

# Registration-Based Encryption: Removing Private-Key Generator from IBE

Sanjam Garg\* Mohammad Hajiabadi†  
Mohammad Mahmoody‡ Ahmadreza Rahimi§

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## Abstract

In this work, we introduce the notion of *registration-based encryption* (RBE for short) with the goal of removing the trust parties need to place in the private-key generator in an IBE scheme. In an RBE scheme, users sample their own public and secret keys. There will also be a “key curator” whose job is only to aggregate the public keys of all the registered users and update the “short” public parameter whenever a new user joins the system. Encryption can still be performed to a particular recipient using the recipient’s identity and any public parameters released subsequent to the recipient’s registration. Decryption requires some auxiliary information connecting users’ public (and secret) keys to the public parameters. Because of this, as the public parameters get updated, a decryptor may need to obtain “a few” additional auxiliary information for decryption. More formally, if  $n$  is the total number of identities and  $\kappa$  is the security parameter, we require the following.

- **Efficiency requirements:** (1) A decryptor only needs to obtain updated auxiliary information for decryption at most  $O(\log n)$  times in its lifetime, (2) each of these updates are computed by the key curator in time  $\text{poly}(\kappa, \log n)$ , and (3) the key curator updates the public parameter upon the registration of a new party in time  $\text{poly}(\kappa, \log n)$ . Properties (2) and (3) require the key curator to have *random* access to its data.
- **Compactness requirements:** (1) Public parameters are always at most  $\text{poly}(\kappa, \log n)$  bits, and (2) the total size of updates a user ever needs for decryption is  $\text{poly}(\kappa, \log n)$  bits.

We present feasibility results for constructions of RBE based on indistinguishably obfuscation. We further provide constructions of *weakly efficient* RBE, in which the registration step is done in  $\text{poly}(\kappa, n)$ , based on CDH, Factoring or LWE assumptions. Note that registration is done only once per identity, and the more frequent operation of generating updates for a user, which can happen more times, still runs in time  $\text{poly}(\kappa, \log n)$ . We leave open the problem of obtaining standard RBE (with  $\text{poly}(\kappa, \log n)$  registration time) from standard assumptions.

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\*[sanjamg@berkeley.edu](mailto:sanjamg@berkeley.edu) Berkeley. Research supported in part from DARPA/ARL SAFEWARE Award W911NF15C0210, AFOSR Award FA9550-15-1-0274, AFOSR YIP Award, DARPA and SPAWAR under contract N66001-15-C-4065, a Hellman Award and research grants by the Okawa Foundation, Visa Inc., and Center for Long-Term Cybersecurity (CLTC, UC Berkeley). The views expressed are those of the author and do not reflect the official policy or position of the funding agencies.

†[mdhajiabadi@berkeley.edu](mailto:mdhajiabadi@berkeley.edu) Berkeley and University of Virginia. Supported by NSF award CCF-1350939 and AFOSR Award FA9550-15-1-0274.

‡[mohammad@virginia.edu](mailto:mohammad@virginia.edu) University of Virginia. Supported by NSF CAREER award CCF-1350939, and two University of Virginia’s SEAS Research Innovation Awards.

§[ahmadreza@virginia.edu](mailto:ahmadreza@virginia.edu) University of Virginia. Supported by NSF award CCF-1350939.

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# 1 Introduction

Public-key encryption [DH76, RSA78, GM82] allows Alice to send Bob private messages without any a-priori shared secrets. However, before Alice can send any messages to Bob, she must obtain Bob’s public key. Enabling Alice to obtain Bob’s public key often requires additional public-key infrastructure and in some cases complex certification authorities; consequently, making implementation of public-key encryption rather cumbersome.

With the goal of simplifying key-management in public-key encryption, Shamir [Sha84] introduced the notion of identity based encryption (IBE). An IBE scheme allows Alice to encrypt her messages to Bob knowing just the identity of Bob and some additional system public parameters. In this setup, Bob can then decrypt Alice’s ciphertexts using an identity-specific secret key that he obtains from the private key generator (PKG). In their celebrated work, Boneh and Franklin [BF01] provided the first construction of IBE using bilinear maps. A long line of subsequent research has provided many other constructions of IBE based on a variety of assumptions [Coc01, DG17]. IBE serves as the basis of several real-world systems (e.g., in systems by Voltage security) to simplify key-management.

Despite its significant advantages, one important limitation of IBE schemes is the so-called *key-escrow* problem. Namely, in an IBE scheme a PKG can generate the identity-specific secret key for any identity. This allows the PKG to arbitrarily decrypt messages that are intended for specific recipients. While in certain applications it is reasonable to place trust in a PKG, doing so is not *always* acceptable. This limitation of IBE often attracts significant criticism and restricts applicability in certain scenarios. In words of Rogaway [Rog15],

*“But this convenience is enabled by a radical change in the trust model: Bob’s secret key is no longer self-selected. It is issued by a trusted authority. That authority knows everyone’s secret key in the system. IBE embeds key-escrow indeed a form of key-escrow where a single entity implicitly holds all secret keys even ones that haven’t yet been issued. [...] Descriptions of IBE don’t usually emphasize the change in trust model. And the key-issuing authority seems never to be named anything like that: it’s just the PKG, for Private Key Generator. This sounds more innocuous than it is, and more like an algorithm than an entity.”*

With the goal of enhancing the applicability of IBE, prior works suggested ways for reducing the level of trust that parties need to place in the PKG. Boneh and Franklin [BF01] suggested the use of multiple PKGs, instead of just one, with the goal of making the trust de-centralized. This idea was further explored in subsequent work (e.g., see [CHSS02, PS08, KG10]). In a different approach, Goyal [Goy07], later followed by Goyal et al. [GLSW08], studied the notion of accountable IBE, which allows users to get their decryption keys from the PKG using a secure key generation protocol. Such schemes provide safeguard against a malicious PKG who might distribute the identity-specific secret key for a particular user to unauthorized parties, as by doing so it risks the possibility of being caught in the future. Another approach to the key escrow problem, studied in [CCV04, Cho09, WQT18], involves settings in which the number of identities is huge, limiting the server’s ability of finding out the receiver identity when it is chosen at random; hence, guaranteeing a form of anonymity. Finally, Al-Riyami and Paterson [ARP03] put forward the notion of “Certificateless” Public Key Cryptography which is a hybrid of IBE and public-key directories, but which, on the down side, does not let the sender use the system as a true IBE, because more information about the user needs to be read from the public-key infrastructure before encrypting a message to them.

None of the above approaches, however, resolve the key-escrow problem entirely, as the PKG (or a collection of several of them) can still decrypt all ciphertexts in the system. Indeed even a trusted PKG may not be able to protect ciphertexts against a subpoena requesting decryption keys. This state of affairs leads us to the main question of this work:

*Can we entirely remove PKG from IBE schemes?*

**A new primitive: registration-based encryption (RBE).** In this work, we pursue a new approach to constructing IBE schemes by introducing a new notion which we call *registration-based encryption*, and which does not suffer from the key-escrow problem. Recall that in traditional IBE schemes, the PKG plays an active role in maintaining the cryptographic secrets corresponding to the public parameters of the system, leading to the key-escrow problem. Deviating from this approach, in our RBE we replace the PKG with a much weaker entity that we call a *key curator*. A key curator does not possess any cryptographic secrets and just plays the role of *aggregating* the public keys of the users.

In more detail, in an RBE scheme each user samples its own public key and secret key and provides its identity and the chosen public key to the key curator.<sup>1</sup> The key curator is now tasked with the goal of curating this new user’s public key in the public parameters. Towards this, the key curator updates the public parameters and publicizes the new public parameters. Thus, unlike traditional IBE schemes, the public parameters in an RBE scheme evolve as new users register in the system. For example, let  $\mathbf{pp}_0, \mathbf{pp}_1, \dots, \mathbf{pp}_n$  be the different instances of the public parameters in the system, where  $\mathbf{pp}_i$  is the public parameter after  $i$  users have registered in the system. Just like an IBE scheme, we require that the size of the public parameter is always small:  $|\mathbf{pp}_i| \leq \text{poly}(\kappa, \log n)$  for  $i \leq n$ , where  $\kappa$  is the security parameter and  $n$  is the number of users in the system.

In an RBE scheme, decryption by a user is performed using its secret key and some *auxiliary information* that connects its public key with system’s public parameters. Note that as new users join the system and public parameters are updated, an update to the auxiliary information connecting a user’s public key to the new public parameters is necessary.<sup>2</sup> However, it would be prohibitive to update each user’s auxiliary information (needed for decryption) after each single registration. Thus, we require that the effect of registration by new users on the previously registered users is minimal. In particular, we require that a registered user needs to query the key curator for auxiliary information connecting its public key to the public parameters at most  $O(\log n)$  times in its lifetime where  $n$  is the total number of registered users. Additionally, we require that the total size of the auxiliary information provided by the key curator needed for any decryption is at most  $\text{poly}(\kappa, \log n)$  for security parameter  $\kappa$ .

