{ot}_{2,\ell} \leftarrow \mathsf{OTS}(1^\lambda, \mu_0, \mu_1, \mathsf{ot}_{1,k})$ if $\beta = 0$, else if $\beta = 1$, it computes $\mathsf{ot}_{2,\ell} \leftarrow \mathsf{OTS}(1^\lambda, \mu_b, \mu_b, \mathsf{ot}_{1,k})$ where $b = \mathsf{OT.Ext}(\mathsf{ot}_{1,k})$. It returns $\mathsf{ot}_{2,k}$ to $\mathcal{B}$.

6. $\mathcal{B}$ sets $\mathsf{ct} = \{\mathsf{ot}_{2,i}\}_{i\in[\ell]}$ and returns ct to $\mathcal{A}$.

7. $\mathcal{A}$ outputs a guess bit β'. $\mathcal{B}$ forwards β' to the OT challenger.

We observe that if the OT challenger samples $\beta = 0$, then $\mathcal{B}$ simulated the distribution $\mathsf{Hyb}_{0.k-1}$, else $\mathsf{Hyb}_{0.k}$ with $\mathcal{A}$. Hence, advantage of $\mathcal{B}$ = $|\Pr(\beta' = 1|\beta = 0) - \Pr(\beta' = 1|\beta = 1)|$ = $|\Pr(\beta' = 1|\mathsf{Hyb}_{0.k-1}) - \Pr(\beta' = 1|\mathsf{Hyb}_{0.k})|$ = ϵ (by assumption). □

Claim B.8. Hyb_1 and Hyb_2 are statistically indistinguishable.

Proof. We observe that by the admissibility of $\mathcal{A}$, we have $x \neq y$, so there exists at least one $j \in [\ell]$ such that $y'[j] \neq x[j]$ where $y'[j] = \mathsf{OT.Ext}(\mathsf{ot}_{1,j})$. Note that s_j is a uniformly random string, as each share of a secret shared message is distributed uniformly at random. Also s_j is not used anywhere because of the change we introduced in Hyb_1 and hence it does not effect the adversary's view. So, we set $s_j := \bigoplus_{i=1, i\neq j}^{\ell} s_i \oplus (m_1 \| 0^{\mathsf{len}'})$, which is again a uniformly random string because of the randomness of s_i's for $i \in [\ell]$ and $i \neq j$. □

$\mathsf{Hyb}_2 \approx_s \mathsf{Hyb}_3$ assuming the statistical sender privacy of the OT scheme and the proof is similar to that of indistinguishability of Hyb_0 and Hyb_1, hence omitted. □

Security against outsiders. We prove that the PCE construction satisfies security against the outsiders using the following theorem.

Theorem B.9. Suppose the OT scheme satisfies receiver privacy (Definition 2.3) and statistical sender-privacy (Definition 2.4).Then the construction of the PCE scheme satisfies security against outsiders (Definition 3.8).

Proof. Recall that to show security against outsiders, we want

$$\mathsf{Enc}(\mathsf{pk}, x, m_0) \approx_c \mathsf{Enc}(\mathsf{pk}, x, m_1)$$

where $\mathsf{pk} \leftarrow \mathsf{Setup}(1^\lambda, y)$ for any database string y. We consider the following two cases.

1. $x \neq y$. Here the proof follows from the similar arguments as in the proof of Theorem B.6.

2. $x = y$. In this case we first replace y with a uniformly random string $\bar{y} \leftarrow \{0,1\}^\ell$ while generating the public key pk, using arguments as in the proof of Theorem B.5. Note that $y = \bar{y}$ with negligible probability $1/2^\ell$, which implies $x \neq \bar{y}$ with all but negligible probability. Now, the proof follows as in the case above.

□

C Missing Details from Section 4

In this section we detail out the missing details from Section 4.

C.1 PCSFE from SSP-OT and Garbled Circuits

We construct a single-key pre constrained static functional encryption scheme $\mathsf{PCSFE} = (\mathsf{Setup}, \mathsf{Enc}, \mathsf{Dec})$ for function family $\mathcal{F} = \{f : \mathcal{X} \to \mathcal{Y}\}$. We consider the boolean representation of the functions in $\mathcal{F}$ using an ℓ bit string.

Building Blocks. We use the following ingredients for our construction.

1. A garbling scheme $\mathsf{GC} = (\mathsf{Garble}, \mathsf{GCEval})$ for universal circuit $U[x]$, with $x \in \mathcal{M}$ hardwired, that takes as input a function from function family $\mathcal{F}$. This can be instantiated from Yao's scheme [Yao82], which can be based on any one way function.
2. A two-message SSP-OT scheme $\mathsf{OT} = (\mathsf{OTR}, \mathsf{OTS}, \mathsf{OTD})$ with input space as the space of labels of the above garbled circuit scheme. This can be instantiated from a wide variety of assumptions as discussed in Section 2.2.

Construction. The construction is similar to that in Section 3.2. We give the full description below for completeness.

$\mathsf{Setup}(1^\lambda, f) \to (\mathsf{pk}, \mathsf{sk})$. The setup algorithm does the following.

- Parse f as an ℓ bit string and let $f[i]$ denote the i-th bit of f.
- For $i = 1, \cdots, \ell$, compute $(\mathsf{ot}_{1,i}, \mathsf{st}_i) \leftarrow \mathsf{OTR}(1^\lambda, f[i])$.
- Output $\mathsf{pk} = \{\mathsf{ot}_{1,i}\}_{i\in[\ell]}$ and $\mathsf{sk}_f = \{f[i], \mathsf{st}_i\}_{i\in[\ell]}$.

$\mathsf{Enc}(\mathsf{pk}, x) \to \mathsf{ct}$. The encryption algorithm does the following.

- Define the circuit $U[x]$, with x hardwired, as follows :
  On input a function f, $U[x](f) = f(x)$.
- Compute $(\tilde{U}, \{\mathsf{lb}_{i,b}\}) \leftarrow \mathsf{Garble}(1^\lambda, U[x])$ for $i \in [\ell], b \in \{0,1\}$.
- Parse $\mathsf{pk} = \{\mathsf{ot}_{1,i}\}_{i\in[\ell]}$ and compute $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, \mathsf{lb}_{i,0}, \mathsf{lb}_{i,1}, \mathsf{ot}_{1,i})$ for all $i \in [\ell]$.
- Output $\mathsf{ct} = (\tilde{U}, \{\mathsf{ot}_{2,i}\}_{i\in[\ell]})$.

$\mathsf{Dec}(\mathsf{sk}, \mathsf{ct}) \to \{m/\perp\}$. The decryption algorithm does the following.

- Parse $\mathsf{sk} = \{f[i], \mathsf{st}_i\}_{i\in[\ell]}$ and $\mathsf{ct} = (\tilde{U}, \{\mathsf{ot}_{2,i}\}_{i\in[\ell]})$.
- For each $i \in [\ell]$, compute $\mathsf{lb}_{i,j} \leftarrow \mathsf{OTD}(1^\lambda, f[i], \mathsf{st}_i, \mathsf{ot}_{2,i})$ where $j \in \{0,1\}$.
- Compute $y \leftarrow \mathsf{GCEval}(\tilde{U}, \{\mathsf{lb}_{i,j}\}_{i\in[\ell]})$.
- Output y.

Correctness. We now show that the above construction satisfies correctness via the following theorem.

Theorem C.1. Suppose the OT scheme and the GC scheme satisfy correctness as defined in Definition 2.2 and Definition 2.20, respectively. Then the above construction satisfies *perfect* correctness as defined in Definition 4.1.

Proof. We note that for any $\mathsf{ct} \leftarrow \mathsf{Enc}(\mathsf{pk}, x)$, we have $\mathsf{ct} = (\tilde{U}, \{\mathsf{ot}_{2,i}\}_{i\in[\ell]})$, where $(\tilde{U}, \{\mathsf{lb}_{i,b}\}) \leftarrow \mathsf{Garble}(1^\lambda, U[x])$ and $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, \mathsf{lb}_{i,0}, \mathsf{lb}_{i,1}, \mathsf{ot}_{1,i})$ for $i \in [\ell]$. By the correctness OT scheme we have for all $i \in [\ell]$, $\mathsf{lb}_{i,f[i]} = \mathsf{OTD}(1^\lambda, f[i], \mathsf{st}_i, \mathsf{ot}_{2,i})$ with probability 1. Also, from the correctness of the GC scheme it follows that $f(x) = U[x](f) \leftarrow \mathsf{GCEval}(\tilde{U}, \{\mathsf{lb}_{i,f[i]}\}_{i\in[\ell]})$ with probability 1.
So we get $\mathsf{Dec}(\mathsf{sk}, \mathsf{ct}) = f(x)$ with probability 1. Hence the above scheme is perfectly correct. □

Function-hiding. This follows directly from the receiver privacy of the underlying OT scheme.

Theorem C.2. Suppose the OT scheme satisfies receiver privacy (Definition 2.3). Then the above construction of the PCSFE scheme satisfies function-hiding (Definition 4.2).

Proof. The proof follows from similar arguments as in Theorem B.1, by replacing the circuits C_0, C_1 with functions f_0, f_1. □

SIM security against malicious authority. This follows from the statistical sender security of the underlying OT scheme and the simulation security of GC scheme.

Theorem C.3. Suppose the OT scheme satisfies statistical sender-privacy (Definition 2.4) and the GC scheme satisfies simulation security (Definition 2.21). Then the above construction of the PCSFE scheme satisfies SIM security against a malicious authority (Definition 4.6).

Proof. To prove the theorem, we first construct the simulator Sim for the security against malicious authority of the PCSFE scheme. Note that the simulator is given $\mathsf{pk}, 1^{|x|}, f(x)$, where $\mathsf{pk} = \{\mathsf{ot}_{1,i}\}_{i\in[\ell]}$ as input. We now provide the description of the simulator Sim.
On input $\mathsf{pk} = \{\mathsf{ot}_{1,i}\}_{i\in[\ell]}, 1^{|x|}, f(x)$

1. Compute $(\tilde{U}, \{\mathsf{lb}_i\}_{i\in[\ell]}) \leftarrow \mathsf{GC.Sim}(1^\lambda, f(x))$.
2. Compute $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, \mathsf{lb}_i, \mathsf{lb}_i, \mathsf{ot}_{1,i})$ for all $i \in [\ell]$.
3. Output $\mathsf{ct} = (\tilde{U}, \{\mathsf{ot}_{2,i}\}_{i\in[\ell]})$.

To prove the security, we consider the following sequence of hybrids.

Hyb_0. This is the real world, i.e., challenge ct is computed by honestly running the Enc algorithm.

Hyb_1. This hybrid is same as the previous hybrid except that the challenger computes $\mathsf{ot}_{2,i}$ differently, i.e., $\mathsf{ot}_{2,i} \leftarrow \mathsf{OTS}(1^\lambda, \mathsf{lb}_{i,b}, \mathsf{lb}_{i,b}, \mathsf{ot}_{1,i})$ for all $i \in [\ell]$, where $b \leftarrow \mathsf{OT.Ext}(\mathsf{ot}_{1,i})$.

Hyb_2. This hybrid is same as the previous hybrid except that the challenger computes the garbled circuit and the labels differently using $\mathsf{GC.Sim}$, i.e., $(\tilde{U}, \{\mathsf{lb}_i\}_{i\in[\ell]}) \leftarrow \mathsf{GC.Sim}(1^\lambda, f(x))$ where $f = f[1]\dots f[\ell]; f[i] \leftarrow \mathsf{OT.Ext}(\mathsf{ot}_{1,i})$ for $i \in [\ell]$.

We note that Hyb_2 is the ideal world.

Indistinguishability of hybrids. The proof is identical to the proof of Theorem 3.12, hence omitted. □

Security against outsiders. This follows from Lemma 4.9.

Unconditional security for NC^1. We note that by using an information theoretic version of Yao's garbled circuit [IK02], efficient for NC^1, in the above construction we can achieve a PCSFE scheme, for class NC^1, with unconditional security against a malicious authority.

Bounded key setting. We observe that it is easy to extend the construction above to bounded multi-key construction by simply preparing more OT instances for f_i. Note that this makes $|\mathsf{pk}|$ linear in the number of functions for which we generate the secret keys.

C.2 PCSFE with Unconditional Security from FHE

In this section we see that by using a slightly stronger assumption we can achieve unconditional security against a malicious authority. We construct a single-key PCSFE scheme for function family $\mathcal{F} = \{f : \mathcal{X} \to \mathcal{Y}\}$. We consider the boolean representation of the functions in $\mathcal{F}$ using an ℓ bit string.

Construction. The construction is similar to that of the pre-constrained encryption scheme from maliciously circuit private FHE in Section 3.3, where we can think of the circuit C implementing the function f, except that the circuit $G[x]$ in the encryption algorithm, on input f, outputs $f(x)$.

Theorem C.4. The construction of PCSFE from maliciously circuit private FHE is correct and satisfies function-hiding, unconditional security against a malicious authority and security against outsiders.

Proof. We note that the proof for the correctness and function-hiding is identical to the proof of Theorem 3.14 and 3.15, respectively.
We prove the security against the authority. First, we construct a simulator Sim for the security against malicious authority of the PCSFE scheme. Note that the simulator is given $\mathsf{pk}, 1^{|x|}, f(x)$, where $\mathsf{pk} = (\mathsf{FHE.pk}, \mathsf{FHE.ct}_f)$ as inputs. The simulator proceeds as follows:

1. Runs the FHE simulator to compute $\mathsf{FHE.ct} \leftarrow \mathsf{FHE.Sim}(\mathsf{FHE.pk}, \mathsf{FHE.ct}_f, f(x))$.
2. Output $\mathsf{ct} = \mathsf{FHE.ct}$.

To prove the security, we consider the following sequence of hybrids.

Hyb_0. This is the real world, i.e., challenge ct is computed by honestly running the Enc algorithm, using the possibly malformed public key pk.

Hyb_1. This hybrid is same as the previous hybrid except that the challenger computes FHE.ct as

$$\mathsf{FHE.ct} \leftarrow \mathsf{FHE.Sim}(\mathsf{FHE.pk}, \mathsf{FHE.ct}_f, G[x](f))$$

where $f = \mathsf{FHE.Ext}(\mathsf{FHE.pk}, \mathsf{FHE.ct}_f)$. We note that this is the ideal world.

The indistinguishability of the above two hybrids follows from the malicious circuit private security of the underlying FHE scheme. Recall that maliciously circuit-private FHE ensures that even for possibly malformed FHE.pk, FHE.ct, $\mathsf{FHE.Eval}(\mathsf{FHE.pk}, G[x], \mathsf{FHE.ct}_f)$ is statistically indistinguishable from
$\mathsf{FHE.Sim}(G[x](f)) = \mathsf{FHE.Sim}(f(x))$. The proof is similar to that of Theorem 3.16, hence omitted.
Also, the security against outsiders follows from Lemma 4.9. □

C.3 Missing Details from Section 4.4.1

Relaxed-SIM security against a semi-malicious authority. We prove the security of our succinct PCSFE scheme from Section 4.4.1.

Theorem C.5 (Restate of Theorem 4.15). Assume that HE scheme is secure against a semi malicious setup , GC is a secure garbled circuit scheme, and RDMPC is secure as per Definition 2.15, 2.21, and 2.12 respectively. Then the above construction of succinct PCSFE scheme satisfies security against a semi malicious authority Definition 4.5.

Proof. To prove the theorem, we first construct the simulator Sim for the security against malicious authority of the PCSFE scheme. Note that the simulator is given $\mathsf{pk} = (\mathsf{key}, h_{\mathsf{str}})$, $1^{|x|}$, and $\mathcal{V}$, where

$$\mathcal{V} = \left\{ f_i(x), f_i, \mathsf{sk}_{f_i} = \left(i, \Delta^{(i)}, \hat{f}_i \right) \right\}_{i \in [Q]}.$$

We begin with setting up few notations. We define sets S_{crr} and S_{dis} as follows.

$$S_{\mathsf{crr}} := \bigcup_{i,i' \in [Q], i \neq i'} \left(\Delta^{(i)} \cap \Delta^{(i')} \right), \quad S_{\mathsf{dis}} := \left(\bigcup_{i \in [Q]} \Delta^{(i)} \right) \setminus S_{\mathsf{crr}}.$$

We now describe the PCSFE ciphertext simulator SIM. On input $\mathsf{pk} = (\mathsf{key}, h_{\mathsf{str}})$, $1^{|x|}$, and $\mathcal{V}$, it runs as follows.

