Evaluating Octic Residue Symbols

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We detail an algorithm for the evaluation of the 8^{th}-power residue symbol. Algorithms for computing r^{th}-power residue symbols have only been devised for $r \in \{2,3,4,5,7\}$. See [8, 2], [7, 2], [6] and [1] for the cases $r = 3$, 4, 5 and 7, respectively. As noted in [1], as r grows, the technical details become increasingly complicated. An excellent account on the octic reciprocity can be found in [4, Chapter 9]. See also [3].

Let $\zeta \coloneqq \zeta_8 = \frac{\sqrt{2}}{2}(1+i)$ be a primitive 8^{th} root of unity. Let also $\epsilon = 1+\sqrt{2} = 1+\zeta+\zeta^{-1}$. The field $\mathbb{Q}(\zeta) = \mathbb{Q}(i,\sqrt{2})$ is biquadratic and its group of units is $\langle \zeta, \epsilon \rangle$. The Galois group of $\mathbb{Q}(\zeta)/\mathbb{Q}$ contains the four automorphisms $\sigma_k \colon \zeta \mapsto \zeta^k$ with $k \in \{1,3,5,7\}$. For an element $\alpha \in \mathbb{Z}[\zeta]$, we write $\alpha_k = \sigma_k(\alpha)$. The (absolute) norm of α is given by $\mathrm{N}(\alpha) = \alpha_1\alpha_3\alpha_5\alpha_7$.

An element $\alpha = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 \in \mathbb{Z}[\zeta]$ is said to be *primary* if $\alpha \equiv 1 \pmod{2+2\zeta}$ or, equivalently, if

$$\begin{cases} a_0 + a_1 + a_2 + a_3 \equiv 1 \pmod{4}\,, \\ a_1 \equiv a_2 \equiv a_3 \equiv 0 \pmod{2}\,. \end{cases}$$

Proof. By definition, α must be such that $(\alpha - 1) \propto 2(1+\zeta)$. Since $1-\zeta^4 = 2$, we have $\frac{(a_0-1)+a_1\zeta+a_2\zeta^2+a_3\zeta^3}{2(1+\zeta)} = \frac{((a_0-1)+a_1\zeta+a_2\zeta^2+a_3\zeta^3)(1-\zeta)(1+\zeta^2)}{4} = \frac{a_0-1+a_1-a_2+a_3}{4} + \frac{-a_0+1+a_1+a_2-a_3}{4}\zeta + \frac{a_0-1-a_1+a_2+a_3}{4}\zeta^2 + \frac{-a_0+1+a_1-a_2+a_3}{4}\zeta^3$. The condition is satisfied provided that $a_0 - 1 + a_1 - a_2 + a_3 \equiv -a_0 + 1 + a_1 + a_2 - a_3 \equiv a_0 - 1 - a_1 + a_2 + a_3 \equiv -a_0 + 1 + a_1 - a_2 + a_3 \equiv 0 \pmod{4}$; that is, $a_0 + a_1 + a_2 + a_3 \equiv 1 \pmod{4}$ and $2a_1 \equiv 2a_2 \equiv 2a_3 \equiv 0 \pmod{4}$. ☐

Proposition 1. *Let $\alpha \in \mathbb{Z}[\zeta]$ such that $(1+\zeta) \nmid \alpha$. Then there is a unit $\upsilon \in \mathbb{Z}[\zeta]$ such that $\alpha = \upsilon\,\alpha^*$ with α^* primary.*

Proof. Let $\alpha = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3$. The condition $(1+\zeta) \nmid \alpha$ implies $a_0 + a_1 + a_2 + a_3 \equiv 1 \pmod{2}$.