**Our results.** We consider two variants of RBE schemes based on the efficiency of the registration and give constructions for both of them. In particular, we construct (standard) RBE using indistinguishability obfuscation, and we construct a “weakly efficient” variant of this primitive based on more standard assumptions.

- *RBE based on IO:* First, we construct (standard) RBE schemes in which the running time of key curator for every new user registration is  $\text{poly}(\kappa, \log n)$  for security parameter  $\kappa$  assuming the key curator has *random* access to its auxiliary information. Other than the desired efficiency

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<sup>1</sup>The key curator will need to verify the identity of the user requesting the registration as it is done by certification authorities in public-key infrastructure.

<sup>2</sup>Note that since the public parameters are small, they cannot contain the public keys of all the registered parties.

itself, one motivation for such minimization in curator’s complexity is that since the work done in each user registration is small, it is then more reasonable to distribute the key curator’s job between the users themselves, removing the need of a dedicated key curator entirely. In such a system, a new user will only need to do a “small” amount of *public* computation to update the public parameters at the time of joining the system. Moreover, any previously registered user could obtain its updated auxiliary information needed for decryption from the public ledger as well. We obtain a feasibility result for this notion based on somewhere statistically binding hash functions [HW15] and indistinguishably obfuscation [BGI+01, GGH+13].

- *RBE with weakly-efficient registration*: Second, we consider a setting where the key curator is allowed to be “weakly efficient”; i.e., the running time of key curator for updating the public parameters as a single new user registers can  $\text{poly}(\kappa, n)$ . We call such RBE schemes weakly efficient and obtain a construction of weakly-efficient RBE based on any *hash garbling* scheme. The notion of hash garbling and its construction has been implicit in prior works [CDG+17, DG17, DG17, DGHM18, BLSV18], and it was shown there that hash garbling can be realized based on CDH, Factoring or LWE assumptions. In this work, we give a formal definition of this primitive (Definition 5.1) and use it to construct RBE.

We leave open constructing RBE with  $\text{poly}(\kappa, \log n)$  registration based on standard assumptions.

**Communication cost of RBE compared with PKE and IBE.** We view RBE as a hybrid between PKE and traditional IBE. PKE schemes are communication heavy for encryptors. In other words, each encryptor must obtain the public keys of each recipient that it sends encrypted messages to. In contrast, IBE schemes remove the need for the communication by the encryptors — specifically, encryptors no longer need to recover the public key of each user separately. However, the decryptor must still obtain its identity-specific secret key via communication with the PKG. Note that since this communication with PKG is only done once, the communication cost of an IBE is much smaller than the communication cost of a PKE. However, this efficiency comes at the cost of the key-escrow problem. Our RBE achieves, in large parts, the communication benefits of IBE without the key-escrow problem. More specifically, in an RBE, the encryptors do not need to recover the public key of each recipient individually. Additionally, a decryptor only needs to interact with the key curator to obtain the relevant updates at most  $\log n$  times in total.

IBE was originally proposed with the goal of simplifying key management in IBE, yet the problem of key-escrow has prevented it from serving as a substitute for PKE — specifically, its applicability remains limited to specialized settings where trust is not a problem. We believe that efficient variants of our RBE constructions could indeed provide an alternative for PKE while also simplifying key management as IBE does.

## 1.1 Technical Overview

Here we describe the high level ideas behind our two constructions. The main challenge in realizing our RBE is to have the key curator gather together public keys of registering users in such a way that no individual’s relation to the public parameter is affected too many times. Doing that is the key for having few necessary updates for decryption. We start by describing how we resolve this challenge using indistinguishability obfuscation (IO). Next, we give our ideas for realizing a (registration) weakly efficient version of this primitive based on standard assumptions such as CDH and Factoring. The IO-based construction, however, remains conceptually simpler.

Our IO based solution is inspired by prior works on using witness encryption [GGSW13], if we interpret the decryption key (i.e., the secret key together with the required auxiliary updates) as a witness that enables decryption. Additionally, both our IO-based and the hash obfuscation based solutions (and in particular their tree-based hashing of the public keys) use ideas developed recently in the context of laconic OT [CDG<sup>+</sup>17] and IBE from the CDH assumption [DG17]. In both of these settings, our contribution is in formalizing the subtle aspects of RBE and then realizing RBE schemes (as mentioned above) using these ideas.

**High level description of our IO-based construction of RBE.** A natural first try for the solution would be for the curator to just Merkle hash together the public keys of all the users in the system (along with their corresponding identities). Here encryption could be performed by an obfuscation of the following program  $P[h, m]$ , with the Merkle hash root  $h$  and the encrypted message  $m$  hardwired. Given input  $(pk, id, pth)$ , the program  $P[h, m]$  outputs an encryption of  $m$  under the public key  $pk$  *only if*  $pth$  is a “Merkle opening” (i.e., the right leaf to root path with siblings) for  $(pk, id)$  as a pair of sibling leaves in the Merkle hash tree with root  $h$ , and it outputs  $\perp$  otherwise. Decryption can proceed naturally with the right Merkle opening as auxiliary information that the key curator needs to provide for decryption. The main issue with this solution is that the Merkle hash root  $h$  changes with every new user registering in the system. Our idea for solving this problem is to maintain *multiple* Merkle hash trees such that any individual user is affected only a bounded number of times. Below, we explain this idea in more detail.

- Public parameters and auxiliary information. At a high level, in our construction, after  $n$  parties have registered, the key curator holds an auxiliary information  $aux_n$  of the following form: it consists of  $\eta$  *full* binary Merkle trees,  $Tree_1, \dots, Tree_\eta$  with corresponding depths  $d_1 > \dots > d_\eta$  and number of leaves  $2^{d_1}, \dots, 2^{d_\eta}$ . The public parameter would be the set of the labels of the *roots* of these trees. Every leaf in either of these trees is either an identity  $id$  or its public key  $pk$  as the sibling of the leaf  $id$ , and every registered identity  $id$  appears exactly once as a leaf. Thus, half of the leaves of these trees contain the strings encoding the registered identities, and for each leaf  $id$ , the sibling leaf contains the public key  $pk$  of  $id$ . So, if there are  $n$  people registered so far in the system, then the total number of leaves in the trees is equal to  $2n$ . Since we stated that  $d_1 > \dots > d_\eta$ , it means that the number of these trees  $\eta$  is at most  $\log(n)$ , simply because  $(d_1, \dots, d_\eta)$  would be the binary representation of number  $2n$ . This point implies that the public parameter is indeed short.

- What is needed for decryption. Even though in general it is more natural to describe encryption first, in our case it is easier to describe the information that is needed for decryption. Each identity  $id$  will hold its own secret key  $sk$  which will be necessary for decryption, but it would need more information for doing so. Indeed, if  $Tree$  is the tree held by the curator that contains (sibling leaves)  $(id, pk)$  in its leaves, then the identity  $id$  needs to know the “Merkle opening” of  $(id, pk)$  to the root of  $Tree$  in order to do any decryption. Since the length of this path is at most the depth of  $Tree$ , which is at most  $\log(n)$ , the total size of the decryption key  $dk$  (which includes  $sk$  and the knowledge of such opening to the root of  $Tree$ ) is at most  $\kappa \cdot \log(n)$ . This makes  $dk$  compact.

- How to encrypt. For simplicity, suppose there is only one tree  $Tree$  held by the key curator and that all the identities are leaves of this tree. The encryptor, knows the public parameter, which is the root  $rt$  of  $Tree$ . For any message  $m$ , the encryptor then sends the *obfuscation* of the following program  $P$ . The program  $P$  takes as input any Merkle opening that contains the path from leaves  $(id, pk)$  to the root  $rt$  of  $Tree$ , and if such opening is given, then  $P$  outputs an encryption of  $m$  under the corresponding registered public key  $pk$ . Since  $id$  is the only identity who knows the

corresponding  $sk$  to the registered  $pk$ , nobody other than  $id$  can decrypt the message  $m$  encrypted that way. When there are *multiple* trees  $Tree_1, \dots, Tree_\eta$  held by the key curator, the ciphertext includes  $\eta$  obfuscations, one for every  $Tree_i$ .

- How to register. When a new party  $id$  joins to register, we first create a single tree  $Tree$  for that party, with  $id, pk$  as its only leaves. But creating too many trees naively increases the length of the public parameter. So, to handle this issue we “merge” the trees every now and then. In particular, upon any registration, so long as there are any two trees  $Tree_1, Tree_2$  of the *same size* held by the key curator, it “merges” them by simply hashing their roots  $rt_1, rt_2$  into a new root  $rt$ . This way, the key curator keeps the invariance property (stated above) that the trees are always full binary trees of different sizes. After doing any such merge, the key curator sends the the generated update of the form  $(rt_1, rt, rt_2)$  to all of the identities that are in *either* of the trees  $Tree_1, Tree_2$ . That is because, the identities in  $Tree_1$  would now need to know  $rt_2$  and the identities in  $Tree_2$  now need the label  $rt_1$  in order to decrypt what is encrypted for them. Alternatively, if the key curator is passive and does not send updates, the users who are in the merged tree  $Tree$  would need to pull their updates whenever they have a ciphertext that they cannot decrypt, realizing that their auxiliary information is outdated.