1. For each $i \in [Q]$, run

$$\left(\left\{\widehat{C}_{i,j}\right\}_{j\in S_{\mathsf{crr}}}, \{\widehat{y}_{i,j}\}_{i\in[Q],j\in\Delta^{(i)}}\right) \leftarrow \mathsf{RDMPC.Sim}_0\left(1^{|C_x|}, S_{\mathsf{crr}}, \{f_i(x), f_i\}_{i\in[Q]}\right)$$

   where C_x is the universal circuit with x hardwired.

2. For all $i \in [Q]$ and $j \in [N]$, do the following.

   – If $j \in S_{\mathsf{crr}}$, do the following.

     (a) Set $\mathsf{L}_{i,j}(\cdot) \coloneqq \mathsf{Local}(\widehat{C}_{i,j}, \cdot)$ and compute

$$\left(\widetilde{L}_{i,j}, \left\{\mathsf{lab}_{i,j,k,b}\right\}_{k\in[\widehat{\ell}],b\in\{0,1\}}\right) \leftarrow \mathsf{GC.Garble}(1^\lambda, \mathsf{L}_{i,j}).$$

     (b) For all $k \in [\widehat{\ell}]$ and $b \in \{0,1\}$, compute

$$\mathsf{HE.ct}_{i,j,k,b} \leftarrow \mathsf{HE.Enc}(\mathsf{key}, (h_{\mathsf{str}}, (i,j,k), b), \mathsf{lab}_{i,j,k,b}).$$

   – If $j \in S_{\mathsf{dis}}$, then there exists an unique i such that $j \in \Delta^{(i)}$. Retrieve the corresponding $i \in [Q]$ and do the following.

     (a) Compute

$$\left(\widetilde{L}_{i,j}, \{\mathsf{lab}_{i,j,k}\}_{k\in[\widehat{\ell}]}\right) \leftarrow \mathsf{GC.Sim}(1^\lambda, \widehat{y}_{i,j}).$$

     (b) For all $k \in [\widehat{\ell}]$ and $b \in \{0,1\}$, compute

$$\mathsf{HE.ct}_{i,j,k,b} \leftarrow \mathsf{HE.Enc}(\mathsf{key}, (h_{\mathsf{str}}, (i,j,k), b), \mathsf{lab}_{i,j,k}).$$

   – If $j \notin S_{\mathsf{crr}} \cup S_{\mathsf{dis}}$, do the following.

     (a) Set $L_{i,j}$ to be a dummy circuit of input length $\widehat{\ell}$, which always outputs $0^{\widehat{s}}$ and compute

$$\left(\widetilde{L}_{i,j}, \left\{\mathsf{lab}_{i,j,k,b}\right\}_{k\in[\widehat{\ell}],b\in\{0,1\}}\right) \leftarrow \mathsf{GC.Garble}(1^\lambda, \mathsf{L}_{i,j}).$$

     (b) For all $k \in [\widehat{\ell}]$ and $b \in \{0,1\}$, compute

$$\mathsf{HE.ct}_{i,j,k,b} \leftarrow \mathsf{HE.Enc}(\mathsf{key}, (h_{\mathsf{str}}, (i,j,k), b), 0^{\mathsf{len}})$$

     where len is the length of the labels.

3. Output the ciphertext

$$\mathsf{ct} = \left(\left\{\widetilde{L}_{i,j}\right\}_{i\in[Q],j\in[N]}, \left\{\mathsf{HE.ct}_{i,j,k,b}\right\}_{i\in[Q],j\in[N],k\in[\widehat{\ell}],b\in\{0,1\}}\right).$$

To prove the security, we consider following hybrids.

Hyb_0. This is the real world. We write the complete game here to set up notations and for easy reference in the later hybrids.

1. The adversary outputs the functions $f_1, \ldots, f_Q$ and the randomness $\Delta^{(1)}, \ldots, \Delta^{(Q)}, r$. Abort and output $\perp$ if either of the following is true.

   – $\left|\Delta^{(i)}\right| \neq n$, for $i \in [Q]$.

– $|S_{\mathsf{crr}}| = \left|\bigcup_{i,i' \in [Q], i \neq i'} \left(\Delta^{(i)} \cap \Delta^{(i')}\right)\right| > t$.

2. The challenger computes $S_i = \{(i,j,k,\hat{f}_i[k])\}_{j \in \Delta^{(i)}, k \in [\widehat{\ell}]}$, and defines S and string str as in the construction. It lets the $\mathsf{key} = r$ and computes $h_{\mathsf{str}} \leftarrow \mathsf{Hash}(\mathsf{key}, \mathsf{str})$. It sets $\mathsf{pk} = (\mathsf{key}, h_{\mathsf{str}})$ and $\mathsf{sk}_{f_i} = \left(i, \Delta^{(i)}, \hat{f}_i\right)$ and returns pk and $\{\mathsf{sk}_{f_i}\}_{i \in [Q]}$ to $\mathcal{A}$.
3. $\mathcal{A}$ outputs the challenge input x. The challenger defines C_x and computes

$$\mathsf{ct} = \left(\left\{\widetilde{L}_{i,j}\right\}_{i \in [Q], j \in [N]}, \left\{\mathsf{HE.ct}_{i,j,k,b}\right\}_{i \in [Q], j \in [N], k \in [\widehat{\ell}], b \in \{0,1\}}\right)$$

as in the construction. It returns ct to $\mathcal{A}$.
4. In the end, $\mathcal{A}$ outputs a guess bit β'.

Hyb_1. In this hybrid we change the way ciphertext is generated. In particular, we change the way $\mathsf{HE.ct}_{i,j,k,b}$ is computed in the following cases.

– If $j \in S_{\mathsf{dis}}$, the challenger computes

$$\mathsf{HE.ct}_{i,j,k,b} \leftarrow \mathsf{HE.Enc}(\mathsf{key}, (h_{\mathsf{str}}, (i,j,k), b), \mathsf{lab}_{i,j,k,\hat{f}_i[k]})$$

where i is the unique index such that $j \in \Delta^{(i)}$.

– If $j \notin S_{\mathsf{crr}} \cup S_{\mathsf{dis}}$, the challenger computes

$$\mathsf{HE.ct}_{i,j,k,b} \leftarrow \mathsf{HE.Enc}(\mathsf{key}, (h_{\mathsf{str}}, (i,j,k), b), 0^{\mathsf{len}})$$

where len is the length of the labels.

Hyb_2. In this hybrid, we further change the way ciphertext is generated. In particular, we change the way $\widetilde{L}_{i,j}$ and $\mathsf{lab}_{i,j,k,b}$ is computed in the following cases.

– If $j \in S_{\mathsf{dis}}$, then there exists an unique i such that $j \in \Delta^{(i)}$. Retrieve the corresponding $i \in [Q]$ and compute

$$(\widetilde{L}_{i,j}, \{\mathsf{lab}_{i,j,k}\}_{k \in [\widehat{\ell}]}) \leftarrow \mathsf{GC.Sim}(1^\lambda, \widehat{y}_{i,j})$$

where $\widehat{y}_{i,j} = \mathsf{Local}(\widehat{C}_{i,j}, \hat{f}_i)$. Then it sets

$$\mathsf{lab}_{i,j,k,\hat{f}_i[k]} \coloneqq \mathsf{lab}_{i,j,k}$$

for $k \in [\widehat{\ell}]$.

– If $j \notin S_{\mathsf{crr}} \cup S_{\mathsf{dis}}$, the challenger sets $L_{i,j}$ to be a dummy circuit of input length $\widehat{\ell}$, which always outputs $0^{\widehat{s}}$ and computes

$$\left(\widetilde{L}_{i,j}, \left\{\mathsf{lab}_{i,j,k,b}\right\}_{k \in [\widehat{\ell}], b \in \{0,1\}}\right) \leftarrow \mathsf{GC.Garble}(1^\lambda, \mathsf{L}_{i,j}).$$

Hyb_3. In this hybrid, we further change the way ciphertext is generated. In particular, we change the way the circuit encoding $\widehat{C}_{i,j}$ and local output encodings $\widehat{y}_{i,j}$ are generated as follows. For each $i \in [Q]$, run

$$\left(\left\{\widehat{C}_{i,j}\right\}_{j \in S_{\mathsf{crr}}}, \{\widehat{y}_{i,j}\}_{i \in [Q], j \in \Delta^{(i)}}\right) \leftarrow \mathsf{RDMPC.Sim}_0\left(1^{|C_x|}, S_{\mathsf{crr}}, \{f_i(x), f_i\}_{i \in [Q]}\right)$$

where C_x is the universal circuit with x hardwired. We note that we only have $\widehat{C}_{i,j}$ for all $j \in S_{\mathsf{crr}}$ and this suffices for the ciphertext generation due to the changes made in Hyb_0 and Hyb_1.
We also note that this is the ideal world with the simulator SIM defined above.

Indistinguishability of hybrids. We now prove that the consecutive hybrids are indistinguishable.

Claim C.6. Assume HE is a semi malicious secure hash encryption scheme, then $\mathsf{Hyb}_0 \approx_c \mathsf{Hyb}_1$.

Proof. We show that if there exists a PPT adversary $\mathcal{A}$ who can distinguish between Hyb_0 and Hyb_1 with non-negligible advantage ϵ, then there exists an adversary $\mathcal{B}$ against the semi malicious secure hash encryption scheme with the same advantage ϵ. The reduction is as follows.

1. The HE challenger samples a bit $\beta \leftarrow \{0,1\}$ and initiates the multi challenge HE security game with $\mathcal{B}$.
2. $\mathcal{B}$ first runs $\mathcal{A}$. $\mathcal{A}$ outputs the functions $f_1, \ldots, f_Q$ and the randomness $\Delta^{(1)}, \ldots, \Delta^{(Q)}, r$. $\mathcal{B}$ aborts and output $\perp$ if $\left|\Delta^{(i)}\right| \neq n$ or $|S_{\mathsf{crr}}| > t$.
3. $\mathcal{B}$ computes $\hat{f}_i$ for $i \in [Q]$, defines the set S and the string str as in the construction. It sends the string str and the randomness r to the HE challenger and gets back the hash key key.
4. $\mathcal{B}$ computes $h_{\mathsf{str}} = \mathsf{Hash}(\mathsf{key}, \mathsf{str})$. It sets $\mathsf{pk} = (\mathsf{key}, h_{\mathsf{str}})$ and $\mathsf{sk}_{f_i} = \left(i, \Delta^{(i)}, \hat{f}_i\right)$ and returns pk and $\{\mathsf{sk}_{f_i}\}_{i\in[Q]}$ to $\mathcal{A}$.
5. $\mathcal{A}$ outputs the challenge input x. $\mathcal{B}$ computes $\left(\widetilde{L}_{i,j}, \left\{\mathsf{lab}_{i,j,k,b}\right\}_{k\in[\widehat{\ell}],b\in\{0,1\}}\right)$ for all $i \in [Q]$ and $j \in [N]$ as in the construction.
6. $\mathcal{B}$ computes $\left\{\mathsf{HE.ct}_{i,j,k,b}\right\}_{i\in[Q],j\in[N],k\in[\widehat{\ell}],b\in\{0,1\}}$ as follows
   - If $j \in S_{\mathsf{dis}}$ and $b \neq \hat{f}_i[k]$, it first retrieves i such that $j \in \Delta^{(i)}$ and then sends the index (i,j,k) and two messages $(\mathsf{lab}_{i,j,k,1-\hat{f}_i[k]}, \mathsf{lab}_{i,j,k,\hat{f}_i[k]})$ to the HE challenger. The challenger computes
   $$\mathsf{HE.ct}_{i,j,k,b} \leftarrow \begin{cases} \mathsf{HE.Enc}(\mathsf{key}, (\mathsf{Hash}(\mathsf{key},\mathsf{str}),(i,j,k),b), \mathsf{lab}_{i,j,k,1-\hat{f}_i[k]}) & \text{if } \beta = 0 \\ \mathsf{HE.Enc}(\mathsf{key}, (\mathsf{Hash}(\mathsf{key},\mathsf{str}),(i,j,k),b), \mathsf{lab}_{i,j,k,\hat{f}_i[k]}) & \text{if } \beta = 1 \end{cases}$$
   and returns $\mathsf{HE.ct}_{i,j,k,b}$ to $\mathcal{B}$.
   - If $j \notin S_{\mathsf{crr}} \cup S_{\mathsf{dis}}$, for each $b \in \{0,1\}$, it sends the index (i,j,k) and two messages $(\mathsf{lab}_{i,j,k,b}, 0^{\mathsf{len}})$ to the HE challenger, where len is the length of the labels. The challenger computes
   $$\mathsf{HE.ct}_{i,j,k,b} \leftarrow \begin{cases} \mathsf{HE.Enc}(\mathsf{key}, (\mathsf{Hash}(\mathsf{key},\mathsf{str}),(i,j,k),b), \mathsf{lab}_{i,j,k,b}) & \text{if } \beta = 0 \\ \mathsf{HE.Enc}(\mathsf{key}, (\mathsf{Hash}(\mathsf{key},\mathsf{str}),(i,j,k),b), 0^{\mathsf{len}}) & \text{if } \beta = 1 \end{cases}$$
   and returns $\mathsf{HE.ct}_{i,j,k,b}$ to $\mathcal{B}$.
   - Else, it computes $\mathsf{HE.ct}_{i,j,k,b}$ as in the construction.
7. $\mathcal{B}$ sends $\mathsf{ct} = \left(\left\{\widetilde{L}_{i,j}\right\}_{i\in[Q],j\in[N]}, \left\{\mathsf{HE.ct}_{i,j,k,b}\right\}_{i\in[Q],j\in[N],k\in[\widehat{\ell}],b\in\{0,1\}}\right)$ to $\mathcal{A}$.
8. $\mathcal{A}$ outputs a guess bit β'.

We observe that if the HE challenger samples $\beta = 0$, then $\mathcal{B}$ simulated the distribution Hyb_0, else Hyb_1 with $\mathcal{A}$. Hence, the advantage of $\mathcal{B} = |\Pr(\beta' = 1|\beta = 0) - \Pr(\beta' = 1|\beta = 1)| = |\Pr(\beta' = 1|\mathsf{Hyb}_0) - \Pr(\beta' = 1|\mathsf{Hyb}_1)| = \epsilon$ (by assumption). □

Claim C.7. Assume GC is a secure garbled circuit scheme, then $\mathsf{Hyb}_1 \approx_c \mathsf{Hyb}_2$.

Proof. We show that if there exists a PPT adversary $\mathcal{A}$ who can distinguish between Hyb_1 and Hyb_2 with non-negligible advantage ϵ, then there exists an adversary $\mathcal{B}$ against the security of GC scheme with the same advantage ϵ. The reduction is as follows.