1. Suppose first that $a_0 \not\equiv a_2 \pmod{2}$ (and thus $a_1 \equiv a_3 \pmod{2}$). Noting that $\alpha \sim \alpha\,\zeta^{-2} = a_2 + a_3\zeta - a_0\zeta^2 - a_1\zeta^3$, we can assume that $a_0 \equiv 1 \pmod{2}$ and $a_2 \equiv 0 \pmod{2}$.
   (a) If $a_1 \equiv a_3 \equiv 0 \pmod{2}$ then $\alpha = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3$ with $a_0 \equiv 1 \pmod{2}$ and $a_1 \equiv a_2 \equiv a_3 \equiv 0 \pmod{2}$.
   (b) If $a_1 \equiv a_3 \equiv 1 \pmod{2}$, we replace α with $\alpha\,\epsilon^{-1}$ and get

$$\alpha\,\epsilon^{-1} = \underbrace{(-a_0 + a_1 - a_3)}_{\equiv 1 \pmod{2}} + \underbrace{(a_0 - a_1 + a_2)}_{\equiv 0 \pmod{2}}\zeta + \underbrace{(a_1 - a_2 + a_3)}_{\equiv 0 \pmod{2}}\zeta^2 + \underbrace{(-a_0 + a_2 - a_3)}_{\equiv 0 \pmod{2}}\zeta^3\ .$$

   By possibly multiplying by $-1 = \zeta^{-4}$ yields a primary element.

2. Suppose now that $a_0 \equiv a_2 \pmod 2$ (and $a_1 \not\equiv a_3 \pmod 2$). Then multiplying α by ζ^{-1} yields $\alpha\,\zeta^{-1} = a_1 + a_2\zeta + a_3\zeta^3 - a_0\zeta^3$. We so obtain a case similar to Case 1.

Consequently, in all cases, α can be expressed as $\alpha = \upsilon\,\alpha^*$ with α^* primary and $\upsilon = \zeta^k\epsilon^l$ for some $0 \le k \le 7$ and $l \in \{0, 1\}$. □

The main result is the octic reciprocity law; see [4, Theorem 9.19].

Theorem 1 (Octic Reciprocity). *Let α and λ be co-prime primary elements of $\mathbb{Z}[\zeta]$. Let N_1, N_2 and N_3 respectively denote the relative norms of the extensions $\mathbb{Q}(\zeta)/\mathbb{Q}(i)$, $\mathbb{Q}(\zeta)/\mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\zeta)/\mathbb{Q}(\sqrt{2})$; and write $\mathrm{N}_1(\alpha) = a(\alpha)^2 + b(\alpha)^2$, $\mathrm{N}_2(\alpha) = c(\alpha)^2 + 2d(\alpha)^2$, $\mathrm{N}_3(\alpha) = e(\alpha)^2 - 2f(\alpha)^2$, and similarly for λ. Then*[1]

$$\left[\frac{\alpha}{\lambda}\right]_8 = \left[\frac{\lambda}{\alpha}\right]_8 (-1)^{\frac{\mathrm{N}(\alpha)-1}{8}\frac{\mathrm{N}(\lambda)-1}{8}} \zeta^{\frac{d(\lambda)f(\alpha)-d(\alpha)f(\lambda)}{4}} \;.$$

Moreover,

$$\left[\frac{1-\zeta}{\alpha}\right]_8 = \zeta^{\frac{5a-5+5b+18d+b^2-2bd+d^4/2}{8}}\,, \qquad \left[\frac{\zeta}{\alpha}\right]_8 = \zeta^{\frac{a-1+4b+2bd+2d^2}{4}}\,,$$

$$\left[\frac{1+\zeta}{\alpha}\right]_8 = \zeta^{\frac{a-1+b+6d+b^2+2bd+d^4/2}{8}}\,, \qquad \left[\frac{\epsilon}{\alpha}\right]_8 = \zeta^{\frac{d-3b-bd-2d^2}{2}}\,,$$

$$\left[\frac{1+\zeta+\zeta^2}{\alpha}\right]_8 = \zeta^{\frac{a-1-2b+2d-2d^2}{4}} \;.$$