To prove security of the above construction, collision-resistance of the used hash function is not enough, and we rely on *somewhere statistically binding* hash functions [HW15] (see Definition 2.3).

**Weakly-efficient construction based on standard assumptions.** In order to replace the use of obfuscation in the above construction, we build on the techniques by Cho, Döttling, Garg, Gupta, Miao, and Polychroniadou [CDG<sup>+</sup>17] and Döttling and Garg [DG17]. We abstract their idea of using hash encryption and garbled circuits as a new primitive that we call *hash garbling*. Use of this abstraction simplifies exposition. A hash garbling scheme consists of algorithms (Hash, HG, HInp).<sup>3</sup> Hash function is a function from  $\{0, 1\}^\ell$  to  $\{0, 1\}^\kappa$ . HG takes as input a secret state  $stt$  and an arbitrary program  $P$  and outputs  $\tilde{P}$ . HInp takes as input a secret state  $stt$  and a value  $y \in \{0, 1\}^\kappa$  and outputs  $\tilde{y}$ . Correctness and security require that  $\tilde{C}, \tilde{y}, x$  can be used to compute  $C(x)$ , but also that they reveal nothing else about  $C$ .

Our construction of RBE from standard assumption is very similar to the IO-based construction except that we replace the use of IO with the less powerful primitive hash garbling. The key challenge in making this switch comes from the fact that hash garbling, unlike IO, cannot process *the entire* root to leaf Merkle opening in one shot. Thus, our construction needs to provide a sequence of hash garblings that traverse the root to leaf path step by step. Therefore, as the tree is being traversed, the hash garblings need to identify whether to go left or to go right. Note that this decision must be taken without any knowledge of what identities are included in the leaves of the left sub-tree and what identities are included in the leaves of the right sub-tree. We resolve this challenge by modifying the Merkle tree in two ways:

1. We ensure that the identities in the leave of any tree are always sorted.
2. In addition to the hashes of its two children, in the computation of the Merkle hash, we also hash the information about the largest identity that is present any leaf of the left subtree at any node. (The latter information allows us to traverse down a Merkle tree using it as a binary search tree.)

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<sup>3</sup>The hash function also has a key setup function which we ignore here for the sake of simplicity.

Using these enhancements over the simple Merkle trees, we can indeed substitute IO with the less powerful primitive of hash garbling, which in turn can be obtained from more standard assumptions. On the down side, this new construction needs to sort the identities for every registration, and in particular the registration cannot run in sublinear time  $\text{poly}(\kappa, \log n)$ . We refer the reader Section 5 for more details on this construction.

## 2 Preliminaries

**Notation.** For a probabilistic algorithm  $A$ , by  $A(x) \rightarrow y$ , we denote the randomized process of running  $A$  on input  $x$  and obtaining the output  $y$ . We use PPT to denote a probabilistic polynomial-time algorithms, where running time is polynomial over the length of their main input (not the random seed). For randomized algorithms  $A_1, A_2, \dots$ , by  $\Pr_{A_1, A_2, \dots}[E]$  we denote the probability of event  $E$  when the randomness is over the algorithms  $A_1, A_2, \dots$  as well. For deterministic algorithms  $A_1, A_2$ , by  $A_1 \equiv A_2$ , we denote that they have the same input-output functionality; namely, for all  $x$  (of the right length, if  $A_1, A_2$  are circuits),  $A_1(x) = A_2(x)$ . For distribution ensembles  $X_n, Y_n$ , by  $X_n \stackrel{c}{\approx} Y_n$  we mean that they are indistinguishable against  $\text{poly}(n)$ -time algorithms. By  $x||y$  we denote the concatenation of the strings  $x, y$ . By  $\text{negl}(\kappa)$  we denote some function that is negligible in input  $\kappa$ ; namely for all  $k$ ,  $\text{negl}(\kappa) \leq O(1/\kappa^k)$ .  $U_n$  denotes the uniform distribution over  $\{0, 1\}^n$ . For algorithm  $A$ , by  $A^B$  we denote an oracle access by  $A$  to oracle  $B$ . By  $A^{[B]}$  we denote  $A$  accessing oracle  $B$  with read and *and write* operations. So, if  $A$  writes  $y$  at location  $x$ , reading a query  $x$  next time will return  $y$ .

**Definition 2.1** (Public key encryption). A public key encryption scheme consists of three PPT algorithms (G, E, D) as follows.

- $G(1^\kappa) \rightarrow (\text{pk}, \text{sk})$ : This algorithm takes a security parameter  $1^\kappa$  as input and outputs a pair of public key  $\text{pk}$  secret key  $\text{sk}$ . Without loss of generality we assume that  $|\text{pk}| = |\text{sk}| = \kappa$ .
- $E(\text{pk}, m) \rightarrow \text{ct}$ : takes a message  $m$  and a public key  $\text{pk}$  as input and outputs a ciphertext  $\text{ct}$ .
- $D(\text{sk}, \text{ct}) \rightarrow m$ : takes a ciphertext  $\text{ct}$  and a secret key  $\text{sk}$  as inputs and outputs a message  $m$ .

The completeness and security properties are defined as follows.

- **Completeness.** The PKE scheme is complete if for every message  $m$ :

$$\Pr_{G, E, D} [D(\text{sk}, E(\text{pk}, m)) = m : (\text{sk}, \text{pk}) \leftarrow G] = 1.$$

- **Semantic Security.** Any PPT Adv wins the following game with probability  $\frac{1}{2} + \text{negl}(\kappa)$ :
  - The challenger generates  $(\text{pk}, \text{sk}) \leftarrow G(1^\kappa)$  and sends  $\text{pk}$  to Adv.
  - The challenger chooses a random bit  $b$  and sends  $c \leftarrow E(\text{pk}, b)$  to Adv.
  - Adv outputs  $b'$  and wins if  $b = b'$ .

**Definition 2.2** (Indistinguishability obfuscation). A uniform PPT algorithm Obf is called an *indistinguishability obfuscator* for a circuit class  $\{\mathcal{C}_\kappa\}_{\kappa \in \mathbb{N}}$  (where each  $\mathcal{C}_\kappa$  is a set indexed by a security parameter  $\kappa$ ) if the following holds:

- **Completeness.** For all security parameters  $\kappa \in \mathbb{N}$  and all circuits  $C \in \mathcal{C}_\kappa$ , we obtain an obfuscation with the same function:

$$\Pr_{\text{Obf}}[C' \equiv C : C' = \text{Obf}(1^\kappa, C)] = 1.$$

- **Security.** For any PPT distinguisher  $D$ , there exists a negligible function  $\text{negl}(\cdot)$  such that for all  $\kappa \in \mathbb{N}$ , for all pairs of functionally equivalent circuits  $C_1 \equiv C_2$  from the same family  $C_1, C_2 \in \mathcal{C}_\kappa$ ,

$$\left| \Pr_{\text{Obf}}[D(1^\kappa, \text{Obf}(1^\kappa, C_1)) = 1] - \Pr_{\text{Obf}}[D(1^\kappa, \text{Obf}(1^\kappa, C_2)) = 1] \right| \leq \text{negl}(\kappa).$$

The next definition is a special case of the definition of somewhere statistically binding (SSB) hash functions introduced by Hubacek and Wichs [HW15] for the blockwise setting. Here we only use two-input blocks.

**Definition 2.3** (SSB hash functions [HW15]). A *somewhere statistically binding hash system* consists of two polynomial time algorithms HGen, Hash.

- $\text{HGen}(1^\kappa, b) \rightarrow \text{hk}$ . This algorithm takes the security parameter  $\kappa$  and an index bit  $b \in \{0, 1\}$ , and outputs a hash key  $\text{hk}$ .
- $\text{Hash}(\text{hk}, x) \rightarrow y$ . This is a deterministic algorithm that takes as input  $x = (x_0, x_1) \in \{0, 1\}^\kappa \times \{0, 1\}^\kappa$  and outputs  $y \in \{0, 1\}^\kappa$ .

We require the following properties for an SSB hashing scheme:

- **Index hiding.** No  $\text{poly}(\kappa)$ -time adversary can distinguish between  $\text{hk}_0$  and  $\text{hk}_1$  by more than  $\text{negl}(\kappa)$ , where  $\text{hk}_b \leftarrow \text{HGen}(1^\kappa, b)$  for  $b \in \{0, 1\}$ .
- **Somewhere statistically binding.** We say that  $\text{hk}$  is statistically binding for index  $i \in \{0, 1\}$ , if there do *not* exist two values  $(x_0, x_1), (x'_0, x'_1) \in \{0, 1\}^\ell \times \{0, 1\}^\ell$  such that  $x_i \neq x'_i$  and  $\text{Hash}(\text{hk}, x) = \text{Hash}(\text{hk}, x')$ . We require that for both  $i \in \{0, 1\}$ ,

$$\Pr_{\text{HGen}}[\text{hk is statistically binding for } i : \text{hk} \leftarrow \text{HGen}(1^\kappa, i)] \geq 1 - \text{negl}(\kappa).$$

### 3 Formal Definition of Registration-Based Encryption

In this section, we formalize the new notion of RBE. After defining the “default” version of RBE, we define weakened forms of this primitive with a specific relaxation in the efficiency requirements. The goal of this relaxation is to base the (relaxed) RBE on more standard assumptions.