1. The GC challenger samples a bit $\beta \leftarrow \{0,1\}$ and initiates the multi challenge GC security game with $\mathcal{B}$.
2. $\mathcal{B}$ first runs $\mathcal{A}$. $\mathcal{A}$ outputs the functions $f_1, \ldots, f_Q$ and the randomness $\Delta^{(1)}, \ldots, \Delta^{(Q)}, r$. $\mathcal{B}$ aborts and output $\perp$ if $\left|\Delta^{(i)}\right| \neq n$ or $|S_{\mathsf{crr}}| > t$.
3. $\mathcal{B}$ computes $\hat{f}_i$ for $i \in [Q]$, defines the set S and the string str as in the construction. $\mathcal{B}$ generates $\mathsf{key} \leftarrow \mathsf{HE.Gen}(1^\lambda, r)$ and computes $h_{\mathsf{str}} = \mathsf{Hash}(\mathsf{key}, \mathsf{str})$. It sets $\mathsf{pk} = (\mathsf{key}, h_{\mathsf{str}})$ and $\mathsf{sk}_{f_i} = \left(i, \Delta^{(i)}, \hat{f}_i\right)$ and returns pk and $\{\mathsf{sk}_{f_i}\}_{i\in[Q]}$ to $\mathcal{A}$.
4. $\mathcal{A}$ outputs the challenge input x. $\mathcal{B}$ defines the circuit C_x as in the construction and computes $(\widehat{C}_{i,1}, \ldots, \widehat{C}_{i,N}) \leftarrow \mathsf{CktEnc}(1^\lambda, 1^\lambda, 1^\ell, C_x)$ for $i \in [Q]$.
5. For all $i \in [Q]$ and $j \in [N]$, $\mathcal{B}$ does the following.
   - If $j \in S_{\mathsf{crr}}$, set $\mathsf{L}_{i,j}(\cdot) := \mathsf{Local}(\widehat{C}_{i,j}, \cdot)$ and compute
   $$\left(\widetilde{L}_{i,j}, \left\{\mathsf{lab}_{i,j,k,b}\right\}_{k\in[\widehat{\ell}],b\in\{0,1\}}\right) \leftarrow \mathsf{GC.Garble}(1^\lambda, \mathsf{L}_{i,j}).$$
   - If $j \in S_{\mathsf{dis}}$, it first retrieves i such that $j \in \Delta^{(i)}$ and then sends the circuit $\mathsf{L}_{i,j}(\cdot) := \mathsf{Local}(\widehat{C}_{i,j}, \cdot)$ and input $\hat{f}_i$ to the GC challenger. The challenger computes
   $$\left(\widetilde{L}_{i,j}, \left\{\mathsf{lab}_{i,j,k,b}\right\}_{k\in[\widehat{\ell}],b\in\{0,1\}}\right) \leftarrow \mathsf{GC.Garble}(1^\lambda, \mathsf{L}_{i,j}) \text{ if } \beta = 0$$
   or
   $$\left(\widetilde{L}_{i,j}, \left\{\mathsf{lab}_{i,j,k}\right\}_{k\in[\widehat{\ell}]}\right) \leftarrow \mathsf{GC.Sim}(1^\lambda, \mathsf{Local}(\widehat{C}_{i,j}, \hat{f}_i)) \text{ if } \beta = 1$$
   and returns $\left(\widetilde{L}_{i,j}, \left\{\mathsf{lab}_{i,j,k,\hat{f}_i[k]}\right\}_{k\in[\widehat{\ell}]}\right)$ to $\mathcal{B}$, where $\mathsf{lab}_{i,j,k,\hat{f}_i[k]} = \mathsf{lab}_{i,j,k}$ for $\beta = 1$.
   - If $j \notin S_{\mathsf{crr}} \cup S_{\mathsf{dis}}$, set $L_{i,j}$ to be a dummy circuit of input length $\widehat{\ell}$, which always outputs $0^{\widehat{s}}$ and compute
   $$\left(\widetilde{L}_{i,j}, \left\{\mathsf{lab}_{i,j,k,b}\right\}_{k\in[\widehat{\ell}],b\in\{0,1\}}\right) \leftarrow \mathsf{GC.Garble}(1^\lambda, \mathsf{L}_{i,j}).$$
6. For all $i \in [Q]$ and $j \in [N]$, $\mathcal{B}$ does the following.
   - If $j \in S_{\mathsf{crr}}$, it computes $\{\mathsf{HE.ct}_{i,j,k,b}\}_{k\in[\widehat{\ell}],b\in\{0,1\}}$ as in the construction.
   - If $j \in S_{\mathsf{dis}}$, it computes $\mathsf{HE.ct}_{i,j,k,b} \leftarrow \mathsf{HE.Enc}(\mathsf{key}, (h_{\mathsf{str}}, (i,j,k), b), \mathsf{lab}_{i,j,k,\hat{f}_i[k]})$ for all $k \in [\widehat{\ell}]$ and $b \in \{0,1\}$.
   - If $j \notin S_{\mathsf{crr}} \cup S_{\mathsf{dis}}$, then for all $k \in [\widehat{\ell}]$ and $b \in \{0,1\}$, it computes $\mathsf{HE.ct}_{i,j,k,b} \leftarrow \mathsf{HE.Enc}(\mathsf{key}, (h_{\mathsf{str}}, (i,j,k), b), 0^{\mathsf{len}})$, where len is the length of the labels.
7. $\mathcal{B}$ sends $\mathsf{ct} = \left(\left\{\widetilde{L}_{i,j}\right\}_{i\in[Q],j\in[N]}, \left\{\mathsf{HE.ct}_{i,j,k,b}\right\}_{i\in[Q],j\in[N],k\in[\widehat{\ell}],b\in\{0,1\}}\right)$ to $\mathcal{A}$.
8. $\mathcal{A}$ outputs a guess bit β'.

First, we note that for $j \notin S_{\mathsf{crr}} \cup S_{\mathsf{dis}}$, the way $\mathcal{B}$ generates $\left(\widetilde{L}_{i,j}, \left\{\mathsf{lab}_{i,j,k,b}\right\}_{k\in[\widehat{\ell}],b\in\{0,1\}}\right)$ does not effect the adversary's view because of the change we introduced in Hyb_1. In particular, we do not use the labels $\mathsf{lab}_{i,j,k,b}$ generated for $j \notin S_{\mathsf{crr}} \cup S_{\mathsf{dis}}$ anywhere in Hyb_1 or Hyb_2 and hence the adversary does not have sufficient information to compute on the garbled circuit $\widetilde{L}_{i,j}$ for the corresponding set of labels. Next, we observe that if the GC challenger samples $\beta = 0$, then $\mathcal{B}$ simulated the distribution Hyb_1 , else Hyb_2 with $\mathcal{A}$. Hence, advantage of $\mathcal{B} = |\Pr(\beta' = 1|\beta = 0) - \Pr(\beta' = 1|\beta = 1)|$ $= |\Pr(\beta' = 1|\mathsf{Hyb}_1) - \Pr(\beta' = 1|\mathsf{Hyb}_2)| = \epsilon$ (by assumption). □

Claim C.8. Assume RDMPC is a secure reusable dynamic MPC scheme, then $\mathsf{Hyb}_2 \approx_c \mathsf{Hyb}_3$.

Proof. We show that if there exists a PPT adversary $\mathcal{A}$ who can distinguish between Hyb_2 and Hyb_3 with non-negligible advantage ϵ, then there exists an adversary $\mathcal{B}$ against the security of RDMPC scheme with the same advantage ϵ. The reduction is as follows.

1. The RDMPC challenger samples a bit $\beta \leftarrow \{0,1\}$ and initiates the multi challenge RDMPC security game with $\mathcal{B}$.
2. $\mathcal{B}$ first runs $\mathcal{A}$. $\mathcal{A}$ outputs the functions $f_1, \ldots, f_Q$ and the randomness $\Delta^{(1)}, \ldots, \Delta^{(Q)}, r$. $\mathcal{B}$ aborts and output $\perp$ if $\left|\Delta^{(i)}\right| \neq n$ or $|S_{\mathsf{crr}}| > t$.
3. $\mathcal{B}$ sends the query bound 1^Q, input length 1^ℓ, where $|f_i| = \ell$ to the challenger. Note that this defines the total number of parties $N = (\lambda, Q)$, number of parties $n = n(\lambda, Q)$ participating in any session, threshold $t = t(\lambda, Q)$. It also sends $S_{\mathsf{crr}}, \Delta^{(1)}, \ldots, \Delta^{(Q)}$ to the challenger.
4. $\mathcal{B}$ sends the functions $f_1, \ldots, f_Q$ as the input encoding query to the RDMPC challenger. The challenger computes and returns $\hat{f}_i \leftarrow \mathsf{InpEnc}(1^\lambda, 1^\ell, f_i)$ for each f_i, where $i \in [Q]$, to $\mathcal{B}$.
5. $\mathcal{B}$ generates $\mathsf{key} \leftarrow \mathsf{HE.Gen}(1^\lambda, \overline{\ell})$ and computes $h_{\mathsf{str}} \leftarrow \mathsf{HE.Hash}(\mathsf{key}, \mathsf{str})$, where the string str is as defined in the construction. It sends $\mathsf{pk} = (\mathsf{key}, h_{\mathsf{str}})$ and $\mathsf{sk}_{f_i} = \left(i, \Delta^{(i)}, \hat{f}_i\right)$, for $i \in [Q]$, to the adversary $\mathcal{A}$.
6. $\mathcal{A}$ outputs the challenge input x. $\mathcal{B}$ does the following.
   (a) It defines the circuit C_x as in the construction and sends the circuit C_x as the challenge for all the input encoding queries to the RDMPC challenger. For each $i \in [Q]$, the challenger does the following:
   - If $\beta = 0$, it computes $(\widehat{C}_{i,1}, \ldots, \widehat{C}_{i,N}) \leftarrow \mathsf{CktEnc}(1^\lambda, 1^\lambda, 1^\ell, C_x)$ for each $i \in [Q]$ and $y_{i,j} := \mathsf{Local}(\widehat{C}_{i,j}, \hat{f}_i)$ for all $i \in [Q]$ and $j \in \Delta^{(i)}$.
   - If $\beta = 1$, it computes
   $$\left(\left\{\widehat{C}_{i,j}\right\}_{j \in S_{\mathsf{crr}}}, \{\widehat{y}_{i,j}\}_{i \in [Q], j \in \Delta^{(i)}}\right) \leftarrow \mathsf{RDMPC.Sim}_0\left(1^{|C_x|}, S_{\mathsf{crr}}, \{f_i(x), f_i\}_{i \in [Q]}\right)$$
   - It returns $\left(\left\{\widehat{C}_{i,j}\right\}_{j \in S_{\mathsf{crr}}}, \{\widehat{y}_{i,j}\}_{i \in [Q], j \in \Delta^{(i)}}\right)$ to $\mathcal{B}$.

   (b) For all $i \in [Q]$ and $j \in [N]$,
   - if $j \in S_{\mathsf{crr}}$, sets $\mathsf{L}_{i,j}(\cdot) := \mathsf{Local}(\widehat{C}_{i,j}, \cdot)$ and computes $(\widetilde{L}_{i,j}, \{\mathsf{lab}_{i,j,k,b}\}_{k \in [\widehat{\ell}], b \in \{0,1\}})$ and $\{\mathsf{HE.ct}_{i,j,k,b}\}_{k \in \widehat{\ell}, b \in \{0,1\}}$ honestly as in the construction.
   - if $j \in S_{\mathsf{dis}}$, then there exists an unique i such that $j \in \Delta^{(i)}$. Retrieve the corresponding $i \in [Q]$ and computes $\left(\widetilde{L}_{i,j}, \left\{\mathsf{lab}_{i,j,k}\right\}_{k \in [\widehat{\ell}]}\right) \leftarrow \mathsf{GC.Sim}(1^\lambda, \widehat{y}_{i,j})$. Then for all $k \in [\widehat{\ell}]$ and $b \in \{0,1\}$, it computes
   $$\mathsf{HE.ct}_{i,j,k,b} \leftarrow \mathsf{HE.Enc}(\mathsf{key}, (h_{\mathsf{str}}, (i,j,k), b), \mathsf{lab}_{i,j,k}).$$
   - if $j \notin S_{\mathsf{crr}} \cup S_{\mathsf{dis}}$, set $L_{i,j}$ to be a dummy circuit of input length $\widehat{\ell}$, which always outputs $0^{\widehat{s}}$ and compute
   $$\left(\widetilde{L}_{i,j}, \left\{\mathsf{lab}_{i,j,k,b}\right\}_{k \in [\widehat{\ell}], b \in \{0,1\}}\right) \leftarrow \mathsf{GC.Garble}(1^\lambda, \mathsf{L}_{i,j}).$$
   Then for all $k \in [\widehat{\ell}]$ and $b \in \{0,1\}$, it computes
   $$\mathsf{HE.ct}_{i,j,k,b} \leftarrow \mathsf{HE.Enc}(\mathsf{key}, (h_{\mathsf{str}}, (i,j,k), b), 0^{\mathsf{len}})$$
   where len is the length of the labels.

(c) $\mathcal{B}$ sends $\mathsf{ct} = \left(\left\{ \widetilde{L}_{i,j} \right\}_{i\in[Q],j\in[N]}, \left\{ \mathsf{HE.ct}_{i,j,k,b} \right\}_{i\in[Q],j\in[N],k\in[\widehat{\ell}],b\in\{0,1\}} \right)$ to $\mathcal{A}$.

7. $\mathcal{A}$ outputs a guess bit β'.

We observe that if the RDMPC challenger samples $\beta = 0$, then $\mathcal{B}$ simulated the distribution Hyb_2 , else Hyb_3 with $\mathcal{A}$. Hence, advantage of $\mathcal{B} = |\Pr(\beta' = 1|\beta = 0) - \Pr(\beta' = 1|\beta = 1)| = |\Pr(\beta' = 1|\mathsf{Hyb}_2) - \Pr(\beta' = 1|\mathsf{Hyb}_3)| = \epsilon$ (by assumption). □

We also note that Hyb_3 is the ideal world where the ciphertext is simulated using the simulator SIM. Hence, the proof. □

C.4 Missing Details from Section 4.4.2

Correctness. We show that the above construction is correct via the following theorem.

Theorem C.9. Assume that the FHE scheme and the SFE scheme satisfies correctness. Then the above construction is correct.

Proof. For any $\mathsf{sk} = (\mathsf{FHE.sk}, \mathsf{SFE.sk}_{f_i})$ and $\mathsf{ct} = \mathsf{FHE.}\widehat{\mathsf{ct}} = \mathsf{FHE.Eval}(\mathsf{FHE.pk}, G[x,r], \mathsf{FHE.ct})$, where $\mathsf{FHE.ct} \leftarrow \mathsf{FHE.Enc}(\mathsf{FHE.pk}, \mathsf{SFE.pk})$, we have

$$\mathsf{FHE.Dec}(\mathsf{FHE.sk}, \mathsf{FHE.}\widehat{\mathsf{ct}}) = G[x,r](\mathsf{SFE.pk}) = \mathsf{SFE.Enc}(\mathsf{SFE.pk}, x; r)$$

with probability 1 by the correctness of the underlying FHE scheme. So we have $y = \mathsf{SFE.Enc}(\mathsf{SFE.pk}, x; r)$ in Step 2 of the decryption algorithm. Next, by the correctness of the underlying SFE scheme, we have

$$\mathsf{SFE.Dec}(\mathsf{SFE.sk}_{f_i}, y) = \mathsf{SFE.Dec}(\mathsf{SFE.sk}_{f_i}, \mathsf{SFE.Enc}(\mathsf{SFE.pk}, x)) = f_i(x)$$

with all but negligible probability. Hence, the correctness follows. □

Security against semi-malicious authority. This follows from the security of the underlying maliciously circuit private FHE and the semi-malicious secure SFE scheme.

Theorem C.10. Assume that FHE is maliciously circuit-private (Definition 2.7) and SFE is (SIM/relaxed-SIM) secure against a semi malicious authority (Definition 4.5). Then the above construction of PCSFE scheme satisfies (SIM/relaxed-SIM) security against a semi malicious authority (Definition 4.5).

Proof. To prove the theorem, we first construct the simulator SIM for the security against the semi-malicious authority of the PCSFE scheme. Note that the simulator takes as input $(\mathsf{pk}, 1^{|x|}, \mathcal{V})$ where

$$\mathcal{V}^{\text{[11]}} = \left\{ f_i(x), f_i, \mathsf{sk}_{f_i} \right\}_{i\in[Q]}.$$

We now provide the description of the simulator SIM. On input $(\mathsf{pk}, 1^{|x|}, \mathcal{V})$, do the following.

1. Parse $\mathsf{pk} = (\mathsf{FHE.pk}, \mathsf{FHE.ct})$ and $\mathsf{sk}_{f_i} = (\mathsf{FHE.sk}, \mathsf{SFE.sk}_{f_i})$ for $i \in [Q]$.
2. Let $\mathcal{V}' = \left\{ f_i(x), f_i, \mathsf{SFE.sk}_{f_i} \right\}_{i\in[Q]}$.
3. Compute $\mathsf{SFE.pk} \leftarrow \mathsf{FHE.Dec}(\mathsf{FHE.sk}, \mathsf{FHE.ct})$.
4. Compute $\mathsf{SFE.ct} \leftarrow \mathsf{SFE.Sim}(\mathsf{SFE.pk}, 1^{|x|}, \mathcal{V}')$.
5. Compute $\mathsf{FHE.}\widehat{\mathsf{ct}} \leftarrow \mathsf{FHE.Sim}(\mathsf{FHE.pk}, \mathsf{FHE.ct}, \mathsf{SFE.ct})$.

[11] If the underlying SFE scheme satisfies SIM security against a semi-malicious authority, then we have $\mathcal{V} = \{f_i(x)\}_{i\in[Q]}$

6. Output $\mathsf{ct} = \mathsf{FHE}.\widehat{\mathsf{ct}}$.

To prove the security, we consider the following sequence of hybrids.

Hyb_0. This is the real world. We write the complete game here to set up notations and for easy reference in the later hybrids.