□

Letting $\alpha = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3$, a direct calculation shows that $\alpha_1\alpha_5 = ({a_0}^2 - {a_2}^2 + 2a_1a_3) + (-{a_1}^2 + {a_3}^2 + 2a_0a_2)i$, $\alpha_1\alpha_3 = ({a_0}^2 - {a_1}^2 + {a_2}^2 - {a_3}^2) + (a_0a_1 + a_0a_3 - a_1a_2 + a_2a_3)\sqrt{-2}$, and $\alpha_1\alpha_7 = ({a_0}^2 + {a_1}^2 + {a_2}^2 + {a_3}^2) + (a_0a_1 - a_0a_3 + a_1a_2 + a_2a_3)\sqrt{2}$ [4, Exerc. 5.21]. This yields $a(\alpha) = {a_0}^2 - {a_2}^2 + 2a_1a_3$, $b(\alpha) = -{a_1}^2 + {a_3}^2 + 2a_0a_2$,[2] $d(\alpha) = a_0a_1 + a_0a_3 - a_1a_2 + a_2a_3$, and $f(\alpha) = a_0a_1 - a_0a_3 + a_1a_2 + a_2a_3$.

As stated, the reciprocity law requires α and λ being primary. Suppose that α is such that $(1+\zeta) \nmid \alpha$, but is not necessarily primary. Then from Proposition 1, we can write $\alpha = \zeta^k\epsilon^l\,\alpha^*$ for some $0 \le k \le 7$ and $l \in \{0, 1\}$, with α^* primary. We note $\alpha^* = \mathrm{primary}(\alpha)$ and $(k, l) = \nu(\alpha)$. Likewise, suppose that λ is such that $(1+\zeta) \nmid \lambda$ and is not necessarily primary. Then $\lambda = \zeta^{k'}\epsilon^{l'}\lambda^*$ with $\lambda^* = \mathrm{primary}(\lambda)$ and $(k', l') = \nu(\lambda)$.

We assume $(1+\zeta) \nmid \lambda$. Putting all together, when $(1+\zeta) \nmid \alpha$, we have:

$$\begin{aligned}
\left[\frac{\alpha}{\lambda}\right]_8 &= \left[\frac{\alpha}{\lambda^*}\right]_8 = \left[\frac{\zeta^k}{\lambda^*}\right]_8 \left[\frac{\epsilon^l}{\lambda^*}\right]_8 \left[\frac{\alpha^*}{\lambda^*}\right]_8 && \text{by Proposition 1}\\
&= \zeta^{\frac{k\left(a(\lambda^*)-1+4b(\lambda^*)+2b(\lambda^*)d(\lambda^*)+2d(\lambda^*)^2\right)}{4}} \zeta^{\frac{l\left(d(\lambda^*)-3b(\lambda^*)-b(\lambda^*)d(\lambda^*)-2d(\lambda^*)^2\right)}{2}} \\
&\qquad \left[\frac{\lambda^*}{\alpha^*}\right]_8 \zeta^{\frac{(\mathrm{N}(\alpha^*)-1)(\mathrm{N}(\lambda^*)-1)}{16} + \frac{d(\lambda^*)f(\alpha^*)-d(\alpha^*)f(\lambda^*)}{4}} && \text{by Theorem 1}\\
&= \left[\frac{\lambda^* \bmod \alpha^*}{\alpha^*}\right]_8 \zeta^{k\,\mathcal{K}(\lambda^*)+l\,\mathcal{L}(\lambda^*)+\mathcal{J}(\alpha^*,\lambda^*) \pmod 8}
\end{aligned}$$

[1] We note that a factor $-\frac{1}{4}$ is missing in the expression given in [4, Theorem 9.19].
[2] The first formula listed in [4, Exerc. 5.21] actually corresponds to $-b$.

where $\mathcal{K}(\lambda^*) = \frac{1}{4}\big[a(\lambda^*) - 1 + 4b(\lambda^*) + 2b(\lambda^*)d(\lambda^*) + 2d(\lambda^*)^2\big]$, $\mathcal{L}(\lambda^*) = \frac{1}{2}\big[d(\lambda^*) - 3b(\lambda^*) - b(\lambda^*)d(\lambda^*) - 2d(\lambda^*)^2\big]$ and $\mathcal{J}(\alpha^*, \lambda^*) = \frac{1}{16}\big[(\mathrm{N}(\alpha^*)-1)(\mathrm{N}(\lambda^*)-1) + 4d(\lambda^*)f(\alpha^*) - 4d(\alpha^*)f(\lambda^*)\big]$. When $(1+\zeta) \mid \alpha$, we have:

$$\begin{aligned}\left[\frac{\alpha}{\lambda}\right]_8 = \left[\frac{\alpha}{\lambda^*}\right]_8 &= \left[\frac{\alpha/(1+\zeta)}{\lambda^*}\right]_8 \left[\frac{1+\zeta}{\lambda^*}\right]_8 \\ &= \left[\frac{\alpha/(1+\zeta)}{\lambda^*}\right]_8 \zeta^{\mathcal{I}(\lambda^*) \pmod 8} && \text{by Theorem 1}\end{aligned}$$

where $\mathcal{I}(\lambda^*) = \frac{1}{8}\big(a(\lambda^*) - 1 + b(\lambda^*) + 6d(\lambda^*) + b(\lambda^*)^2 + 2b(\lambda^*)d(\lambda^*) + d(\lambda^*)^4/2\big)$. These two observations lead to Algorithm 1.

Algorithm 1: Computing $\left[\frac{\alpha}{\lambda}\right]_8$

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Data: α, λ ∈ Z[ζ] with α and λ co-prime, and (1 + ζ) ∤ λ
Result: [α/λ]_8 ∈ {±1, ±i, ±ζ, ±iζ}
λ ← primary(λ); j ← 0
while N(α) ≠ 1 do
    if (1 + ζ) | α then
        α ← α/(1 + ζ)
        j ← j + I(λ) (mod 8)
    else
        (k, l) ← ν(α); α ← primary(α)
        j ← j + k K(λ) + l L(λ) + J(α, λ) (mod 8)
        (α, λ) ← (λ mod α, α)
    end
end
(k, l) ← ν(α); α ← primary(α)
[u_0, u_1, u_2, u_3] ← α mod 8; k ← k + u_0 − 1; l ← l + u_3
j ← j + k K(λ) + l L(λ) (mod 8)
return ζ^j
```

At the end of the while-loop, α is transformed into a primary unit, say υ^*. Letting $\upsilon^* \bmod 8 = u_0 + u_1\zeta + u_2\zeta^2 + u_3\zeta^3 := [u_0, u_1, u_2, u_3]$, it turns out that the possible values are $[1,0,0,0]$, $[1,4,0,4]$, $[5,6,0,2]$, $[5,2,0,6]$, respectively corresponding to $\left[\frac{\upsilon^*}{\lambda^*}\right]_8 = \left[\frac{1}{\lambda^*}\right]_8$, $\left[\frac{\epsilon^4}{\lambda^*}\right]_8$, $\left[\frac{\zeta^4\epsilon^2}{\lambda^*}\right]_8$, $\left[\frac{\zeta^4\epsilon^6}{\lambda^*}\right]_8$.

As a reminder, a ring R is said *norm-Euclidean* or *Euclidean with respect to the norm* N if for every $\alpha, \beta \in R$, $\beta \neq 0$, there exist $\eta, \rho \in R$ such that $\alpha = \beta\,\eta + \rho$ and $\mathrm{N}(\rho) < \mathrm{N}(\beta)$. The correctness of the algorithm is a consequence of the fact that $\mathbb{Z}[\zeta]$ is norm-Euclidean [5]: when α is replaced by $\lambda \bmod \alpha$, its norm decreases. Also, when α is divided by $(1+\zeta)$, its norm is divided by 2 since $\mathrm{N}(1+\zeta) = 2$. Therefore, in all cases, the norm of α is decreasing and eventually becomes 1.

Remark 1. Letting $\alpha = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3$, the condition $(1+\zeta) \mid \alpha$ simply amounts to verify whether $a_0 + a_1 + a_2 + a_3 \equiv 0 \pmod 2$; in this case, $\alpha/(1+\zeta) = \frac{1}{2}(a_0 + a_1 - a_2 + a_3) + \frac{1}{2}(-a_0 + a_1 + a_2 - a_3)\zeta + \frac{1}{2}(a_0 - a_1 + a_2 + a_3)\zeta^2 + \frac{1}{2}(-a_0 + a_1 - a_2 + a_3)\zeta^3$.

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