We start by defining the syntax of the default notion of RBE. We will then discuss the required compactness, completeness, and security properties.

**Definition 3.1** (Syntax of RBE). A *registration-based encryption* (RBE for short) scheme consists of PPT algorithms (Gen, Reg, Enc, Upd, Dec) working as follows. The Reg and Upd algorithms are performed by the key curator, which we call KC for short.

- **Generating common random string.** Some of the subroutines below will need a common random string  $\text{crs}$ , which could be sampled publicly using some public randomness beacon.  $\text{crs}$  of length  $\text{poly}(\kappa)$  is sampled at the beginning, for the security parameter  $\kappa$ .
- **Key generation.**  $\text{Gen}(1^\kappa) \rightarrow (\text{pk}, \text{sk})$ : The randomized algorithm  $\text{Gen}$  takes as input the security parameter  $1^\kappa$  and outputs a pair of public/secret keys  $(\text{pk}, \text{sk})$ . Note that these are only *public* and *secret* keys, not the *encryption* or *decryption* keys. The key generation algorithm is run by any honest party locally who wants to register itself into the system.
- **Registration.**  $\text{Reg}^{\text{aux}}(\text{crs}, \text{pp}, \text{id}, \text{pk}) \rightarrow \text{pp}'$ : The deterministic<sup>4</sup> algorithm  $\text{Reg}$  takes as input the common random string  $\text{crs}$ , current public parameter  $\text{pp}$ , a registering identity  $\text{id}$  and a public key  $\text{pk}$  (supposedly for the identity  $\text{id}$ ), and it outputs  $\text{pp}'$  as the updated public parameters. The  $\text{Reg}$  algorithm uses *read and write* oracle access to  $\text{aux}$  which will be updated into  $\text{aux}'$  during the process of registration.<sup>5</sup> (The system is initialized with public parameters  $\text{pp}$  and auxiliary information  $\text{aux}$  set to  $\perp$ .)
- **Encryption.**  $\text{Enc}(\text{crs}, \text{pp}, \text{id}, \text{m}) \rightarrow \text{ct}$ : The randomized algorithm  $\text{Enc}$  takes as input the common random string  $\text{crs}$ , a public parameter  $\text{pp}$ , a recipient identity  $\text{id}$  and a plaintext message  $\text{m}$  and outputs a ciphertext  $\text{ct}$ .
- **Update.**  $\text{Upd}^{\text{aux}}(\text{pp}, \text{id}) \rightarrow \text{u}$ : The deterministic algorithm  $\text{Upd}$  takes as input the current information  $\text{pp}$  stored at the KC and an identity  $\text{id}$ , has *read only* oracle access to  $\text{aux}$  and generates an *update* information  $\text{u}$  that can help  $\text{id}$  to decrypt its messages.<sup>6</sup>
- **Decryption.**  $\text{Dec}(\text{sk}, \text{u}, \text{ct})$ : The deterministic decryption algorithm  $\text{Dec}$  takes as input a secret key  $\text{sk}$ , an update information  $\text{u}$ , and a ciphertext  $\text{ct}$ , and it outputs a message  $\text{m} \in \{0, 1\}^*$  or in  $\{\perp, \text{GetUpd}\}$ . The special symbol  $\perp$  indicates a syntax error, while  $\text{GetUpd}$  indicates that more recent update information (than  $\text{u}$ ) might be needed for decryption.

**Remark 3.2** (Key curator is transparent). We emphasize that in the definition above the KC has no secret state. In fact, the registration and update operations are both *deterministic*. This makes KC’s job fully auditable. Even the generation of the  $\text{crs}$  (that is done before KC takes control of the server’s information) only needs common *random* strings (as opposed to a common *reference* string), so that can be generated using public randomness beacon as well.

We will now first describe the *completeness*, *compactness*, *efficiency* properties (under the completeness definition) and then we will describe the *security* properties. Both definitions are based on a security game that involves an “adversary” that tries to break the security, completeness, compactness, or efficiency properties by controlling how the identities (including the target/challenge identity) are registered and when the encryptions and decryptions happen.

<sup>4</sup>In our constructions, the algorithms  $\text{Reg}$ ,  $\text{Upd}$  and  $\text{Reg}$  are deterministic, and this feature makes our KC transparent (see Remark 3.2), so we keep the default definition based on deterministic version of these subroutines.

<sup>5</sup>This is the step that needs the identity of the registering  $\text{id}$  to be verified. This verification step is similar to IBE and its details are outside scope of this work.

<sup>6</sup>Looking ahead, we will aim for schemes that require the identity  $\text{id}$  to launch this request as rarely as possible. However, we note that this information  $\text{u}$  does not need to be kept secret for the security of the scheme, and any user can request this update without its identity being checked.

**Definition 3.3** (Completeness, compactness, and efficiency of RBE). For any interactive *computationally unbounded* adversary  $\text{Adv}$  that still has a limited  $\text{poly}(\kappa)$  round complexity, consider the following game  $\text{Comp}_{\text{Adv}}(\kappa)$  between  $\text{Adv}$  and a challenger  $\text{Chal}$ .

1. **Initialization.**  $\text{Chal}$  sets  $\text{pp} = \perp$ ,  $\text{aux} = \perp$ ,  $\text{u} = \perp$ ,  $\mathcal{D} = \emptyset$ ,  $\text{id}^* = \perp$ ,  $t = 0$ ,  $\text{crs} \leftarrow U_{\text{poly}(\kappa)}$  and sends the sampled  $\text{crs}$  to  $\text{Adv}$ .
2. Till  $\text{Adv}$  continues (which is at most  $\text{poly}(\kappa)$  steps), proceed as follows. At every iteration,  $\text{Adv}$  chooses exactly one of the actions below to be performed.
  - (a) **Registering new (non-target) identity.**  $\text{Adv}$  sends some  $\text{id} \notin \mathcal{D}$  and  $\text{pk}$  to  $\text{Chal}$ .  $\text{Chal}$  registers  $(\text{id}, \text{pk})$  by letting  $\text{pp} := \text{Reg}^{\text{aux}}(\text{crs}, \text{pp}, \text{id}, \text{pk})$  and  $\mathcal{D} := \mathcal{D} \cup \{\text{id}\}$ .
  - (b) **Registering the target identity.** If  $\text{id}^*$  was chosen by  $\text{Adv}$  already (i.e.,  $\text{id}^* \neq \perp$ ), skip this step. Otherwise,  $\text{Adv}$  sends some  $\text{id}^* \notin \mathcal{D}$  to  $\text{Chal}$ .  $\text{Chal}$  then samples  $(\text{pk}^*, \text{sk}^*) \leftarrow \text{Gen}(1^\kappa)$ , updates  $\text{pp} := \text{Reg}^{\text{aux}}(\text{crs}, \text{pp}, \text{id}^*, \text{pk}^*)$ ,  $\mathcal{D} := \mathcal{D} \cup \{\text{id}^*\}$ , and sends  $\text{pk}^*$  to  $\text{Adv}$ .
  - (c) **Encrypting for the target identity.** If  $\text{id}^* = \perp$  then skip this step. Otherwise,  $\text{Chal}$  sets  $t = t + 1$ , then  $\text{Adv}$  sends some  $m_t \in \{0, 1\}^*$  to  $\text{Chal}$  who then sets  $m'_t := m_t$  and sends back a corresponding ciphertext  $\text{ct}_t \leftarrow \text{Enc}(\text{crs}, \text{pp}, \text{id}^*, m_t)$  to  $\text{Adv}$ .
  - (d) **Decryption by target identity.**  $\text{Adv}$  sends a  $j \in [t]$  to  $\text{Chal}$ .  $\text{Chal}$  then lets  $m'_j = \text{Dec}(\text{sk}^*, \text{u}, \text{ct}_j)$ . If  $m'_j = \text{GetUpd}$ , then  $\text{Chal}$  obtains the update  $\text{u} = \text{Upd}^{\text{aux}}(\text{pp}, \text{id}^*)$  and then lets  $m'_j = \text{Dec}(\text{sk}^*, \text{u}, \text{ct}_j)$ .
3. The adversary  $\text{Adv}$  wins the game if there is some  $j \in [t]$  for which  $m'_j \neq m_j$ .

Let  $n = |\mathcal{D}|$  be the number of identities registered till a specific moment. We require the following properties to hold for any  $\text{Adv}$  (as specified above) and for *all* the moments (and so for all the values of  $\mathcal{D}$  and  $n = |\mathcal{D}|$  as well) during the game  $\text{Comp}_{\text{Adv}}(\kappa)$ .