1. The adversary outputs the functions $f_1, \ldots, f_Q$ and the randomness $\mathsf{FHE}.r_1$, $\mathsf{FHE}.r_2$, and $\mathsf{SFE}.r$ corresponding to the randomness used in the $\mathsf{FHE.KeyGen}$, $\mathsf{FHE.Enc}$ and $\mathsf{SFE.Setup}$ algorithms respectively.
2. The challenger computes $(\mathsf{FHE.pk}, \mathsf{FHE.sk}) \leftarrow \mathsf{FHE.KeyGen}(1^\lambda; \mathsf{FHE}.r_1)$, $(\mathsf{SFE.pk}, \mathsf{SFE.sk}_{f_1}, \ldots, \mathsf{SFE.sk}_{f_Q}) \leftarrow \mathsf{SFE.Setup}(1^\lambda, f_1, \ldots, f_Q; \mathsf{SFE}.r)$, and $\mathsf{FHE.ct} \leftarrow \mathsf{FHE.Enc}(\mathsf{FHE.pk}, \mathsf{SFE.pk}; \mathsf{FHE}.r_2)$. It returns $\mathsf{pk} = (\mathsf{FHE.pk}, \mathsf{FHE.ct})$ and $\mathsf{sk}_{f_i} = (\mathsf{FHE.sk}, \mathsf{SFE.sk}_{f_i})$ for $i \in [Q]$ to the adversary.
3. The adversary outputs the challenge input x. The challenger computes $\mathsf{ct} = \mathsf{FHE}.\widehat{\mathsf{ct}}$ as in the construction and returns ct to the adversary.
4. In the end, the adversary outputs a guess bit β'.

Hyb_1. In this hybrid we change the way ciphertext is generated. In particular, we change the way $\mathsf{FHE}.\widehat{\mathsf{ct}}$ is computed. The challenger computes $\mathsf{FHE}.\widehat{\mathsf{ct}} \leftarrow \mathsf{FHE.Sim}(\mathsf{FHE.pk}, \mathsf{FHE.ct}, \mathsf{SFE.ct})$, where $\mathsf{SFE.ct} \leftarrow \mathsf{SFE.Enc}(\mathsf{pk}, x)$.

Hyb_2. In this hybrid we further change the way ciphertext is generated. In particular, we change the way $\mathsf{SFE.ct}$ is computed. The challenger computes $\mathsf{SFE.ct} \leftarrow \mathsf{SFE.Sim}(\mathsf{SFE.pk}, 1^{|x|}, \mathcal{V}')$, where $\mathcal{V}' = \left\{ f_i(x), f_i, \mathsf{SFE.sk}_{f_i} \right\}_{i \in [Q]}$. We also note that this is the ideal world with the simulator SIM defined above.

Indistinguishability of hybrids. We now show that the consecutive hybrids are indistinguishable.

Claim C.11. Assume that FHE satisfies malicious circuit privacy, then $\mathsf{Hyb}_0 \approx_s \mathsf{Hyb}_1$.

Proof. We show that if there exists an unbounded adversary $\mathcal{A}$ who can distinguish between Hyb_0 and Hyb_1 with non-negligible advantage ϵ, then there exists an unbounded adversary $\mathcal{B}$ against the malicious circuit privacy security of FHE scheme with the same advantage ϵ. The reduction is as follows.

1. $\mathcal{B}$ first runs $\mathcal{A}$. $\mathcal{A}$ outputs the functions $f_1, \ldots, f_Q$ and the randomness $\mathsf{FHE}.r_1$, $\mathsf{FHE}.r_2$, and $\mathsf{SFE}.r$.
2. $\mathcal{B}$ computes $(\mathsf{FHE.pk}, \mathsf{FHE.sk}) \leftarrow \mathsf{FHE.KeyGen}(1^\lambda; \mathsf{FHE}.r_1)$, $(\mathsf{SFE.pk}, \mathsf{SFE.sk}_{f_1}, \ldots, \mathsf{SFE.sk}_{f_Q}) \leftarrow \mathsf{SFE.Setup}(1^\lambda, f_1, \ldots, f_Q; \mathsf{SFE}.r)$, and $\mathsf{FHE.ct} \leftarrow \mathsf{FHE.Enc}(\mathsf{FHE.pk}, \mathsf{SFE.pk}; \mathsf{FHE}.r_2)$. It returns $\mathsf{pk} = (\mathsf{FHE.pk}, \mathsf{FHE.ct})$ and $\mathsf{sk}_{f_i} = (\mathsf{FHE.sk}, \mathsf{SFE.sk}_{f_i})$ for $i \in [Q]$ to $\mathcal{A}$.
3. $\mathcal{A}$ outputs the challenge input x. $\mathcal{B}$ defines the circuit $G[x, r]$ as in the construction and sends $\mathsf{FHE.pk}$, $\mathsf{FHE.ct}$, and $G[x, r]$ to the FHE challenger. The FHE challenger computes $\mathsf{SFE.pk} = \mathsf{FHE.Ext}(\mathsf{FHE.pk}, \mathsf{FHE.ct})$, samples $\beta \leftarrow \{0,1\}$ and returns $\mathsf{FHE}.\widehat{\mathsf{ct}}_\beta$ to $\mathcal{B}$, where $\mathsf{FHE}.\widehat{\mathsf{ct}}_0 \leftarrow \mathsf{FHE.Eval}(\mathsf{FHE.pk}, G[x, r], \mathsf{FHE.ct})$ and $\mathsf{FHE}.\widehat{\mathsf{ct}}_1 \leftarrow \mathsf{FHE.Sim}(\mathsf{FHE.pk}, \mathsf{FHE.ct}, \mathsf{SFE.Enc}(\mathsf{SFE.pk}, x))$ (since $G[x, r](\mathsf{SFE.pk}) = \mathsf{Enc}(\mathsf{SFE.pk}, x; r)$).
4. $\mathcal{B}$ sets and forwards $\mathsf{ct} = \mathsf{FHE}.\widehat{\mathsf{ct}}_\beta$ to $\mathcal{A}$.
5. In the end, $\mathcal{A}$ outputs a bit β'. $\mathcal{B}$ sends β' to the FHE challenger.

We observe that if the FHE challenger samples $\beta = 0$, then $\mathcal{B}$ simulated Hyb_0, else Hyb_1 with $\mathcal{A}$. Hence, advantage of $\mathcal{B} = |\Pr(\beta' = 1|\beta = 0) - \Pr(\beta' = 1|\beta = 1)| = |\Pr(\beta' = 1|\mathsf{Hyb}_0) - \Pr(\beta' = 1|\mathsf{Hyb}_1)| = \epsilon$ (by assumption). □

Claim C.12. Assume that SFE satisfies relaxed-SIM security against a semi malicious authority. Then $\mathsf{Hyb}_1 \approx_c \mathsf{Hyb}_2$.

Proof. We show that if there exists a PPT adversary $\mathcal{A}$ who can distinguish between Hyb_1 and Hyb_2 with non-negligible advantage ϵ, then there exists a PPT adversary $\mathcal{B}$ against the malicious circuit privacy security of SFE scheme with the same advantage ϵ. The reduction is as follows.

1. $\mathcal{B}$ first runs $\mathcal{A}$. $\mathcal{A}$ outputs the functions $f_1, \ldots, f_Q$ and the randomness $\mathsf{FHE}.r_1$, $\mathsf{FHE}.r_2$, and $\mathsf{SFE}.r$.
2. $\mathcal{B}$ sends the functions $f_1, \ldots, f_Q$ and the randomness $\mathsf{SFE}.r$ to the SFE challenger and gets back $(\mathsf{SFE.pk}, \mathsf{SFE.sk}_{f_1}, \ldots, \mathsf{SFE.sk}_{f_Q})$.
3. $\mathcal{B}$ generates $(\mathsf{FHE.pk}, \mathsf{FHE.sk}) \leftarrow \mathsf{FHE.KeyGen}(1^\lambda; \mathsf{FHE}.r_1)$, and computes $\mathsf{FHE.ct} \leftarrow \mathsf{FHE.Enc}(\mathsf{FHE.pk}, \mathsf{SFE.pk}; \mathsf{FHE}.r_2)$. It returns $\mathsf{pk} = (\mathsf{FHE.pk}, \mathsf{FHE.ct})$ and $\mathsf{sk}_{f_i} = (\mathsf{FHE.sk}, \mathsf{SFE.sk}_{f_i})$ for $i \in [Q]$ to $\mathcal{A}$.
4. $\mathcal{A}$ outputs the challenge input x. $\mathcal{B}$ forwards x to the SFE challenger as the challenge ciphertext. The SFE challenger samples a bit $\beta \leftarrow \{0,1\}$ and returns $\mathsf{SFE.ct}_\beta$ to $\mathcal{B}$ where $\mathsf{SFE.ct}_0 \leftarrow \mathsf{SFE.Enc}(\mathsf{SFE.pk}, x)$ and $\mathsf{SFE.ct}_1 \leftarrow \mathsf{SFE.Sim}(\mathsf{SFE.pk}, 1^{|x|}, \mathcal{V}')$ where $\mathcal{V}' = \left\{f_i(x), f_i, \mathsf{SFE.sk}_{f_i}\right\}_{i\in[Q]}$.
5. $\mathcal{B}$ computes $\mathsf{FHE}.\widehat{\mathsf{ct}} \leftarrow \mathsf{FHE.Sim}(\mathsf{FHE.pk}, \mathsf{FHE.ct}, \mathsf{SFE.ct}_\beta)$ and forwards $\mathsf{ct} = \mathsf{FHE}.\widehat{\mathsf{ct}}$ to $\mathcal{A}$.
6. In the end, $\mathcal{A}$ outputs a bit β'. $\mathcal{B}$ sends β' to the SFE challenger.

We observe that if the SFE challenger samples $\beta = 0$, then $\mathcal{B}$ simulated Hyb_1, else Hyb_2 with $\mathcal{A}$. Hence, advantage of $\mathcal{B} = |\Pr(\beta' = 1|\beta = 0) - \Pr(\beta' = 1|\beta = 1)| = |\Pr(\beta' = 1|\mathsf{Hyb}_1) - \Pr(\beta' = 1|\mathsf{Hyb}_2)| = \epsilon$ (by assumption). □

□

Security against the outsiders. This follows from Lemma 4.9.

C.5 Succinct Hash Encryption with Semi-Malicious security in ROM.

In this section, we provide a construction of hash encryption satisfying semi-malicious security. To begin, we observe that the construction of HE from LWE by Döttling et al. [DGHM18] already satisfies semi-*honest* security. To make it succinct and semi-malicious secure in the ROM, we make the following two modifications:

1. Succinctness: We use a pseudorandom generator PRG to generate key.
2. Semi-malicious security: Let H be a hash function modelled as the random oracle. Let R be the (potentially bad) randomness sampled by the adversary in the semi-malicious game. Then, the key of HE is set R, using which the parties compute $H(\mathsf{PRG}(R))$.

We also note that for our purpose we need a hash encryption scheme with hash domain $\{-1,0,1\}^m$ satisfying semi-malicious security. We use the HE scheme for binary hash domain from [DGHM18] as a black box to construct our hash encryption scheme with hash domain $\{-1,0,1\}^m$ satisfying semi-malicious security.

Building blocks. We use the following ingredients for our construction.

1. A pseudorandom generator $\mathsf{PRG} : \{0,1\}^\lambda \to \{0,1\}^*$.
2. A random oracle $H : \{0,1\}^* \to \mathbb{Z}_p^{2m\times\lambda}$.
3. A hash encryption scheme $\mathsf{HE} = (\mathsf{HE.Gen}, \mathsf{HE.Hash}, \mathsf{HE.Enc}, \mathsf{HE.Dec})$ satisfying semi-honest security for the hash domain $\{0,1\}^{2m}$ and message space $\mathcal{M}$. This can be instantiated from LWE assumption (Theorem 2.18).
4. A mapping $\phi : \mathcal{X} \to \{0,1\}^{2m}$, where, $\mathcal{X} \subseteq \{-1,0,1\}^m$ and $\phi(x_1 \ldots x_m) = y_1 \ldots y_{2m}$ where $y_{2i-1} = y_{2i} = 0$ if $x_i = 0$, $y_{2i-1} = y_{2i} = 1$ if $x_i = 1$, and $y_{2i-1} = 1, y_{2i} = 0$ if $x_i = -1$.

Construction. We describe our construction below.

$\mathsf{Gen}(1^\lambda, m) \to \mathsf{key}$. The setup algorithm does the following.

1. Sample $r \leftarrow \{0,1\}^\lambda$.
2. Compute $\mathbf{A} = H(\mathsf{PRG}(r))$ and set $\mathsf{HE.key} = \mathbf{A}$.
3. Output $\mathsf{key} = r$ and set the CRS as $\mathsf{HE.key}$.

$\mathsf{Hash}(\mathsf{key}, x \in \{-1,0,1\}^m) \to h$. The hash algorithm does the following.

1. Parse $\mathsf{key} = r$ and compute $\mathsf{HE.key} := H(\mathsf{PRG}(r))$.
2. Let $\phi(x) = y$ and compute $\mathsf{HE.hash} \leftarrow \mathsf{HE.Hash}(\mathsf{HE.key}, y)$.
3. Output $h = \mathsf{HE.hash}$.

$\mathsf{Enc}(\mathsf{key}, (h, i \in [m], c \in \{-1,0,1\}), \mu)$. The encryption algorithm does the following.

1. Parse $\mathsf{key} = r$, $h = \mathsf{HE.hash}$ and compute $\mathsf{HE.key} := H(\mathsf{PRG}(r))$.
2. Sample random $s_0 \leftarrow \mathcal{M}$ and set $s_1 = \mu \oplus s_0$.
3. Let $\phi(c) = c_0c_1$ and compute $\mathsf{HE.ct}_0 \leftarrow \mathsf{HE.Enc}(\mathsf{HE.key}, (\mathsf{HE.hash}, 2i-1, c_0), s_0)$ and $\mathsf{HE.ct}_1 \leftarrow \mathsf{HE.Enc}(\mathsf{HE.key}, (\mathsf{HE.hash}, 2i, c_1), s_1)$.
4. Output $\mathsf{ct} = (\mathsf{HE.ct}_0, \mathsf{HE.ct}_1)$.

$\mathsf{Dec}(\mathsf{key}, x, \mathsf{ct}) \to \{0,1\}$. The decryption algorithm does the following.

1. Parse $\mathsf{key} = r$ and compute $\mathsf{HE.key} := H(\mathsf{PRG}(r))$.
2. Let $\phi(x) = y$.
3. Parse $\mathsf{ct} = (\mathsf{HE.ct}_0, \mathsf{HE.ct}_1)$ and compute $s_0' \leftarrow \mathsf{HE.Dec}(\mathsf{HE.key}, y, \mathsf{HE.ct}_0)$ and $s_1' \leftarrow \mathsf{HE.Dec}(\mathsf{HE.key}, y, \mathsf{HE.ct}_1)$.
4. Output $\mu' = s_0 \oplus s_1$.

Collusion resistance. First we note that ϕ is a one to one mapping, i.e. if $x_1 \neq x_2 \implies \phi(x_1) \neq \phi(x_2)$. This along with the fact that $\mathsf{HE.Hash}$ is a collusion resistant hash, implies that the Hash algorithm in the above construction is a collusion resistant hash for hash domain $\{-1,0,1\}^m$.

Correctness. For any $\mathsf{Enc}(\mathsf{key}, (h,i,c), \mu) = \mathsf{ct} = (\mathsf{HE.ct}_0, \mathsf{HE.ct}_1)$, where $\mathsf{HE.ct}_0 \leftarrow \mathsf{HE.Enc}(\mathsf{HE.key}, (\mathsf{HE.hash}, 2i-1, c_0), s_0)$ and $\mathsf{HE.ct}_1 \leftarrow \mathsf{HE.Enc}(\mathsf{HE.key}, (\mathsf{HE.hash}, 2i, c_1), s_1)$ and for any x such that $\mathsf{Hash}(\mathsf{key}, x) = h = \mathsf{HE.Hash}(\mathsf{HE.key}, y)$, where $y = \phi(x)$, we observe that with all but negligible probability

1. if $y_{2i-1} = c_0$,

$$\mathsf{HE.Dec}(\mathsf{HE.key}, y, \mathsf{HE.ct}_0) = \mathsf{HE.Dec}(\mathsf{HE.key}, y, \mathsf{HE.Enc}(\mathsf{HE.key}, (\mathsf{HE.hash}, 2i-1, c_0), s_0)) = s_0$$

2. and if $y_{2i} = c_1$

$$\mathsf{HE.Dec}(\mathsf{HE.key}, y, \mathsf{HE.ct}_1) = \mathsf{HE.Dec}(\mathsf{HE.key}, y, \mathsf{HE.Enc}(\mathsf{HE.key}, (\mathsf{HE.hash}, 2i, c_1), s_1)) = s_1$$

by the correctness of the underlying HE scheme. Now, if $x_i = c$, it implies that $\phi(x_i) = \phi(c) \implies y_{2i-1}y_{2i} = c_0c_1$. So we recover s_0, s_1 in Step 3 of the decryption algorithm and $s_0 \oplus s_1 = \mu$ in the Step 4 by the correctness of secret sharing. Hence, the correctness follows.

Semi-malicious security. We show that the above construction of a hash encryption scheme is secure via following theorem.

Theorem C.13. Assume that HE is a secure hash encryption scheme for hash domain $\{0,1\}^{2m}$ in the semi-honest setting (Definition 2.15). Then the above construction is a secure hash encryption scheme in the semi-malicious setting for hash domain $\{-1,0,1\}^m$.

Proof. Recall that to prove the security we need to show that

$$\mathsf{Enc}(\mathsf{key},(h,i,c_i),\mu_0) \approx_c \mathsf{Enc}(\mathsf{key},(h,i,c_i),\mu_1)$$

where $h = \mathsf{Hash}(\mathsf{key},x)$ for some $x \in \{-1,0,1\}^m$ and $x_i \neq c_i$ for some $c_i \in \{-1,0,1\}$.
We consider the following sequence of hybrid games.

Hyb_0. This is the real world where the ciphertext is computed for message μ_0, We write the complete game to setup notations and for easy reference in the later hybrids.

1. The adversary outputs $x \in \{-1,0,1\}^m$ and a randomness r.
2. The challenger computes $y = \phi(x)$, generates $\mathbf{A} \leftarrow \mathsf{HE.Gen}(\mathsf{HE.key}, 1^{2m})$ and sets $\mathbf{A} := H(\mathsf{PRG}(r))$. It sets $\mathsf{key} = r$.
3. The adversary outputs an index $i \in [m]$, two messages μ_0, μ_1 and a $c_i \in \{-1,0,1\}$ such that $x_i \neq c_i$. The challenger does the following.
   (a) It computes $\phi(c_i) = d_{2i-1}d_{2i}$, where $d_{2i-1}, d_{2i} \in \{0,1\}$.
   (b) It samples $s_0^{(2i-1)}, s_1^{(2i-1)} \leftarrow \mathcal{M}$ and sets $s_0^{(2i)} = \mu_0 \oplus s_0^{(2i-1)}$, $s_1^{(2i)} = \mu_1 \oplus s_1^{(2i-1)}$ ($s_b^{(\mathsf{ind})}$ denotes the share of message μ_b at position ind).
   (c) It computes $\mathsf{HE.ct}_{(2i-1)} \leftarrow \mathsf{HE.Enc}(\mathsf{HE.key}, (\mathsf{HE.Hash}(\mathsf{HE.key}, y), 2i-1, d_{2i-1}), s_0^{(2i-1)})$ and $\mathsf{HE.ct}_{(2i)} \leftarrow \mathsf{HE.Enc}(\mathsf{HE.key}, (\mathsf{HE.Hash}(\mathsf{HE.key}, y), 2i, d_{2i}), s_0^{(2i)})$.
4. The challenger returns $\mathsf{ct} = (\mathsf{HE.ct}_{(2i-1)}, \mathsf{HE.ct}_{(2i)})$ to the adversary.
5. The adversary outputs a guess bit β.

Hyb_1. In this hybrid we change the way the ciphertext is generated. In particular for $j \in \{2i-1, 2i\}$ such that $y_j \neq d_j$ (since $x_i \neq c_i$, then by the definition of ϕ, it implies there exists such an index j), the challenger computes $\mathsf{HE.ct}_{(j)} \leftarrow \mathsf{HE.Enc}(\mathsf{HE.key}, (\mathsf{HE.Hash}(\mathsf{HE.key}, y), j, d_j, s_1^{(j)})$.

Hyb_2. This hybrid is same as the previous hybrid except that we set $s_0^{(j')} := s_1^{(j)} \oplus \mu_1$, where $j' \in \{2i-1, 2i\}$ such that $j' \neq j$.
We note that this hybrid is the real world where the ciphertext is computed for message μ_1.

Indistinguishability of hybrids. Now we show that the above consecutive hybrids are indistinguishable.

Claim C.14. Assume that HE is a secure hash encryption scheme in the semi-honest setting. Then $\mathsf{Hyb}_0 \approx_c \mathsf{Hyb}_1$.

Proof. We show that if there exists a PPT adversary $\mathcal{A}$ who can distinguish between the two hybrids with non-negligible advantage ϵ, then there exists a PPT adversary $\mathcal{B}$ against the semi-honest security of the underlying HE scheme with the same advantage ϵ. The reduction is as follows.

1. $\mathcal{B}$ first runs $\mathcal{A}$. The adversary outputs $x \in \{-1,0,1\}^m$ and a randomness r.
2. $\mathcal{B}$ computes $y = \phi(x)$ and sends y to the HE challenger. The challenger generates $\mathbf{A} \leftarrow \mathsf{HE.Gen}(\mathsf{HE.key}, 1^{2m})$ and returns the $\mathsf{HE.key} = \mathbf{A}$ to $\mathcal{B}$.
3. $\mathcal{B}$ sets the $H(\mathsf{PRG}(r)) = \mathbf{A}$ and returns $\mathbf{A}$ to the adversary $\mathcal{A}$.

4. $\mathcal{A}$ outputs an index $i \in [m]$, two messages μ_0, μ_1 and a $c_i \in \{-1, 0, 1\}$ such that $x_i \neq c_i$. $\mathcal{B}$ does the following.
   - It computes $\phi(c_i) = d_{2i-1}d_{2i}$, where $d_{2i-1}, d_{2i} \in \{0, 1\}$.
   - It samples $s_0^{(2i-1)}, s_1^{(2i-1)} \leftarrow \mathcal{M}$ and sets $s_0^{(2i)} = \mu_0 \oplus s_0^{(2i-1)}$, $s_1^{(2i)} = \mu_1 \oplus s_1^{(2i-1)}$.
   - For $j \in \{2i-1, 2i\}$ such that $y_j \neq d_j$, $\mathcal{B}$ sends $(j, (s_0^{(j)}, s_1^{(j)}), d_j)$ as the ciphertext challenge to the HE challenger. The HE challenger samples a bit $\beta \leftarrow \{0, 1\}$ and computes $\mathsf{HE.ct}_{(j)} \leftarrow \mathsf{HE.Enc}(\mathsf{HE.key}, (\mathsf{HE.Hash}(\mathsf{HE.key}, y), j, d_j), s_\beta^{(j)})$ and returns $\mathsf{HE.ct}_{(j)}$ to $\mathcal{B}$.
   - Let $j' \in \{2i-1, 2i\}$ s.t $j' \neq j$. $\mathcal{B}$ computes $\mathsf{HE.ct}_{(j')} \leftarrow \mathsf{HE.Enc}(\mathsf{HE.key}, (\mathsf{HE.Hash}(\mathsf{HE.key}, y), j', d_{j'}), s_0^{(j')})$.
   - $\mathcal{B}$ sets $\mathsf{ct} = (\mathsf{HE.ct}_{(j)}, \mathsf{HE.ct}_{(j')})$ if $j = 2i-1$, else it sets $\mathsf{ct} = (\mathsf{HE.ct}_{(j')}, \mathsf{HE.ct}_{(j)})$ and sends ct to $\mathcal{A}$.
5. In the end $\mathcal{A}$ outputs a bit β'. $\mathcal{B}$ forwards β' to the HE challenger.

We observe that if the HE challenger samples $\beta = 0$, then $\mathcal{B}$ simulated the real world where μ_0 is encrypted , else the real world where μ_1 is encrypted with $\mathcal{A}$. Hence, advantage of $\mathcal{B}$ = $|\Pr(\beta' = 1|\beta = 0) - \Pr(\beta' = 1|\beta = 1)|$ = $|\Pr(\beta' = 1|\mathsf{Hyb}_0) - \Pr(\beta' = 1|\mathsf{Hyb}_1)|$ = ϵ (by assumption). □

Claim C.15. Hyb_1 and Hyb_2 are statistically indistinguishable.

Proof. We note that $s_0^{(j')}$ is a uniformly random string, since each share of μ_0 is distributed uniformly at random. Also, since the other share of μ_0, which is $s_0^{(j)}$, does not appear in Hyb_1 and Hyb_2, we can set $s_0^{(j')} = s_1^{(j)} \oplus \mu_1$, which is again a uniformly random string due to the randomness of $s_1^{(j)}$ and the guarantee of secret sharing. We also note that now $\mathsf{HE.ct}_{(j')}$ encrypts the message $s_1^{(j)} \oplus \mu_1$. □

□

D Missing details from Section 5

In this section we show that pre-constrained input obfuscation (PCIO) is equivalent to pre-constrained static functional encryption (PCSFE).

D.1 From PCSFE to PCIO.

We construct a PCIO scheme, for circuit family $\mathcal{C} = \{C : \mathcal{X} \to \mathcal{Y}\}$, from a PCSFE = (PCSFE.Setup, PCSFE.Enc, PCSFE.Dec) scheme as defined in Section 4.

$\mathsf{Setup}(1^\lambda, \mathcal{S}) \to (\mathsf{pk}, \mathsf{ek})$**.** The setup algorithm does the following.

- Parse $\mathcal{S} = (x_1, \ldots, x_Q)$.
- Let $U[x]$ be a universal circuit, which on input a circuit C, outputs $C(x)$.
- Run $(\mathsf{fe.pk}, \mathsf{fe.sk}_1, \ldots, \mathsf{fe.sk}_Q) \leftarrow \mathsf{PCSFE.Setup}(1^\lambda, U[x_1], \ldots, U[x_Q])$.
- Output $\mathsf{pk} := \mathsf{fe.pk}$ and $\mathsf{ek} := (\mathsf{fe.sk}_1, \ldots, \mathsf{fe.sk}_Q)$. (We assume that $\mathsf{fe.sk}_i$ includes x_i.)

$\mathcal{O}(\mathsf{pk}, C) \to \widetilde{C}$**.** The obfuscating algorithm does the following.

- Parse $\mathsf{pk} = \mathsf{fe.pk}$.
- Run $\mathsf{fe.ct} \leftarrow \mathsf{PCSFE.Enc}(\mathsf{fe.pk}, C)$.
- Output $\widetilde{C} := \mathsf{fe.ct}$.

$\mathsf{Eval}(\mathsf{ek}, \widetilde{C}, x) \to y$**.** The evaluation algorithm does the following.

– Parse $\mathsf{ek} = (\mathsf{fe.sk}_1, \ldots, \mathsf{fe.sk}_Q)$ and $\widetilde{C} = \mathsf{fe.ct}$.
– Find $\mathsf{fe.sk}_i$ corresponding to x. If there is no such sk_i, output $\perp$.
– Otherwise, run and output $y \leftarrow \mathsf{PCSFE.Dec}(\mathsf{fe.sk}_i, \mathsf{fe.ct})$.

Correctness. We show the correctness of our PCIO scheme via the following theorem.

Theorem D.1. Suppose PCSFE satisfies correctness as defined in Definition 4.1. Then the above construction satisfies correctness as defined in Definition 5.1.

Proof. From the correctness of PCSFE, it holds that $y = U[x_i](C)$ where $y = \mathsf{PCSFE.Dec}(\mathsf{fe.sk}_i, \mathsf{fe.ct})$. By the definition of $U[x]$, it holds that $y = C(x_i)$. Next, if $x \in \mathcal{X} \setminus \mathcal{S}$, there is no sk_i corresponding to x in ek. In this case, the decryption algorithm outputs $\perp$. Thus, the theorem holds. □

Input-set-hiding. We show that the above construction satisfies the input-set-hiding property.

Theorem D.2. Suppose PCSFE satisfies the function-hiding property as defined in Definition 4.2. Then the above construction satisfies input-set-hiding as defined in Definition 5.2.

Proof. We construct an adversary $\mathcal{B}$ of the function-hiding game for PCSFE by using an adversary $\mathcal{A}$ of the input-set-hiding game for PCIO. $\mathcal{B}$ works as follows.

1. When $\mathcal{A}$ sends $\mathcal{S}_0 = (x_1^{(0)}, \ldots, x_Q^{(0)})$ and $\mathcal{S}_1 = (x_1^{(1)}, \ldots, x_Q^{(1)})$, $\mathcal{B}$ sets $f_i^{(b)} := U[x_i^{(b)}]$ for $b \in \{0,1\}$ and sends $:= (f_1^{(0)}, f_Q^{(1)}), \ldots, (f_Q^{(0)}, f_Q^{(1)})$ to its challenger.
2. $\mathcal{B}$ receives $\mathsf{fe.pk}$ from its challenger.
3. $\mathcal{B}$ sends $\mathsf{pk} := \mathsf{fe.pk}$ to $\mathcal{A}$.
4. $\mathcal{B}$ outputs whatever $\mathcal{A}$ outputs.

$\mathcal{B}$ perfectly simulates the input-set-hiding game for $\mathcal{A}$. Thus, if $\mathcal{A}$ breaks the input-set-hiding property, $\mathcal{B}$ also breaks the function-hiding game. This completes the proof. □

Security against malicious authority. We show that the above construction satisfies security against malicious authority.

Theorem D.3. Suppose PCSFE satisfies security SIM security (resp. indistinguishability) against malicious authority. Then the above construction satisfies VBB security (resp. indistinguishability) against malicious authority. In addition, if PCSFE has unconditional (SIM or IND) security, the above construction also has unconditional (VBB or IND) security.

We focus on the proof for the simulation-based security. The proof for the indistinguishability-based definition is similar to the simulation-based one, and we omit it.

Proof. We define $\mathrm{Ext}(1^\lambda, \mathsf{pk})$ as follows.

- Parse $\mathsf{pk} = \mathsf{fe.pk}$.
- Run $(f_1, \ldots, f_Q) \leftarrow \mathsf{PCSFE.Ext}(1^\lambda, \mathsf{fe.pk})$.
- Interpret f_i as a universal circuit $U[x_i']$ for all $i \in [Q]$.
- Output $(x_1', \ldots, x_Q')$.

We also define $\mathsf{Sim}(1^\lambda, 1^{|C|}, \mathsf{pk}, C(x_1'), \ldots, C(x_Q'))$ as follows.

- Parse $\mathsf{pk} = \mathsf{fe.pk}$.

- Run $\mathsf{fe.ct}_1 \leftarrow \mathsf{PCSFE.Sim}(\mathsf{fe.pk}, 1^{|C|}, C(x'_1), \ldots, C(x'_Q))$.
- Output $\widetilde{C} := \mathsf{fe.ct}_1$.

We construct an adversary $\mathcal{B}$ of the SIM security for PCSFE by using an adversary $\mathcal{A}$ of the VBB security for PCIO. $\mathcal{B}$ works as follows.

1. When $\mathcal{A}$ sends pk and C, $\mathcal{B}$ sends $\mathsf{fe.pk} := \mathsf{pk}$ and $x := C$ to its challenger.
2. $\mathcal{B}$ receives a challenge ciphertext fe.ct from its challenger.
3. $\mathcal{B}$ sends $\widetilde{C} := \mathsf{fe.ct}$ to $\mathcal{A}$
4. $\mathcal{B}$ outputs whatever $\mathcal{A}$ outputs.

If $\beta = 0$, $\mathcal{B}$ receives $\mathsf{fe.ct}_0 \leftarrow \mathsf{PCSFE.Enc}(\mathsf{fe.pk}, C)$. If $\beta = 1$, $\mathcal{B}$ receives $\mathsf{fe.ct}_1 \leftarrow \mathsf{PCSFE.Sim}(\mathsf{fe.pk}, 1^{|C|}, U[x'_1](C), \ldots, U[x'_Q](C))$ where $(U[x'_1], \ldots U[x'_Q]) \leftarrow \mathsf{PCSFE.Ext}(1^\lambda, \mathsf{FE.pk})$. So, $\mathcal{B}$ perfectly simulates the VBB security game for $\mathcal{A}$. Thus, if $\mathcal{A}$ breaks the VBB security, $\mathcal{B}$ also breaks the SIM security. This completes the proof. □

D.2 From PCIO to PCSFE.

It is easy to see that we can construct PCSFE from PCIO. Let $\mathsf{PCIO} := \mathsf{PCIO}.(\mathsf{Setup}, \mathcal{O}, \mathsf{Eval})$ be a PCIO scheme.

$\mathsf{Setup}(1^\lambda, f_1, \ldots, f_Q) \to (\mathsf{pk}, \mathsf{sk}_{f_1}, \ldots, \mathsf{sk}_{f_Q})$**.** The setup algorithm does the following.