- **Completeness.**  $\Pr[\text{Adv wins in } \text{Comp}_{\text{Adv}}(\kappa)] = \text{negl}(\kappa)$ .
- **Compactness of public parameters and updates.**  $|\text{pp}|, |\text{u}|$  are both  $\leq \text{poly}(\kappa, \log n)$ .
- **Efficiency of runtime of registration and update.** The running time of each invocation of  $\text{Reg}$  and  $\text{Upd}$  algorithms is at most  $\text{poly}(\kappa, \log n)$ . (This implies the compactness property.)
- **Efficiency of the number of updates.** The *total* number of invocations of  $\text{Upd}$  for identity  $\text{id}^*$  in Step 2d of the game  $\text{Comp}_{\text{Adv}}(\kappa)$  is at most  $O(\log n)$  for every  $n$  during  $\text{Comp}_{\text{Adv}}(\kappa)$ .

**Remark 3.4** (Other definitions based on quantifying compactness and efficiency parameters). Even though Definition 3.3 requires compactness and efficiency requirements using function  $c(\kappa, n) \leq \text{poly}(\kappa, \log n)$ , one can consider a more general definition that uses different (e.g., sublinear) functions to obtain various versions of RBE. In general, one can consider  $(c_1, \dots, c_5)$ -RBE schemes where  $c_i$ 's are functions of  $(\kappa, n)$ , and that functions  $c_1, c_2$  describe the compactness requirements (of public-key and updates), and functions  $c_3, c_4, c_5$  describe the efficiency requirements.

The following definition instantiates the general quantified definition of Remark 3.4 by relaxing the efficiency of the registration and keeping the other efficiency and compactness requirements to be as needed for Definition 3.3.

**Definition 3.5** (WE-RBE). A *registration weakly efficient RBE* (or WE-RBE for short) is defined similarly to Definition 3.3, where the specified  $\text{poly}(\kappa, \log n)$  runtime efficiency of the registration algorithm is not required anymore, but instead we require the registration time to be  $\text{poly}(\kappa, n)$ .

**Remark 3.6** (Denial of service attacks using fake ciphertexts). A class of malicious adversaries that are *not* captured by Definition 3.3 can potentially launch a “denial of service” attack against the efficiency of the decryption procedure as follows. Specifically, such malicious completeness adversary (that can also be seen as a form of “environment”) can cause an honest user to request too many updates by continually providing it with fake ciphertexts that seem to require an update for decryption. Here, we propose a generic approach for dealing with this issue. We can generalize the RBE primitive and allow the KC to have a secret state. This will take away the appealing transparency feature of the KC, but it will instead allow the KC to sign the public parameters, and those signed public parameters can then be included in the ciphertexts. Doing this will allow the decryption algorithm to detect fake ciphertexts that (maliciously) indicate that the population has grown beyond the last update, and that new update is needed for recent decryptions.

**Security.** For security, we require that no PPT adversary should be able to distinguish between encryptions of two messages (of equal lengths) made to a user who has registered honestly into the system, even if the adversary colludes and obtains the secret keys of all the other users. This is formalized by the adversary specifying a challenge identity and distinguishing between encryptions made to that identity. In order to prevent the adversary from winning trivially, we require that the adversary does not know any secret key for a public key registered for the challenge identity.

We present the formal definition only for the case of bit encryption, but any scheme achieving this level of security can be extended to arbitrary length messages using independent bit-by-bit encryption and a standard hybrid argument.

**Definition 3.7** (Security of RBE). For any interactive PPT adversary  $\text{Adv}$ , consider the following game  $\text{Sec}_{\text{Adv}}(\kappa)$  between  $\text{Adv}$  and a challenger  $\text{Chal}$ . (Steps that are different from the completeness definition are denoted with purple stars ( $\star\star$ ). Specifically, Steps 2c and 2d from Definition 3.3 are replaced by Step 3 below. Additionally, Step 3 from Definition 3.3 is replaced by Step 4 below.)

1. **Initialization.**  $\text{Chal}$  sets  $\text{pp} = \perp$ ,  $\text{aux} = \perp$ ,  $\mathcal{D} = \emptyset$ ,  $\text{id}^* = \perp$ ,  $\text{crs} \leftarrow U_{\text{poly}(\kappa)}$  and sends the sampled  $\text{crs}$  to  $\text{Adv}$ .
2. Till  $\text{Adv}$  continues (which is at most  $\text{poly}(\kappa)$  steps), proceed as follows. At every iteration,  $\text{Adv}$  chooses exactly one of the actions below to be performed.
  - (a) **Registering new (non-target) identity.**  $\text{Adv}$  sends some  $\text{id} \notin \mathcal{D}$  and  $\text{pk}$  to  $\text{Chal}$ .  $\text{Chal}$  registers  $(\text{id}, \text{pk})$  by letting  $\text{pp} := \text{Reg}^{\text{aux}}(\text{crs}, \text{pp}, \text{id}, \text{pk})$  and  $\mathcal{D} := \mathcal{D} \cup \{\text{id}\}$ .
  - (b) **Registering the target identity.** If  $\text{id}^*$  was chosen by  $\text{Adv}$  already (i.e.,  $\text{id}^* \neq \perp$ ), skip this step. Otherwise,  $\text{Adv}$  sends some  $\text{id}^* \notin \mathcal{D}$  to  $\text{Chal}$ .  $\text{Chal}$  then samples  $(\text{pk}^*, \text{sk}^*) \leftarrow \text{Gen}(1^\kappa)$ , updates  $\text{pp} := \text{Reg}^{\text{aux}}(\text{crs}, \text{pp}, \text{id}^*, \text{pk}^*)$ ,  $\mathcal{D} := \mathcal{D} \cup \{\text{id}^*\}$ , and sends  $\text{pk}^*$  to  $\text{Adv}$ .
3. ( $\star\star$ ) **Encrypting for the target identity.** If no  $\text{id}^*$  was chosen by  $\text{Adv}$  before (i.e.,  $\text{id}^* = \perp$ ) then  $\text{Adv}$  first sends some  $\text{id}^* \notin \mathcal{D}$  to  $\text{Chal}$ . Next,  $\text{Chal}$  generates  $\text{ct} \leftarrow \text{Enc}(\text{crs}, \text{pp}, \text{id}^*, b)$ , where  $b \leftarrow \{0, 1\}$  is a random bit, lets  $\mathcal{D} = \mathcal{D} \cup \{\text{id}^*\}$ , and sends  $\text{ct}$  to  $\text{Adv}$ .
4. ( $\star\star$ ) The adversary  $\text{Adv}$  outputs a bit  $b'$  and wins the game if  $b = b'$ .

We call an RBE scheme secure if  $\Pr[\text{Adv} \text{ wins in } \text{Sec}_{\text{Adv}}(\kappa)] < \frac{1}{2} + \text{negl}(\kappa)$  for any PPT  $\text{Adv}$ .

**Equivalence to other definitions.** One might consider a seemingly stronger security definition in which the adversary chooses its challenge identity from a *set* of previously chosen identities for which it does *not* know the keys. However, since the adversary can *guess* its own selection with probability  $1/\text{poly}(\kappa)$ , that definition becomes equivalent to Definition 3.7 above. Another seemingly stronger definition would allow the adversary to register even more identities after receiving the challenge ciphertext (and before answering the challenge), however this is again an equivalent definition as the information distributed in this extra step is simulatable by the adversary and thus not helpful to her.

**Choosing a registered or an unregistered identity.** Here we note a subtle aspect of Definition 3.7. If the adversary chooses Step 2b, it means that it is attacking a target identity that is registered in the system. Otherwise, the adversary shall choose the target identity in Step 3, which means that the attacked target identity is not even registered in the system. In both cases, we require that the adversary has negligible advantage in guessing the encrypted bit.

**Why not giving update oracle to adversary?** In Definition 3.7, we did not provide explicit oracle access to Upd subroutine for the adversary. The reason is that the adversary receives the  $\text{crs}$ , chooses the identities and receives the public keys. Moreover, KC is deterministic, has no secret state, and all the inputs it receives in maintaining the auxiliary information is  $\text{crs}$ , identities, and the public-keys. Therefore, throughout the attack, the adversary knows the exact state of  $(\text{pp}, \text{aux})$  hold by the key curator, and thus it can run the update operation itself. However, if one considers a KC with a *secret state* (perhaps for the goal of signing the public parameters as discussed in Remark 3.6) then the corresponding security definition shall give the adversary oracle access to the update subroutine.

**Remark 3.8** (Unauthorized registration of an identity). A malicious KC  $K^*$ , not following the protocol as modeled in the security game of Definition 3.7 can generate a pair of keys  $(\text{pk}, \text{sk})$  on its own and register  $\text{pk}$  on behalf of an identity  $\text{id}$ . By that,  $K^*$  can read messages that are subsequently encrypted to the identity  $\text{id}$ . Here we describe two approaches to tackle this problem.