- Set $\mathcal{S} := (f_1, \ldots, f_Q)$
- Run $(\mathsf{pcio.pk}, \mathsf{pcio.ek}) \leftarrow \mathsf{PCIO.Setup}(1^\lambda, \mathcal{S})$.
- Output $\mathsf{pk} := \mathsf{pcio.pk}$ and $\mathsf{sk}_{f_i} := (\mathsf{pcio.ek}, f_i)$ for all $i \in [Q]$.

$\mathsf{Enc}(\mathsf{pk}, x) \to \mathsf{ct}$**.** The encryption algorithm does the following.

- Parse $\mathsf{pk} = \mathsf{pcio.pk}$.
- Run $\widetilde{U} \leftarrow \mathsf{PCIO}.\mathcal{O}(\mathsf{pcio.pk}, U[x])$ where $U[x]$ is a universal circuit that takes as input f, and outputs $f(x)$.
- Output $\mathsf{ct} := \widetilde{U}$.

$\mathsf{Dec}(\mathsf{sk}_{f_i}, \mathsf{ct}) \to y$**.** The decryption algorithm does the following.

- Parse $\mathsf{sk}_{f_i} = (\mathsf{pcio.ek}, f_i)$ and $\mathsf{ct} = \widetilde{U}$.
- Compute and output $\mathsf{PCIO.Eval}(\mathsf{pcio.ek}, \widetilde{U}, f_i)$.

Correctness. Correctness of the above scheme follows from the correctness of the PCSFE scheme.

Theorem D.4. Suppose PCIO satisfies correctness as defined in Definition 5.1. Then the above construction satisfies correctness as defined in Definition 4.1.

Input-set-hiding. The input-set-hiding property of the above scheme follows from the function-hiding of the PCSFE scheme.

Theorem D.5. Suppose PCIO satisfies input-set-hiding as defined in Definition 5.2. Then the above construction satisfies the function-hiding property as defined in Definition 4.2.

Security against malicious authority. This follows from the security against malicious authority of the PCSFE scheme.

Theorem D.6. Suppose PCIO satisfies VBB security (resp. indistinguishability) against malicious authority. Then the above construction satisfies security SIM security (resp. indistinguishability) against malicious authority. In addition, if PCIO has unconditional (VBB or IND) security, the above construction also has unconditional (SIM or IND) security.

The proofs of Theorems D.4 to D.6 are almost the same as those of Theorems D.1 to D.3, so we omit them.

E Missing Details from Section 6

In this section, we provide our definition and construction for pre-constrained group signatures.

E.1 Definition

A pre-constrained group signature (PCGS) scheme for a circuit family $\mathcal{C} = \{C : \{0,1\}^n \to \{0,1\}\}$ for $n = n(\lambda)$, a message space $\mathcal{M} = \{0,1\}^n$, an identity space $\mathcal{ID}$ consists of five algorithms $(\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Sign}, \mathsf{Verify}, \mathsf{Open})$ with the following syntax.

$\mathsf{Setup}(1^\lambda, C) \to (\mathsf{mpk}, \mathsf{msk})$. The setup algorithm, run by the group manager GM, takes as input the security parameter λ and a circuit $C \in \mathcal{C}$, and outputs a master public key mpk and a master secret key msk.

$\mathsf{KeyGen}\langle \mathsf{GM}(\mathsf{msk}), U\rangle \to (\mathsf{id}, \mathsf{sk}_{\mathsf{id}})$. This is an interactive protocol between the group manager GM with msk and the user U. It delivers an identity $\mathsf{id} \in \mathcal{ID}$ to both GM and U and a user secret signing key $\mathsf{sk}_{\mathsf{id}}$ to U.

$\mathsf{Sign}(\mathsf{mpk}, \mathsf{sk}_{\mathsf{id}}, m) \to \sigma$. The signing algorithm takes as input the master public key mpk, the user signing key $\mathsf{sk}_{\mathsf{id}}$ and a message m, and outputs a signature σ.

$\mathsf{Verify}(\mathsf{mpk}, m, \sigma) \to \{0,1\}$. The verification algorithm takes as input the master public key mpk, a message m and a signature σ, and outputs a bit indicating accept or reject.

$\mathsf{Open}(\mathsf{msk}, \sigma) \to \{\mathsf{id}, \perp\}$. The opening algorithm on input the master secret key msk and a signature σ outputs either an identity $\mathsf{id} \in \mathcal{ID}$ or $\perp$.

Definition E.1 (Correctness). A PCGS scheme is said to be correct if for any $C \in \mathcal{C}$, $m \in \mathcal{M}$, the following holds

$$\Pr[\mathsf{Verify}(\mathsf{mpk}, m, \mathsf{Sign}(\mathsf{mpk}, \mathsf{sk}_{\mathsf{id}}, m)) = 1] \geq 1 - \mathsf{negl}(\lambda)$$

where $(\mathsf{mpk}, \mathsf{msk}) \leftarrow \mathsf{Setup}(1^\lambda, C)$ and $(\mathsf{id}, \mathsf{sk}_{\mathsf{id}}) \leftarrow \mathsf{KeyGen}\langle \mathsf{GM}(\mathsf{msk}), U\rangle$.

Definition E.2 (Constraint-Hiding). A PCGS scheme is said to satisfy constraint-hiding security if for any PPT adversary $\mathcal{A}$, the following holds

$$\Pr\left[\beta' = \beta : \begin{array}{l} C_0, C_1 \leftarrow \mathcal{A}; \\ \beta \leftarrow \{0,1\}; (\mathsf{mpk}_\beta, \mathsf{msk}) \leftarrow \mathsf{Setup}(1^\lambda, C_\beta); \\ \beta' \leftarrow \mathcal{A}(\mathsf{mpk}_\beta) \end{array}\right] \leq \frac{1}{2} + \mathsf{negl}(\lambda)$$

where $\mathcal{A}$ is admissible if $|C_0| = |C_1|$ and $C_0, C_1 \in \mathcal{C}$.

Definition E.3 (Traceability). For a PCGS scheme and an adversary $\mathcal{A}$, let us define the traceability experiment as follows.

1. $\mathcal{A}$ outputs a challenge circuit $\mathcal{C}$.
2. The challenger generates $(\mathsf{mpk}, \mathsf{msk}) \leftarrow \mathsf{Setup}(1^\lambda, C)$. It sends mpk to $\mathcal{A}$.

3. $\mathcal{A}$ can make the following queries to the challenger.

   (a) $\mathsf{H\text{-}KeyGen}$. $\mathcal{A}$ can request the joining of an user U to the group. The challenger executes the KeyGen protocol where it acts both as the group manager and the user. The challenger receives an $\mathsf{id} \in \mathcal{ID}$ and the respective signing key $\mathsf{sk}_{\mathsf{id}}$. It forwards id to $\mathcal{A}$ and maintains a set of these identities H_{id}. It also maintains a list H_{key} of the identity and the respective signing key pair $(\mathsf{id}, \mathsf{sk}_{\mathsf{id}})$ for the queried identities.

   (b) $\mathsf{C\text{-}KeyGen}$. $\mathcal{A}$ can request to join the group as an user U. Then the challenger and the adversary executes the KeyGen protocol where the challenger acts as the group manager and $\mathcal{A}$ as the user U. The challenger and $\mathcal{A}$ receives an $\mathsf{id} \in \mathcal{ID}$ and $\mathcal{A}$ additionally receives the signing key $\mathsf{sk}_{\mathsf{id}}$. The challenger maintains a set of these identities C_{id}.

   (c) Sign. $\mathcal{A}$ can request a signature on message m. It sends (m, id) to the challenger, where the id is the user identity corresponding to some user in the group. If $\mathsf{id} \notin H_{\mathsf{id}}$, the challenger outputs $\perp$ else it returns $\sigma \leftarrow \mathsf{Sign}(\mathsf{mpk}, \mathsf{sk}_{\mathsf{id}}, m)$ to $\mathcal{A}$. The challenger maintains a list S_{id} of (m, id) for which it returns the signature to $\mathcal{A}$.

   (d) Open. $\mathcal{A}$ can request to open a signature σ. The challenger runs $\mathsf{Open}(\mathsf{msk}, \sigma)$ and forwards the output to $\mathcal{A}$.

4. $\mathcal{A}$ outputs a message m^* and a signature σ^*.

$\mathcal{A}$ wins if the following conditions hold

1. $\mathsf{Verify}(\mathsf{mpk}, m, \sigma) = 1$ and $C(m) = 1$.
2. $\mathsf{id} \leftarrow \mathsf{Open}(\mathsf{msk}, \sigma) \wedge ((m, \mathsf{id}) \notin S_{\mathsf{id}}) \wedge (\mathsf{id} \notin C_{\mathsf{id}})$.

We say that a PCGS scheme is traceable if for any PPT adversary $\mathcal{A}$, the probability that $\mathcal{A}$ wins the traceability experiment is $\mathsf{negl}(\lambda)$.

Definition E.4 (Unframeability). For a PCGS scheme and an adversary $\mathcal{A}$, let us define the unframeability experiment as follows.

1. $\mathcal{A}$ outputs a circuit $C \in \mathcal{C}$.
2. On input $1^\lambda, C$, the challenger generates $(\mathsf{mpk}, \mathsf{msk}) \leftarrow \mathsf{Setup}(1^\lambda, C)$. It sends $(\mathsf{mpk}, \mathsf{msk})$ to $\mathcal{A}$.
3. $\mathcal{A}$ can make the $\mathsf{H\text{-}KeyGen}$ and Sign queries to the challenger as defined in Definition E.3. The challenger maintains the list H_{id} and S_{id} for the respective queries.
4. $\mathcal{A}$ outputs a message m^* and a signature σ^*.

$\mathcal{A}$ wins if the following conditions hold

1. $\mathsf{Verify}(\mathsf{mpk}, m, \sigma) = 1$.
2. $\mathsf{id} \leftarrow \mathsf{Open}(\mathsf{msk}, \sigma) \wedge ((m, \mathsf{id}) \notin S_{\mathsf{id}}) \wedge (\mathsf{id} \in H_{\mathsf{id}})$.

We say that a PCGS scheme satisfies unframeability if for any PPT adversary $\mathcal{A}$, the probability that $\mathcal{A}$ wins the unframeability experiment is $\mathsf{negl}(\lambda)$.

Definition E.5 (Client-Authority Anonymity against Malicious Authority). For a PCGS scheme and an adversary $\mathcal{A}$, let us define the experiment for anonymity against malicious authority $\mathsf{Expt}^{\mathsf{CAA}}_{\beta,\mathcal{A}}(1^\lambda)$ as follows.

1. $\mathcal{A}$ outputs the challenge master public key mpk^*, a message m^* and two identities $(\mathsf{id}^*_0, \mathsf{id}^*_1)$ and the respective signing keys $(\mathsf{sk}^*_{\mathsf{id}_0}, \mathsf{sk}^*_{\mathsf{id}_1})$.
2. The challenger samples a bit $\beta \leftarrow \{0,1\}$ and computes $\sigma_\beta \leftarrow \mathsf{Sign}(\mathsf{mpk}^*, \mathsf{sk}^*_{\mathsf{id}_\beta}, m^*)$ and returns σ_β to $\mathcal{A}$.

3. $\mathcal{A}$ outputs a bit β^* as the output of the experiment.

We say that an adversary $\mathcal{A}$ is *admissible* if (i) $C \in \mathcal{C}$ where $C \leftarrow \mathrm{Ext}(1^\lambda, \mathsf{mpk}^*)$, and (ii) $C(m^*) = 0$.
We define the advantage of $\mathcal{A}$ in the above experiment as

$$\mathsf{Adv}^{\mathsf{CAA}}_{\mathcal{A}}(\lambda) = \left|\Pr\left[\mathsf{Expt}^{\mathsf{CAA}}_{0,\mathcal{A}}(1^\lambda) = 1\right] - \Pr\left[\mathsf{Expt}^{\mathsf{CAA}}_{1,\mathcal{A}}(1^\lambda) = 1\right]\right|.$$

We say that a PCGS scheme satisfies anonymity against malicious authority if there exists an (possibly inefficient) extractor Ext such that for any admissible PPT adversary $\mathcal{A}$, $\mathsf{Adv}^{\mathsf{CAA}}_{\mathcal{A}}(\lambda) \leq \mathsf{negl}(\lambda)$.

Definition E.6 (Unconditional Client-Authority Anonymity against Malicious Authority). We say that a PCGS scheme satisfies unbounded anonymity against a malicious authority if for *any* (unbounded) admissible adversary $\mathcal{A}$, $\mathsf{Adv}^{\mathsf{CAA}}_{\mathcal{A}}(\lambda)$ (as defined in Definition E.5) is negligible in the security parameter.

Definition E.7 (Client-Authority Unlinkability against Malicious Authority). For a PCGS scheme and an adversary $\mathcal{A}$, let us define the experiment for unlinkability against malicious authority $\mathsf{Expt}^{\mathsf{CAU}}_{\beta,\mathcal{A}}(1^\lambda)$ as follows.

1. $\mathcal{A}$ outputs the challenge master public key mpk^*, two messages (m^*_0, m^*_1) and two identities $(\mathsf{id}^*_0, \mathsf{id}^*_1)$ and the respective signing keys $(\mathsf{sk}^*_{\mathsf{id}_0}, \mathsf{sk}^*_{\mathsf{id}_1})$.
2. The challenger computes $\sigma_0 \leftarrow \mathsf{Sign}(\mathsf{mpk}, \mathsf{sk}^*_{\mathsf{id}_0}, m^*_0)$, samples a bit $\beta \leftarrow \{0,1\}$ and computes $\sigma_1 \leftarrow \mathsf{Sign}(\mathsf{mpk}^*, \mathsf{sk}^*_{\mathsf{id}_\beta}, m^*_1)$ and returns (σ_0, σ_1) to $\mathcal{A}$.
3. $\mathcal{A}$ outputs a bit β^* as the output of the experiment.

We say that an adversary $\mathcal{A}$ is *admissible* if (i) $C \in \mathcal{C}$ where $C \leftarrow \mathrm{Ext}(1^\lambda, \mathsf{mpk}^*)$, and (ii) $C(m^*_0) = C(m^*_1) = 0$.
We define the advantage of $\mathcal{A}$ in the above experiment as

$$\mathsf{Adv}^{\mathsf{CAU}}_{\mathcal{A}}(\lambda) = \left|\Pr\left[\mathsf{Expt}^{\mathsf{CAU}}_{0,\mathcal{A}}(1^\lambda) = 1\right] - \Pr\left[\mathsf{Expt}^{\mathsf{CAU}}_{1,\mathcal{A}}(1^\lambda) = 1\right]\right|.$$

We say that a PCGS scheme satisfies unlinkability against malicious authority if there exists an (possibly inefficient) extractor Ext such that for any admissible PPT adversary $\mathcal{A}$, $\mathsf{Adv}^{\mathsf{CAU}}_{\mathcal{A}}(\lambda) \leq \mathsf{negl}(\lambda)$.

Definition E.8 (Unconditional Client-Authority Unlinkability against Malicious Authority). We say that a PCGS scheme satisfies unbounded unlinkability against a malicious authority if for *any* (unbounded) admissible adversary $\mathcal{A}$, $\mathsf{Adv}^{\mathsf{CAU}}_{\mathcal{A}}(\lambda)$ (as defined in Definition E.7) is negligible in the security parameter.

Definition E.9 (Client-Client Anonymity). For a PCGS scheme and an adversary $\mathcal{A}$, let us define the experiment for client-client anonymity $\mathsf{Expt}^{\mathsf{CCA}}_{\beta,\mathcal{A}}(1^\lambda)$ as follows.

1. $\mathcal{A}$ outputs a circuit $C \in \mathcal{C}$.
2. The challenger generates $(\mathsf{mpk}, \mathsf{msk}) \leftarrow \mathsf{Setup}(1^\lambda, C)$. It sends the master public key mpk to $\mathcal{A}$.
3. $\mathcal{A}$ can make the H-KeyGen, C-KeyGen and Sign queries to the challenger as defined in Definition E.3. The challenger maintains the list H_{id}, H_{key}, C_{id} and S_{id} for the respective queries.
4. $\mathcal{A}$ outputs a message m^* and two user identities $(\mathsf{id}_0, \mathsf{id}_1)$. If $\mathsf{id}_0 \vee \mathsf{id}_1 \in C_{\mathsf{id}}$ then abort, else the challenger samples $\beta \leftarrow \{0,1\}$ and computes $\sigma_\beta \leftarrow \mathsf{Sign}(\mathsf{mpk}, \mathsf{sk}_{\mathsf{id}_\beta}, m^*)$, where $\mathsf{sk}_{\mathsf{id}_\beta}$ is the signing key corresponding to id_β from the list H_{key}, and returns this to the adversary.
5. $\mathcal{A}$ outputs a bit β' as the output of the experiment.