1. **Bootstrapping public-key directories.** RBE schemes could be launched with respect to an external public-key directory  $D$ . Namely, only public-keys in  $D$  could be registered for matching identities. This way, a malicious key curator  $K^*$  can only register the *actual* public keys of the identities, and thus it is not able to decrypt the messages encrypted to them. Moreover, by also including (public) verification keys of the signatures by the identities in the public-key directory  $D$ , we can even prevent  $K^*$  from successfully registering any identities in the RBE scheme without having their permission (even by using their real public keys) as follows. Whenever the public parameter  $\text{pp}$  is updated, a signature of  $\text{pp}$  by the registering identity is added to the public auxiliary  $\text{aux}$ . This way, a public auditor can detect a fake registration.
2. **Proof of Knowledge.** An alternative method to prevent fake identity registrations is to use a similar approach to the one mentioned above, but replace the signature with a zero-knowledge proof of knowledge of an actual certificate from some trusted party (e.g., their driving licence information) that validates the ownership of an identity.

## 4 IO-Based Construction of RBE

In this section we present a formal construction of (efficient) RBE based on indistinguishability obfuscation and SSB hash functions (see Section 2 for formal definitions of the standard primitives used). We first describe the construction along the line of Definition 3.1 and then will prove its completeness, compactness, and security based on Definitions 3.3 and 3.7. We will then describe minor modifications that make the construction efficient according to Definition 3.5 (basically by not producing the updates in the registration).

**Notation on binary trees.** In our construction below,  $\text{Tree}$  is always a *full* binary tree (with  $2^i$  leaves for some  $i$ ), where the label of each node in  $\text{Tree}$  is calculated as the “hash” of its left and right children. We define the size of a tree  $\text{Tree}$  as the number of its *leaves*, denoted by  $\text{size}(\text{Tree})$  (so if  $\text{size}(\text{Tree}) = s$ , the total number of nodes will be  $2s - 1$ ), and we denote the root of  $\text{Tree}$  as  $\text{rt}(\text{Tree})$ , and we use  $\text{d}(\text{Tree})$  to refer to the depth of  $\text{Tree}$ . Since we assume that  $\text{Tree}$  is always a full tree, we always have  $2^{\text{d}(\text{Tree})} = \text{size}(\text{Tree})$ . When it is clear from the context, we use  $\text{rt}$  and  $\text{d}$  to denote the root and the depth of  $\text{Tree}$ .

**Simplifying assumption on lengths.** We note that without loss of generality, we can assume that public keys, secret keys and identities are all of the length security parameter  $\kappa$ .

**Construction 4.1** (RBE from IO and SSB Hashing). We will use an IO scheme  $(\text{Obf}, \text{Eval})$  and a SSB hash function system  $(\text{Hash}, \text{HGen})$  and a PKE scheme  $(\text{G}, \text{E}, \text{D})$ . Using them, we show how to implement the subroutines of RBE according to Definition 3.1.

- $\text{Stp}(1^\kappa) \rightarrow (\text{pp}_0, \text{aux}_0)$ . This algorithm outputs  $\text{pp}_0 = (\text{hk}_1, \dots, \text{hk}_\kappa)$  where each  $\text{hk}_i$  is sampled from  $\text{HGen}(1^\kappa, 0)$  and  $\text{aux} = \emptyset$  is empty.
- $\text{Reg}^{\text{aux}}(\text{pp}_n, \text{id}, \text{pk}) \rightarrow \text{pp}_{n+1}$ . This algorithm works as follows:
  1. Parse  $\text{aux} := ((\text{Tree}_1, \dots, \text{Tree}_\eta), (\text{id}_1, \dots, \text{id}_n))$  where the trees have corresponding depths  $\text{d}_1 > \text{d}_2 > \dots > \text{d}_\eta$ , and  $(\text{id}_1, \dots, \text{id}_n)$  is the order by which the current identities have registered.<sup>7</sup>
  2. Parse  $\text{pp}_n$  as a sequence  $((\text{hk}_1, \dots, \text{hk}_\kappa), (\text{rt}_1, \text{d}_1), \dots, (\text{rt}_\eta, \text{d}_\eta))$  where  $\text{rt}_i \in \{0, 1\}^\kappa$  represents the root of  $\text{Tree}_i$ , and  $\text{d}_i$  represents the depth of  $\text{Tree}_i$ .
  3. Create new tree  $\text{Tree}_{\eta+1}$  with leaves  $\text{id}, \text{pk}$  and set its root as  $\text{rt}_{\eta+1} := \text{Hash}(\text{hk}_1, \text{id} || \text{pk})$  and thus its depth would be  $\text{d}_{\eta+1} = 1$ .
  4. Let  $\mathcal{T} = \{\text{Tree}_1, \dots, \text{Tree}_{\eta+1}\}$ . (We will keep changing  $\mathcal{T}$  in steps below.)
  5. While there are two different trees  $\text{Tree}_L, \text{Tree}_R \in \mathcal{T}$  of the same depth  $\text{d}$ , same size  $s = 2^{\text{d}}$  (as our trees are always full binary trees), and roots  $\text{rt}_L, \text{rt}_R$ , do the following.
    - (a) Let  $\text{Tree}$  be a new tree of depth  $\text{d} + 1$  that contains  $\text{Tree}_L$  as its left subtree,  $\text{Tree}_R$  as its right subtree, and  $\text{rt} = \text{Hash}(\text{hk}_{\text{d}+1}, \text{rt}_L || \text{rt}_R)$  as its root.
    - (b) Remove both of  $\text{Tree}_L, \text{Tree}_R$  from  $\mathcal{T}$  and add  $\text{Tree}$  to  $\mathcal{T}$  instead.

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<sup>7</sup>Keeping this list is not necessary, but simplifies the presentation of the updates.

6. Let  $\mathcal{T} := (\text{Tree}_1, \dots, \text{Tree}_\zeta)$  be the final set of trees with depths  $d'_1 > \dots > d'_\zeta$  and roots  $rt'_1, \dots, rt'_\zeta$ . Set  $pp_{n+1}$  and  $aux$  as follows:

$$pp_{n+1} := ((hk_1, \dots, hk_\kappa), (rt'_1, d'_1), \dots, (rt'_\zeta, d'_\zeta)) \text{ and}$$

$$aux := (\mathcal{T}, (id_1, \dots, id_n, id_{n+1} = id)).$$

- $\text{Enc}(pp, id, m) \rightarrow ct$ : First parse  $pp := ((hk_1, \dots, hk_\kappa), (rt_1, d_1), \dots, (rt_\eta, d_\eta))$ . Generate programs  $P_1, \dots, P_\eta$  where each program  $P_i$  works as follows:

**Hardwired values:**  $rt_i, d_i, (hk_1, \dots, hk_{d_i}), m, id, r$  (the randomness)

**Input:**  $pth$

1. Parse  $pth := [(h_0^0, h_0^1), (h_1^0, h_1^1, b_1) \dots, (h_{d_i-1}^0, h_{d_i-1}^1, b_{d_i-1}), rt]$ .
2. If  $rt_i \neq rt$ , then output  $\perp$ .
3. If  $id \neq h_0^0$ , then output  $\perp$ .
4. If  $rt = \text{Hash}(hk_{d_i}, h_{d_i-1}^0 || h_{d_i-1}^1)$  and  $h_j^{b_j} = \text{Hash}(hk_j, h_{j-1}^0 || h_{j-1}^1)$  for all  $j \in [d_i - 1]$ , then output  $E(h_0^1, m; r)$  by using  $h_0^1$  as the public key and  $r$  as the randomness, otherwise output  $\perp$ .

Then, output  $ct := (pp, \text{Obf}(P_1), \dots, \text{Obf}(P_\eta))$  where  $\text{Obf}$  is IO obfuscation.

- $\text{Upd}^{\text{aux}}(pp, id) \rightarrow u$ : Letting  $aux := (\text{Tree}_1, \dots, \text{Tree}_\zeta)$  and letting  $i$  be the index of the tree that holds  $id$ , return the whole Merkle opening of the path that leads to  $id$  in  $\text{Tree}_i$ .
- $\text{Dec}(sk, u, ct) \rightarrow m$ : Parse  $ct = (pp, \bar{P}_1, \dots, \bar{P}_\eta)$ . Form  $m_i = \text{Dec}_{sk}(\bar{P}_i(u))$  for each program  $\bar{P}_i$ . Output the first  $m_i \neq \perp$ .

**Theorem 4.2.** *The RBE of Construction 4.1 satisfies the compactness, completeness properties according to Definition 3.3 and security according to Definition 3.7.*

In the rest of this section, we prove Theorem 4.2. Along the way, we describe the modifications that are needed to Construction 4.1 to make it efficient according to Definition 3.5.

## 4.1 Proofs of Completeness, Compactness and Efficiency

Completeness is straightforward. Below we sketch why compactness holds.

**Compactness of public parameters and updates.** The public parameter's format is of the form  $pp = ((hk_1, \dots, hk_\kappa), (rt_1, d_1), \dots, (rt_\eta, d_\eta))$  where  $rt_i \in \{0, 1\}^\kappa$ . Also, the identities are of length  $\kappa$ , so the depth of each tree is at most  $\kappa$  bits. It only remains to show that the *number* of trees at any moment is at most  $\log(n)$ . This is because the trees are *full* binary trees (of size  $2^{d_i}$ ) and the size of the trees are always different (otherwise, the registration step keeps merging them). Therefore,  $\eta \leq \log(n)$ , and so the length of the  $pp_n$  will be at most  $O(\kappa^2 + \kappa \cdot \log(n))$ . In fact, we can optimize this length to be at most  $O(\kappa \cdot \log(n))$  by only generating the hash keys when needed (i.e., when the registered population reaches  $2^k$ , we will generate  $hk_k$  and put it in the public parameter). Compactness of updates is trivial.

**Efficiency of runtime of registration and update.** The efficiency of registration follows from the fact that the total number of merges is at most  $\log n$ . The efficiency of update runtime can also be easily guaranteed by using an appropriate data structure that maps a given identity to the leafs containing it in each tree (e.g., we can use a Trie data structure for this purpose to get such list in minimal time over the input length).

All other measures of efficiency either follows trivially, or by the  $\log(n)$  upperbound on the number of merges.

## 4.2 Proof of Security

We now prove the security of Construction 4.1. We start by giving intuition about the security proof for a simple case. We will then give a detailed proof for the general case.

### 4.2.1 Simple Case of One User

Consider the case in which only one user has registered, and that the adversary wants to distinguish between encryptions of  $m \in \{0, 1\}$  made to that user. Let  $id^*$  be the identity of the user who has registered, and let  $(pk^*, sk^*) \leftarrow G(1^\kappa)$  be the pair of public/secret keys that the challenger  $\text{Chal}$  produced at the time of registration as per Definition 3.7. Since we have only one user, the public parameter is  $pp := \text{Hash}(hk, id^* || pk^*)$ , where  $hk \leftarrow \text{HGen}(1^\kappa, 0)$ . Recall that w.l.o.g., we have  $|id^*| = |pk^*| = |pp| = \kappa$ .

An encryption of a bit  $m \in \{0, 1\}$  to identity  $id^*$  is an IO obfuscation of the circuit  $P$  in Figure 1.

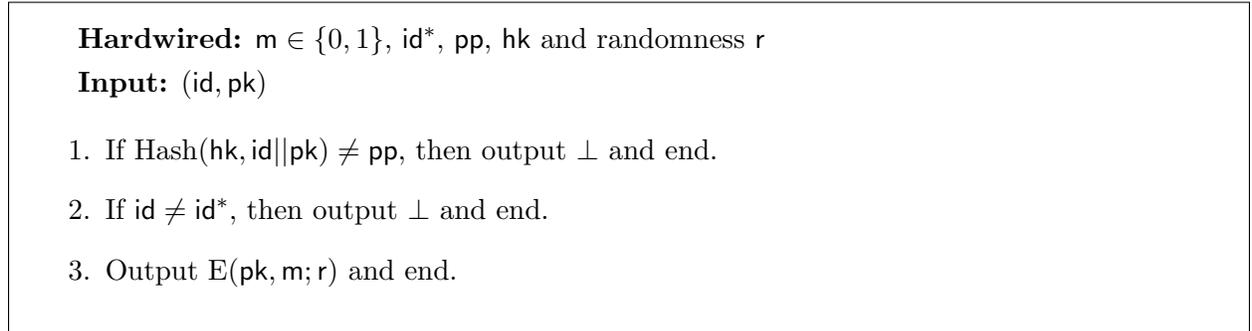


Figure 1: Circuit  $P$  used for encryption of  $m$  to identity  $id^*$

**Theorem 4.3** (Security). *For any  $id^*$  we have*

$$\text{Obf}(P[0, id^*, pp, hk, r]) \stackrel{c}{\approx} \text{Obf}(P[1, id^*, pp, hk, r]), \quad (1)$$

for  $(pk^*, sk^*) \leftarrow G(1^\kappa)$ ,  $hk \leftarrow \text{HGen}(1^\kappa, 0)$ ,  $pp := \text{Hash}(hk, id^* || pk^*)$ ,  $r \leftarrow \{0, 1\}^*$ .

**Roadmap for the proof of Theorem 4.3.** We first alter the circuit  $P$  to obtain a circuit  $P_1$ , which works similarly except that  $P_1$  checks whether or not its given input path is exactly  $(id^*, pk^*)$  (i.e., the already registered identity along with its public key); if not,  $P_1$  will return  $\perp$ , even if the two leaves  $(id, pk)$  correctly hashe to  $pp$ . If yes,  $P_1$  will encrypt the hardwired bit  $m$  under the public key  $pk^*$  and the hardwired randomness  $r$ . The circuit  $P_1$  is defined in Figure 2.

Equipped with this new circuit  $P_1$ , first in Lemma 4.4 we show that under  $P_1$  we may switch the underlying hardwired plaintext bit  $m$  from 0 to 1 while keeping the obfuscations of the resulting circuits indistinguishable. Then, in Lemma 4.5 we will show that for any fixed plaintext bit  $m$ , the obfuscations of  $P$  and  $P_1$  are indistinguishable. Lemmas 4.4 and 4.5 together imply Theorem 4.3.

We start by defining the circuit  $P_1$ , which is a modified version of  $P$ .

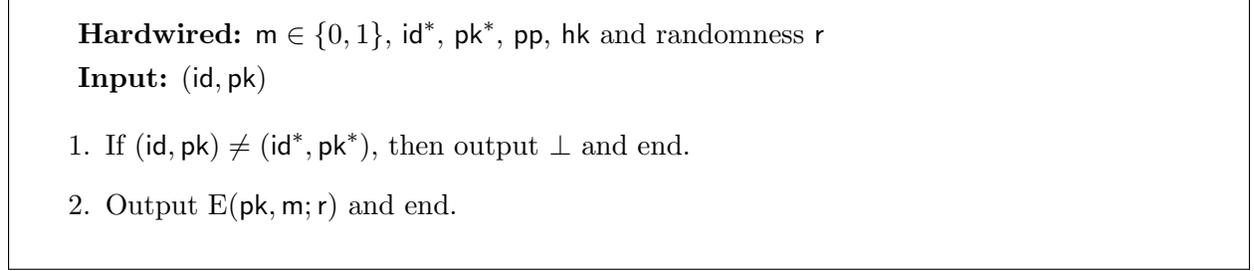


Figure 2: Circuit  $P_1$

We now formally show that under  $P_1$  we may switch the underlying plaintext bit while keeping their obfuscations indistinguishable.

**Lemma 4.4.** *For any  $id^*$  and  $hk$  we have*

$$\text{Obf}(P_1[0, id^*, pk^*, pp, hk, r]) \stackrel{c}{\approx} \text{Obf}(P_1[1, id^*, pk^*, pp, hk, r]), \quad (2)$$

where  $(pk^*, sk^*) \leftarrow G(1^\kappa)$ ,  $r \leftarrow \{0, 1\}^*$  and  $pp := \text{Hash}(hk, id^* || pk^*)$ .

*Proof.* Fix  $id^*$  and  $hk$ . We slightly change the circuit  $P_1$  into a circuit  $P_2$ , so that the circuit  $P_2$ , instead of getting  $m$ ,  $pk^*$  and  $r$  hardwired into itself, it gets the resulting ciphertext  $c^*$  hardwired, and it will return this ciphertext if the check inside the program holds. This new circuit  $P_2$  is defined in Figure 3.

Notice that for all fixed  $m \in \{0, 1\}$ ,  $id^*$ ,  $pk^*$ ,  $r$  and  $pp := \text{Hash}(hk, id^* || pk^*)$ ,

$$\text{Obf}(P_1[m, id^*, pk^*, pp, hk, r]) \stackrel{c}{\approx} \text{Obf}(P_2[id^*, pp, hk, c^*]), \quad (3)$$

where  $c^* := E(pk^*, m; r)$ . The reason behind Equation 3 is that the underlying two circuits are functionally equivalent, and so their obfuscations must be computationally indistinguishable by the property of IO.

We now show that under  $P_2$  we may switch the hardwired ciphertext from an encryption of zero to one, by relying on semantic security of the PKE. Formally,

$$\text{Obf}(P_2[id^*, pp, hk, c_0^*]) \stackrel{c}{\approx} \text{Obf}(P_2[id^*, pp, hk, c_1^*]), \quad (4)$$

for  $(pk^*, sk^*) \leftarrow G(1^\kappa)$ ,  $c_0^* \leftarrow E(pk^*, 0)$ ,  $c_1^* \leftarrow E(pk^*, 1)$ ,  $pp := \text{Hash}(hk, id^* || pk^*)$ . Equation 4 directly follows from the semantic security of the underlying public-key encryption scheme. Finally, note that Equations 4 and 3 imply Equation 2 of the lemma, and so we are done.  $\square$

**Hardwired:**  $\text{id}^*$ ,  $\text{pp}$ ,  $\text{hk}$  and  $\text{c}^*$

**Input:**  $(\text{id}, \text{pk})$

1. If  $(\text{id}, \text{pk}) \neq (\text{id}^*, \text{pk}^*)$ , then output  $\perp$  and end.
2. Output  $\text{c}^*$  and end.