We define the advantage of $\mathcal{A}$ in the above experiment as

$$\mathsf{Adv}^{\mathsf{CCA}}_{\mathcal{A}}(\lambda) = \left|\Pr\left[\mathsf{Expt}^{\mathsf{CCA}}_{0,\mathcal{A}}(1^\lambda) = 1\right] - \Pr\left[\mathsf{Expt}^{\mathsf{CCA}}_{1,\mathcal{A}}(1^\lambda) = 1\right]\right|.$$

We say that a PCGS scheme satisfies client-client anonymity if for any PPT adversary $\mathcal{A}$, $\mathsf{Adv}^{\mathsf{CCA}}_{\mathcal{A}}(\lambda) \leq \mathsf{negl}(\lambda)$.

Definition E.10 (Client-Client Unlinkability). For a PCGS scheme and an adversary $\mathcal{A}$, let us define the experiment for client-client unlinkability $\mathsf{Expt}^{\mathsf{CCU}}_{\beta,\mathcal{A}}(1^\lambda)$ as follows.

1. $\mathcal{A}$ outputs a circuit $C \in \mathcal{C}$.
2. The challenger generates $(\mathsf{mpk}, \mathsf{msk}) \leftarrow \mathsf{Setup}(1^\lambda, C)$. It sends the master public key mpk to $\mathcal{A}$.
3. $\mathcal{A}$ can make the H-KeyGen, C-KeyGen and Sign queries to the challenger as defined in Definition E.3. The challenger maintains the list H_{id}, H_{key}, C_{id} and S_{id} for the respective queries.
4. $\mathcal{A}$ outputs messages m_0, m_1 and two user identities $\mathsf{id}_0, \mathsf{id}_1$. If $\mathsf{id}_0 \in C_{\mathsf{id}}$ or $\mathsf{id}_1 \in C_{\mathsf{id}}$ then abort, else the challenger computes $\sigma_0 \leftarrow \mathsf{Sign}(\mathsf{mpk}, \mathsf{sk}_{\mathsf{id}_0}, m_0)$, samples $\beta \leftarrow \{0,1\}$ and computes $\sigma_1 \leftarrow \mathsf{Sign}(\mathsf{mpk}, \mathsf{sk}_{\mathsf{id}_\beta}, m_1)$, where $\mathsf{sk}_{\mathsf{id}_\beta}$ is the signing key corresponding to id_β from the list H_{key}. It returns (σ_0, σ_1) to $\mathcal{A}$.
5. $\mathcal{A}$ outputs a bit β' as the output of the experiment.

We define the advantage of $\mathcal{A}$ in the above experiment as

$$\mathsf{Adv}^{\mathsf{CCU}}_{\mathcal{A}}(\lambda) = \left|\Pr\left[\mathsf{Expt}^{\mathsf{CCU}}_{0,\mathcal{A}}(1^\lambda) = 1\right] - \Pr\left[\mathsf{Expt}^{\mathsf{CCU}}_{1,\mathcal{A}}(1^\lambda) = 1\right]\right|.$$

We say that a PCGS scheme satisfies client-client unlinkability if for any PPT adversary $\mathcal{A}$, $\mathsf{Adv}^{\mathsf{CCU}}_{\mathcal{A}}(\lambda) \leq \mathsf{negl}(\lambda)$.

E.2 Construction

Our construction of PCGS follows from the compiler provided by [BGJP23] and is secure in ROM. As we will see below, we can use our LWE based PCE scheme achieving unconditional security together with dual-mode NIZK to obtain unconditional anonymity against malicious authority.

Building Blocks We use the following ingredients for our construction.

1. A special PCE encryption scheme $\mathsf{PCE} = (\mathsf{PCE.Setup}, \mathsf{PCE.Enc}, \mathsf{PCE.Dec})$ as defined in Definition 3.10. This can be instantiated from a variety of assumptions as described in Sections 3.2 and 3.3 and Appendix B.2.
2. A dual-mode NIZK proof system $\mathsf{ZK} = (\mathsf{ZK.Hsetup}, \mathsf{ZK.Bsetup}, \mathsf{ZK.Prove}, \mathsf{ZK.Verify})$ satisfying statistical zero-knowledge in. This can be instantiated from LWE (Corollary A.11).
3. A one-way relation $\mathcal{R} = (\mathcal{R}.\mathsf{Gen}, \mathcal{R}.\mathsf{Sample})$ for the relation $\mathcal{R}$.
4. A digital signature scheme $\mathcal{S} = (\mathcal{S}.\mathsf{Setup}, \mathcal{S}.\mathsf{Sign}, \mathcal{S}.\mathsf{Verify})$.
5. A random oracle H.

Construction. We now describe the construction of PCGS scheme, adapted from [BGJP23].

$\mathsf{Setup}(1^\lambda, C) \to (\mathsf{mpk}, \mathsf{msk})$. The setup algorithm does the following.

- Generate $\mathcal{R}.\mathsf{pp} \leftarrow \mathcal{R}.\mathsf{Gen}(1^\lambda)$, $(\mathsf{PCE.pk}, \mathsf{PCE.sk}) \leftarrow \mathsf{PCE.Gen}(1^\lambda, C)$ and $(\mathcal{S}.\mathsf{vk}, \mathcal{S}.\mathsf{sk}) \leftarrow \mathcal{S}.\mathsf{Gen}(1^\lambda)$.
- Output $\mathsf{mpk} = (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk})$ and $\mathsf{msk} = (\mathsf{PCE.sk}, \mathcal{S}.\mathsf{sk})$.

$\mathsf{KeyGen}\langle \mathsf{GM}(\mathsf{msk}), U\rangle \to (\mathsf{id}, \mathsf{sk}_{\mathsf{id}})$. The key generation protocol is as follows.

- The user U samples $s \leftarrow \{0,1\}^r$ and computes an instance-witness pair $(\mathsf{id}, w) = \mathcal{R}.\mathsf{Sample}(\mathcal{R}.\mathsf{pp}; s)$. It sends id to the group manager GM.
- The group manager parses $\mathsf{msk} = (\mathsf{PCE.sk}, \mathcal{S}.\mathsf{sk})$ and computes $\sigma_{\mathsf{id}} \leftarrow \mathcal{S}.\mathsf{Sign}(\mathcal{S}.\mathsf{sk}, \mathsf{id})$ and returns it to the user.

– The user sets $\mathsf{sk}_{\mathsf{id}} = (s, \sigma_{\mathsf{id}})$.

$\mathsf{Sign}(\mathsf{mpk}, \mathsf{sk}_{\mathsf{id}}, m) \to \sigma$. The signing algorithm does the following.

– Parse $\mathsf{mpk} = (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk})$ and $\mathsf{sk}_{\mathsf{id}} = (s, \sigma_{\mathsf{id}})$.
– Compute $(\mathsf{id}, w) = \mathcal{R}.\mathsf{Sample}(\mathcal{R}.\mathsf{pp}; s)$.
– Sample $r \leftarrow \{0,1\}^\lambda$ and compute $\mathsf{ct} = \mathsf{PCE.Enc}(\mathsf{PCE.pk}, m, \mathsf{id}; r)$.
– Let $\mathsf{crs} \leftarrow H(m, \mathsf{ct})$, and compute $\pi \leftarrow \mathsf{ZK.Prove}(\mathsf{crs}, (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk}, m, \mathsf{ct}), (\mathsf{id}, s, w, \sigma_{\mathsf{id}}, r))$ for the relation that checks that:
  1. $\mathsf{ct} = \mathsf{PCE.Enc}(\mathsf{PCE.pk}, m, \mathsf{id}; r)$.
  2. $(\mathsf{id}, w) = \mathcal{R}.\mathsf{Sample}(\mathcal{R}.\mathsf{pp}; s)$.
  3. $\mathcal{S}.\mathsf{Verify}(\mathcal{S}.\mathsf{vk}, \mathsf{id}, \sigma_{\mathsf{id}})$
– Output $\sigma = (\mathsf{ct}, \mathsf{crs}, \pi)$.

$\mathsf{Verify}(\mathsf{mpk}, m, \sigma) \to \{0,1\}$. The verification algorithm does the following.

– Parse $\mathsf{mpk} = (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk})$ and $\sigma = (\mathsf{ct}, \mathsf{crs}, \pi)$.
– Check $H(m, \mathsf{ct}) = \mathsf{crs}$. If true, output $\mathsf{ZK.Verify}(\mathsf{crs}, (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk}, m, \mathsf{ct}), \pi)$.

$\mathsf{Open}(\mathsf{msk}, \sigma) \to \{\mathsf{id}, \perp\}$. The open algorithm does the following.

– Parse $\mathsf{msk} = (\mathsf{PCE.sk}, \mathcal{S}.\mathsf{sk})$ and $\sigma = (\mathsf{ct}, \mathsf{crs}, \pi)$.
– Output $\mathsf{PCE.Dec}(\mathsf{PCE.sk}, \mathsf{ct})$.

Correctness. We now show that the above construction is correct via the following theorem.

Theorem E.11. Suppose that $\mathcal{S}$ is a correct digital signature scheme and ZK satisfies completeness (Definition A.1). Then the above construction of PCGS satisfies correctness (Definition E.1).

Proof. We observe that for $\sigma = (\mathsf{ct}, \mathsf{crs}, \pi)$, we have $\mathsf{ct} = \mathsf{PCE.Enc}(\mathsf{PCE.pk}, m, \mathsf{id}; r)$, $\mathsf{crs} = H(m, \mathsf{ct})$ and $\pi \leftarrow \mathsf{ZK.Prove}(\mathsf{crs}, (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk}, m, \mathsf{ct}), (\mathsf{id}, s, w, \sigma_{\mathsf{id}}, r))$, where $(\mathsf{id}, w) = \mathcal{R}.\mathsf{Sample}(\mathcal{R}.\mathsf{pp}; s)$ and $\sigma_{\mathsf{id}} \leftarrow \mathcal{S}.\mathsf{Sign}(\mathcal{S}.\mathsf{sk}, \mathsf{id})$.
By the correctness of the signature scheme, we have with all but negligible probability, $\mathcal{S}.\mathsf{Verify}(\mathcal{S}.\mathsf{vk}, \mathsf{id}, \sigma_{\mathsf{id}}) = 1$. Hence, the completeness of ZK scheme implies $\mathsf{ZK.Verify}(\mathsf{crs}, (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk}, m, \mathsf{ct}), \pi) = 1$ with all but negligible probability.
So, $\mathsf{Verify}(\mathsf{mpk}, m, \sigma) = (\mathsf{crs} = H(m, \mathsf{ct})) \wedge (\mathsf{ZK.Verify}(\mathsf{crs}, (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk}, m, \mathsf{ct}), \pi)) = 1$ with all but negligible probability . Hence the above construction satisfies correctness. □

Instantiation Below we describe the properties inherited by the resulting PCGS scheme for each instantiation of PCE scheme that we construct.

1. If we use PCE as in Section 3.2, we achieve anonymity and unlinkability against a malicious authority for general circuits. For circuits in NC^1, we can achieve unconditional anonymity and unlinkability against a malicious authority.

2. If we PCE as in Section 3.3, we can achieve unconditional anonymity and unlinkability against a malicious authority for general circuits.

3. If we use PCE for database as in Appendix B.2, we can achieve unconditional anonymity against a malicious authority, but not unlinkability.

Next, we prove the security properties of the above PCGS scheme.

Constraint-Hiding. This follows immediately from the constraint-hiding property of the underlying PCE scheme. We prove it formally using the following theorem.

Theorem E.12. Suppose that PCE scheme satisfies constraint-hiding (Definition 3.2). Then the above construction of PCGS satisfies constraint-hiding (Definition E.2).

Proof. Recall that to show constraint-hiding, we want

$$\{\mathsf{mpk} \mid \mathsf{mpk} \leftarrow \mathsf{Setup}(1^\lambda, C_0)\} \approx_c \{\mathsf{mpk} \mid \mathsf{mpk} \leftarrow \mathsf{Setup}(1^\lambda, C_1)\}$$

for any circuits $C_0, C_1 \in \mathcal{C}$. We show that if there exists a PPT adversary $\mathcal{A}$ who can distinguish between the above two distributions with non-negligible advantage ϵ, then there exists a PPT adversary $\mathcal{B}$ against the constraint-hiding of the PCE scheme with the same advantage ϵ. The reduction is as follows.

1. $\mathcal{B}$ first runs $\mathcal{A}$. $\mathcal{A}$ outputs the challenge circuits C_0, C_1 such that $C_0, C_1 \in \mathcal{C}$.
2. $\mathcal{B}$ sends C_0 and C_1 to the PCE challenger as challenge circuits. The PCE challenger samples $\beta \leftarrow \{0,1\}$, generates $(\mathsf{PCE.pk}, \mathsf{PCE.sk}) \leftarrow \mathsf{PCE.Setup}(1^\lambda, C_\beta)$, and returns $\mathsf{PCE.pk}$ to $\mathcal{B}$.
3. $\mathcal{B}$ generates $\mathcal{R}.\mathsf{pp} \leftarrow \mathcal{R}.\mathsf{Gen}(1^\lambda)$ and $(\mathcal{S}.\mathsf{vk}, \mathcal{S}.\mathsf{sk}) \leftarrow \mathcal{S}.\mathsf{Gen}(1^\lambda)$. It forwards $\mathsf{mpk} = (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk})$ to $\mathcal{A}$.
4. In the end $\mathcal{A}$ outputs a bit β'. $\mathcal{B}$ forwards β' to the PCE challenger.

We observe that if the PCE challenger samples $\beta = 0$, then $\mathcal{B}$ simulated the distribution $D_0 = \{\mathsf{mpk} | \mathsf{mpk} \leftarrow \mathsf{Setup}(1^\lambda, C_0)\}$, else $D_1 = \{\mathsf{mpk} | \mathsf{mpk} \leftarrow \mathsf{Setup}(1^\lambda, C_1)\}$ with $\mathcal{A}$. Hence, advantage of $\mathcal{B} = |\Pr(\beta' = 1 | \beta = 0) - \Pr(\beta' = 1 | \beta = 1)| = |\Pr(\beta' = 1 | D_0) - \Pr(\beta' = 1 | D_1)| = \epsilon$ (by assumption). □

Traceability. We show that the above scheme satisfies traceability using the following theorem.

Theorem E.13. Suppose that PCE scheme satisfies perfect correctness (Definition 3.1) and security against outsiders (Definition 3.8), ZK satisfies statistical zero-knowledge in the hiding mode (Definition A.3) and knowledge extraction in the binding mode (Definition A.5), one way relation $\mathcal{R}$ is secure (Definition A.13) and the signature scheme $\mathcal{S}$ satisfies EUF-CMA security. Then the above construction of PCGS satisfies traceability (Definition E.3) in the random oracle model.

Proof. The proof is identical to the proof of traceability in Theorem 4, [BGJP23] and is hence omitted. □

Unframeability. We show that the above scheme satisfies unframeability using the following theorem.

Theorem E.14. Suppose that PCE scheme satisfies perfect correctness (Definition 3.1) and security against outsiders (Definition 3.8), ZK satisfies statistical zero-knowledge with common random string in the hiding mode (Definition A.3) and knowledge extraction in the binding mode (Definition A.5), and the one way relation $\mathcal{R}$ is secure (Definition A.13). Then the above construction of PCGS satisfies unframeability (Definition E.4) in the random oracle model.

Proof. The proof is identical to the proof of unframeability in Theorem 4, [BGJP23] and is hence omitted. □

Client-Authority Anonymity against Malicious Authority.

Theorem E.15. Suppose that PCE scheme satisfies (computational/unconditional) security against malicious authority (Definition 3.7) and ZK satisfies statistical zero-knowledge with common random string in the hiding mode (Definition A.3). Then the above construction of PCGS satisfies (computational/unconditional) client-authority anonymity against a malicious authority (Definition E.6) in the random oracle model.

Proof. Recall that in the client-authority anonymity against a malicious authority, we want to show that

$$\mathsf{Sign}(\mathsf{mpk}, \mathsf{sk}_{\mathsf{id}_0}, m) \approx_s \mathsf{Sign}(\mathsf{mpk}, \mathsf{sk}_{\mathsf{id}_1}, m)$$

where $C(m) = 0$, for the circuit C associated with the, possibly malformed, master public key mpk.
Here we let the extractor Ext of PCGS scheme to be the extractor of the underlying PCE scheme, say PCE.Ext, which is secure against a malicious authority. The proof proceeds via the following sequence of hybrid games between the challenger and an unbounded adversary $\mathcal{A}$.

Hyb_0. This is the real world with $\beta = 0$, i.e., the challenge signature is computed using the signing key $\mathsf{sk}_{\mathsf{id}_0}$ associated with the identity id_0. We write the complete game here to set up the notations and easy reference in later hybrids.

1. $\mathcal{A}$ outputs the challenge master public key mpk, a message m and two identities $(\mathsf{id}_0, \mathsf{id}_1)$ and the respective signing keys $(\mathsf{sk}_{\mathsf{id}_0}, \mathsf{sk}_{\mathsf{id}_1})$.
2. The challenger parses $\mathsf{mpk} = (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk})$ and $\mathsf{sk}_{\mathsf{id}_0} = (s_0, \sigma_{\mathsf{id}_0})$. It computes $\mathsf{ct} = \mathsf{PCE.Enc}(\mathsf{PCE.pk}, m, \mathsf{id}_0; r_0)$, $\mathsf{crs} \leftarrow H(m, \mathsf{ct})$ and proof $\pi \leftarrow \mathsf{ZK.Prove}(\mathsf{crs}, (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk}, m, \mathsf{ct}), (\mathsf{id}_0, s_0, w_0, \sigma_{\mathsf{id}_0}, r_0))$ as in the Sign algorithm of the construction. It returns $\sigma = (\mathsf{ct}, \mathsf{crs}, \pi)$ to $\mathcal{A}$.
3. $\mathcal{A}$ outputs a bit β as the output of the experiment.

Hyb_1. This hybrid is same as the previous hybrid except that the random oracle H is lazily sampled, i.e., the challenger generates (crs, π) using $\mathsf{ZK.Sim}$ on the instance $(\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk}, m, \mathsf{ct})$ and sets $H(m, \mathsf{ct}) = \mathsf{crs}$.

Hyb_2. This hybrid is same as the previous hybrid except that the challenger computes ct differently as $\mathsf{ct} \leftarrow \mathsf{PCE.Enc}(\mathsf{PCE.pk}, m, \mathsf{id}_1)$.

Hyb_3. This hybrid is same as the previous hybrid except that the challenger generates (crs, π) differently . It computes $\mathsf{crs} \leftarrow H(m, \mathsf{ct})$ and the proof $\pi \leftarrow \mathsf{ZK.Prove}(\mathsf{crs}, (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk}, m, \mathsf{ct}), (\mathsf{id}_1, s_1, w_1, \sigma_{\mathsf{id}_1}, r_1))$ as in the Sign algorithm of the construction.
This is the real world with $\beta = 1$.

Indistinguishability of hybrids. We now show that the above consecutive hybrids are indistinguishable.

Claim E.16. Assume that ZK satisfies statistical zero-knowledge with common random string, then $\mathsf{Hyb}_0 \approx_s \mathsf{Hyb}_1$.

Proof. We show that if there exists an unbounded adversary $\mathcal{A}$ who can distinguish between Hyb_0 and Hyb_1 with non-negligible advantage ϵ, then there exists an unbounded adversary $\mathcal{B}$ against the statistical zero-knowledge of ZK scheme with the same advantage ϵ. The reduction is as follows.

1. $\mathcal{B}$ first runs $\mathcal{A}$. $\mathcal{A}$ outputs the challenge master public key mpk, a message m and two identities $(\mathsf{id}_0, \mathsf{id}_1)$ and the respective signing keys $(\mathsf{sk}_{\mathsf{id}_0}, \mathsf{sk}_{\mathsf{id}_1})$.
2. $\mathcal{B}$ parses $\mathsf{mpk} = (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk})$ and $\mathsf{sk}_{\mathsf{id}_0} = (s_0, \sigma_{\mathsf{id}_0})$ and does the following
   - Computes $(\mathsf{id}_0, w_0) = \mathcal{R}.\mathsf{Sample}(\mathcal{R}.\mathsf{pp}; s_0)$.
   - Samples $r_0 \leftarrow \{0,1\}^\lambda$ and computes $\mathsf{ct} = \mathsf{PCE.Enc}(\mathsf{PCE.pk}, m, \mathsf{id}; r_0)$.
   - Sets $x = (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk}, m, \mathsf{ct})$ and $w = (\mathsf{id}_0, s_0, w_0, \sigma_{\mathsf{id}_0}, r_0)$ and sends (x, w) to the ZK challenger. The ZK challenger generates $\mathsf{crs}_0 \leftarrow \mathsf{Setup}(1^\lambda)$ and computes $\pi_0 \leftarrow \mathsf{ZK.Prove}(\mathsf{crs}, x, w)$. It also computes $(\mathsf{crs}_1, \pi_1) \leftarrow \mathsf{ZK.Sim}(1^\lambda, x)$. It samples a bit $\beta \leftarrow \{0,1\}$ and returns $(\mathsf{crs}_\beta, \pi_\beta)$ to $\mathcal{B}$.
   - $\mathcal{B}$ sets $\mathsf{crs}_\beta = H(m, \mathsf{ct})$ and forwards $(\mathsf{ct}, \mathsf{crs}_\beta, \pi_\beta)$ to $\mathcal{A}$.
3. In the end, $\mathcal{A}$ outputs a bit β'. $\mathcal{B}$ sends β' to the ZK challenger.

We observe that if the ZK challenger samples $\beta = 0$, then $\mathcal{B}$ simulated Hyb_0, else Hyb_1 with $\mathcal{A}$. Hence, advantage of $\mathcal{B}$ $= |\Pr(\beta' = 1|\beta = 0) - \Pr(\beta' = 1|\beta = 1)| = |\Pr(\beta' = 1|\mathsf{Hyb}_0) - \Pr(\beta' = 1|\mathsf{Hyb}_1)| = \epsilon$ (by assumption). □

Claim E.17. Assume that PCE satisfies unconditional security against malicious authority, then $\mathsf{Hyb}_1 \approx_s \mathsf{Hyb}_2$.

Proof. We show that if there exists an unbounded adversary $\mathcal{A}$ who can distinguish between Hyb_1 and Hyb_2 with non-negligible advantage ϵ, then there exists an unbounded adversary $\mathcal{B}$ against the malicious authority security of the PCE scheme with the same advantage ϵ. The reduction is as follows.

1. $\mathcal{B}$ first run $\mathcal{A}$. $\mathcal{A}$ outputs the challenge master public key mpk, a message m and two identities $(\mathsf{id}_0, \mathsf{id}_1)$ and the respective signing keys $(\mathsf{sk}_{\mathsf{id}_0}, \mathsf{sk}_{\mathsf{id}_1})$.
2. $\mathcal{B}$ parses $\mathsf{mpk} = (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk})$, $\mathsf{sk}_{\mathsf{id}_0} = (s_0, \sigma_{\mathsf{id}_0})$, $\mathsf{sk}_{\mathsf{id}_1} = (s_1, \sigma_{\mathsf{id}_1})$, and does the following
   - It computes $(\mathsf{id}_0, w_0) = \mathcal{R}.\mathsf{Sample}(\mathcal{R}.\mathsf{pp}; s_0)$ and $(\mathsf{id}_1, w_1) = \mathcal{R}.\mathsf{Sample}(\mathcal{R}.\mathsf{pp}; s_1)$.
   - It sets $(x_0, m_0) = (m, \mathsf{id}_0)$, $(x_1, m_1) = (m, \mathsf{id}_1)$, and sends $\mathsf{PCE.pk}, (x_0, m_0), (x_1, m_1)$ to the PCE challenger as the challenge query. The challenger samples a bit $\beta \leftarrow \{0,1\}$, computes and returns $\mathsf{ct} \leftarrow \mathsf{PCE.Enc}(\mathsf{PCE.pk}, x_\beta, m_\beta)$ to $\mathcal{B}$.
   - It computes $(\mathsf{crs}, \pi) \leftarrow \mathsf{ZK.Sim}(1^\lambda, (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk}, m, \mathsf{ct}))$, sets $H(m, \mathsf{ct}) = \mathsf{crs}$ and sends $(\mathsf{ct}, \mathsf{crs}, \pi)$ to $\mathcal{A}$.
3. In the end, $\mathcal{A}$ outputs a bit β'. $\mathcal{B}$ sends β' to the PCE challenger.

We observe that if the PCE challenger samples $\beta = 0$, then $\mathcal{B}$ simulated Hyb_1, else Hyb_2 with $\mathcal{A}$. Hence, advantage of $\mathcal{B} = |\Pr(\beta' = 1|\beta = 0) - \Pr(\beta' = 1|\beta = 1)| = |\Pr(\beta' = 1|\mathsf{Hyb}_0) - \Pr(\beta' = 1|\mathsf{Hyb}_1)| = \epsilon$ (by assumption).

Admissibility of $\mathcal{B}$ Observe that $\mathcal{B}$ sends PCE.pk corresponding to a circuit C, and $(x_0, m_0) = (m, \mathsf{id}_0)$ and $(x_1, m_1) = (m, \mathsf{id}_1)$ as the challenge query to the PCE challenger. It follows from the admissibility of $\mathcal{A}$ that $C(m) = 0$ and hence $C(x_0) = C(x_1) = C(m) = 0$, as desired. □

Claim E.18. Assume that ZK satisfies statistical zero-knowledge with common random string, then $\mathsf{Hyb}_2 \approx_s \mathsf{Hyb}_3$.

Proof. The proof of this claim follows the same steps as the proof of Claim E.16, hence omitted. □

□

Client-Authority Unlinkability against Malicious Authority. [12]

Theorem E.19. Suppose that PCE scheme satisfies (computational/unconditional) security against malicious authority (Definition 3.7) and ZK satisfies statistical zero-knowledge with common random string in the hiding mode (Definition A.3). Then the above construction of PCGS satisfies (computational/unconditional) client-authority unlinkability against a malicious authority (Definition E.7) in the random oracle model.

Proof. Recall that in the client-authority unlinkability against a malicious authority, we want to show that

$$\mathsf{Sign}(\mathsf{mpk}, \mathsf{sk}_{\mathsf{id}_0}, m_0) \approx_s \mathsf{Sign}(\mathsf{mpk}, \mathsf{sk}_{\mathsf{id}_b}, m_1)$$

where $b \in \{0,1\}$ and $C(m_0) = 0 = C(m_1)$, for the circuit C associated with the, possibly malformed, master public key mpk.
Here we let the extractor Ext of PCGS scheme to be the extractor of the underlying PCE scheme, say PCE.Ext, which is secure against a malicious authority. We prove the above for $b = 0$ and $b = 1$ separately.

1. $\mathsf{Sign}(\mathsf{mpk}, \mathsf{sk}_{\mathsf{id}_0}, m_0) \approx_s \mathsf{Sign}(\mathsf{mpk}, \mathsf{sk}_{\mathsf{id}_0}, m_1)$.
   The proof proceeds via the following sequence of hybrid games between the challenger and an unbounded adversary $\mathcal{A}$.

[12] We note that this property does not hold when we use our construction of PCE for database (Appendix B.2) as it doesn't satisfy attribute-hiding.

Hyb_0. This is the real world with $\beta = 0$, i.e., the challenge signature is computed for message m_0 using the signing key $\mathsf{sk}_{\mathsf{id}_0}$ associated with the identity id_0. We write the complete game here to set up the notations and easy reference in later hybrids.

(a) $\mathcal{A}$ outputs the challenge master public key mpk, two messages m_0, m_1 and two identities $(\mathsf{id}_0, \mathsf{id}_1)$ and the respective signing keys $(\mathsf{sk}_{\mathsf{id}_0}, \mathsf{sk}_{\mathsf{id}_1})$.

(b) The challenger parses $\mathsf{mpk} = (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk})$ and $\mathsf{sk}_{\mathsf{id}_0} = (s_0, \sigma_{\mathsf{id}_0})$.
It computes $\mathsf{ct} = \mathsf{PCE.Enc}(\mathsf{PCE.pk}, m_0, \mathsf{id}_0; r_0)$, $\mathsf{crs} \leftarrow H(m_0, \mathsf{ct})$ and proof $\pi \leftarrow \mathsf{ZK.Prove}(\mathsf{crs}, (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk}, m_0, \mathsf{ct}), (\mathsf{id}_0, s_0, w_0, \sigma_{\mathsf{id}_0}, r_0))$ as in the Sign algorithm of the construction. It returns $\sigma = (\mathsf{ct}, \mathsf{crs}, \pi)$ to $\mathcal{A}$.

(c) $\mathcal{A}$ outputs a bit β as the output of the experiment.

Hyb_1. This hybrid is same as the previous hybrid except that the random oracle H is lazily sampled, i.e., the challenger generates (crs, π) using $\mathsf{ZK.Sim}$ on the instance $(\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk}, m_0, \mathsf{ct})$ and sets $H(m_0, \mathsf{ct}) = \mathsf{crs}$.

Hyb_2. This hybrid is same as the previous hybrid except that the challenger computes ct differently as $\mathsf{ct} \leftarrow \mathsf{PCE.Enc}(\mathsf{PCE.pk}, m_1, \mathsf{id}_0)$ and the corresponding proof using ZK simulator as $(\mathsf{crs}, \pi) \leftarrow \mathsf{ZK.Sim}(\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk}, m_1, \mathsf{ct})$.

Hyb_3. This hybrid is same as the previous hybrid except that the challenger generates (crs, π) differently. It computes $\mathsf{crs} \leftarrow H(m_1, \mathsf{ct})$ and the proof $\pi \leftarrow \mathsf{ZK.Prove}(\mathsf{crs}, (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk}, m_1, \mathsf{ct}), (\mathsf{id}_0, s_0, w_0, \sigma_{\mathsf{id}_0}, r_0))$ as in the Sign algorithm of the construction.
This is the real world with $\beta = 1$.

Indistinguishability of hybrids. We note that $\mathsf{Hyb}_0 \approx_s \mathsf{Hyb}_1$ using the statistical zero-knowledge with common random string of the underlying ZK scheme and the proof follows the same steps as that of Claim E.16.
$\mathsf{Hyb}_1 \approx_s \mathsf{Hyb}_2$ using the unconditional security of the underlying PCE scheme against a malicious authority and the proof follows the same steps as that of Claim E.17. We note that here the reduction $\mathcal{B}$ sends $(x_0, m_0) = (m_0, \mathsf{id}_0)$, $(x_1, m_1) = (m_1, \mathsf{id}_0)$ as the challenge query to the PCE challenger and we have $C(m_0) = C(m_1) = 0$, by the admissibility of $\mathcal{A}$, so $\mathcal{B}$ is admissible.
$\mathsf{Hyb}_2 \approx_s \mathsf{Hyb}_3$ using the similar arguments as of $\mathsf{Hyb}_0 \approx_s \mathsf{Hyb}_1$.

2. $\mathsf{Sign}(\mathsf{mpk}, \mathsf{sk}_{\mathsf{id}_0}, m_0) \approx_s \mathsf{Sign}(\mathsf{mpk}, \mathsf{sk}_{\mathsf{id}_1}, m_1)$. The indistinguishability of these two distribution follows from the similar sequence of hybrids as above except the following changes:

- In Hyb_2 we generate ct differently as $\mathsf{ct} \leftarrow \mathsf{PCE.Enc}(\mathsf{PCE.pk}, m_1, \mathsf{id}_1)$.
- In Hyb_3 we generate the proof as $\pi \leftarrow \mathsf{ZK.Prove}(\mathsf{crs}, (\mathcal{R}.\mathsf{pp}, \mathsf{PCE.pk}, \mathcal{S}.\mathsf{vk}, m_1, \mathsf{ct}), (\mathsf{id}_1, s_1, w_1, \sigma_{\mathsf{id}_1}, r_1))$.

□

Client-Client Anonymity.

Theorem E.20. Suppose that PCE scheme satisfies security against outsiders (Definition 3.8) and ZK satisfies statistical zero-knowledge in the hiding mode (Definition A.3). Then the above construction of PCGS satisfies client-client anonymity (Definition E.9) in the random oracle model.

Proof. The proof is identical to the proof of client-client anonymity in Theorem 4, [BGJP23] and is hence omitted. □

Client-Client Unlinkability. [13]

Theorem E.21. Suppose that PCE scheme satisfies security against outsiders (Definition 3.8) and ZK satisfies statistical zero-knowledge with common random string in the hiding mode (Definition A.3). Then the above construction of PCGS satisfies client-client unlinkability (Definition E.10) in the random oracle model.

Proof. The proof is identical to the proof of client-client unlinkability in Theorem 4, [BGJP23] and is hence omitted. □

[13] We note that this property does not hold when we use our construction of PCE for database (Appendix B.2) as it doesn't satisfy attribute-hiding.