Figure 3: Circuit  $P_2$

We now show that for any fixed plaintext  $m \in \{0, 1\}$ , the obfuscations of the two circuits  $P$  and  $P_1$  are computationally indistinguishable.

**Lemma 4.5.** *For fixed  $m \in \{0, 1\}$ ,  $\text{id}^* \in \{0, 1\}^\kappa$ ,  $\text{pk}^* \in \{0, 1\}^\kappa$  and randomness  $r$ , it holds that*

$$\text{Obf}(P[m, \text{id}^*, \text{pp}, \text{hk}, r]) \stackrel{c}{\approx} \text{Obf}(P_1[m, \text{id}^*, \text{pk}^*, \text{pp}, \text{hk}, r]), \quad (5)$$

where  $\text{hk} \leftarrow \text{HGen}(1^\kappa, 0)$  and  $\text{pp} := \text{Hash}(\text{hk}, \text{id}^* || \text{pk}^*)$ .

*Proof.* Let a hash key  $\text{hk}_1$  be sampled as follows:  $\text{hk}_1 \leftarrow \text{HGen}(1^\kappa, 1)$ . We show that Equation 5 will hold if  $\text{hk}$  is replaced with  $\text{hk}_1$ . This will complete our proof because by the index hiding property of  $(\text{HGen}, \text{Hash})$  we know  $\text{hk} \stackrel{c}{\approx} \text{hk}_1$ . Thus, it only remains to prove

$$\text{Obf}(P[m, \text{id}^*, \text{pk}^*, \text{pp}, \text{hk}_1, r]) \stackrel{c}{\approx} \text{Obf}(P_1[m, \text{id}^*, \text{pk}^*, \text{pp}, \text{hk}_1, r]), \quad (6)$$

where  $\text{hk}_1 \leftarrow \text{HGen}(1^\kappa, 1)$  and  $\text{pp} := \text{Hash}(\text{hk}_1, \text{id}^* || \text{pk}^*)$ . To prove Equation 6 we claim that the underlying two circuits are functionally equivalent; namely,

$$P[m, \text{id}^*, \text{pk}^*, \text{pp}, \text{hk}_1, r] \equiv P_1[m, \text{id}^*, \text{pk}^*, \text{pp}, \text{hk}_1, r]. \quad (7)$$

Note that by security definition of IO, Equation 7 implies Equation 6, and thus we just need to prove Equation 7. To prove equivalence of the circuits, assume to the contrary that there exists an input  $(\text{id}, \text{pk})$  for which we have  $P(\text{id}, \text{pk}) \neq P_1(\text{id}, \text{pk})$ . (Here for better readability we dropped the hardwired values.) By simple inspection, we can see that we have  $P(\text{id}, \text{pk}) \neq P_1(\text{id}, \text{pk})$  iff all the following conditions hold:

1.  $\text{Hash}(\text{hk}_1, (\text{id}, \text{pk})) = \text{pp}$ ; and
2.  $\text{id} = \text{id}^*$ ; and
3.  $\text{pk} \neq \text{pk}^*$ .

This, however, is a contradiction. By the somewhere statistical binding property of  $(\text{HGen}, \text{Hash})$  and by the fact that  $\text{hk}_1 \leftarrow \text{HGen}(1^\kappa, 1)$ , Conditions 1 and 2 imply  $\text{pk} = \text{pk}^*$ , a contradiction to Condition 3.  $\square$

### 4.2.2 General Case of Multiple Users

We will prove our security for the case in which at the time of encryption, we only have one tree (of any arbitrary depth). This is without loss of generality for the following reason. Recall that for encryption, if we have  $m$  roots, we obfuscate a circuit individually for each root. Suppose at the time of encryption, we have  $m$  trees with respective roots  $rt_1, \dots, rt_m$ . Then, between the two main hybrids which correspond to an encryption of zero and an encryption of one, we may consider  $m$  intermediate hybrids, where under the  $i$ th hybrid we encrypt 0 under the roots  $\{rt_1, \dots, rt_i\}$  and we encrypt 1 under the roots  $\{rt_{i+1}, \dots, rt_m\}$ . Thus, using a hybrid argument, the result will follow.

**Roadmap of the security proof.** We will define four hybrids, where the first hybrid corresponds to an encryption of bit 0 and the last hybrid corresponds to an encryption of bit 1. We will prove that the views of the adversary in each of the two adjacent hybrids are computationally indistinguishable.

**High-level proof sketch.** Let *Tree* be the underlying tree at the time of encryption. An encryption of a bit  $m$  to an identity  $id$  corresponds to an IO obfuscation of a circuit  $P$ , which takes as input a path, and which will release an encryption of  $m$  under a public key given as a leaf of the path, if the given path is “valid.” As a hybrid, we will consider a circuit  $P_1$ , which does all the checks that are already performed by  $P$ , but which also does the following: if the given path is not *present* in the tree, then  $P_1$  will return  $\perp$ , even if the path is valid. We will show that for any fixed bit  $m$ , if we encrypt  $m$  by obfuscating either the circuit  $P$  or  $P_1$ , the result will be indistinguishable. We will make use of the somewhere statistical binding and index hiding of the underlying hash function in order to prove this. Now under an obfuscation of  $P_1$ , one may easily switch the hardwired plaintext bit. The reason is that since under  $P_1$ , a given input path to the circuit must be present in the tree, and since the challenge identity  $id^*$  is registered only once (say under a public key  $pk$ ), one may consider a related circuit which, instead of hardwiring a plaintext bit  $m$ , it hardwires into itself an encryption  $c \leftarrow E(pk, m)$ . The rest follows by semantic security of the PKE scheme.

We now go over the formal proof. We start by defining some notation.

**Notation.** Consider a path  $pth := [(id, pk), (h_1^0, h_1^1, b_1), \dots, (h_{t-1}^0, h_{t-1}^1, b_{t-1}), rt]$  where  $rt$  is the root and  $id$  and  $pk$  are the two leaves and  $b_1, \dots, b_{t-1} \in \{\text{left}, \text{right}\}$ . For a tree *Tree* of depth  $t$ , we write  $pth \subseteq \text{Tree}$  if  $pth$  is a valid path in *Tree* in the usual sense. The procedure  $\text{Valid}(hk_1, \dots, hk_t, pth)$  checks if the given path is a ‘valid path’ according to the given hash keys  $hk_1, \dots, hk_t$  then it output  $\top$ , otherwise outputs  $\perp$ . For a path  $pth$  and interger  $i$  we write  $\text{Last}(pth, i)$  to refer to the last  $i$  node “elements” in  $pth$ . Note that we do not consider the left-or-right bits as part of this counting. For example, letting  $pth$  be as above,

$$\text{Last}(pth, 5) = ((h_{t-2}^0, h_{t-1}^1, b_{t-2}), (h_{t-1}^0, h_{t-1}^1, b_{t-1}), rt).$$

We also extend the notation  $\subseteq$  given above to define  $\text{Last}(pth, i) \subseteq \text{Tree}$  in the straightforward way.

**Hardwired:**  $m \in \{0, 1\}$ ,  $id^*$ ,  $rt$ ,  $hk_1, \dots, hk_t$  and randomness  $r$

**Input:**  $pth := [(id, pk), (h_1^0, h_1^1, b_1), \dots, (h_{t-1}^0, h_{t-1}^1, b_{t-1}), rt']$

1. If  $\text{id} \neq \text{id}^*$ ,  $\text{rt} \neq \text{rt}'$  or  $\text{Valid}(\text{hk}_1, \dots, \text{hk}_t, \text{pth}) \neq \top$ , then output  $\perp$  and end.
2. Output  $E(\text{pk}, \text{m}; r)$ .

Figure 4: Circuit P

**Circuit  $P_1$**

**Hardwired:**  $m \in \{0, 1\}$ ,  $\text{id}^*$ ,  $\text{pth}^*$ ,  $\text{rt}$ ,  $\text{hk}_1, \dots, \text{hk}_t$  and randomness  $r$

**Input:**  $\text{pth} := [(\text{id}, \text{pk}), (h_1^0, h_1^1, b_1), \dots, (h_{t-1}^0, h_{t-1}^1, b_t), \text{rt}']$

1. If  $\text{pth} = \text{pth}^*$ , then output  $E(\text{pk}, \text{m}; r)$  and end.
2. Else, output  $\perp$  and end.

Figure 5: Circuit  $P_1$

**Notation used in hybrids.** We will write  $\text{id}^* \leftarrow \text{Adv}(\text{hk}_1, \dots, \text{hk}_\kappa)$  to mean that the adversary  $\text{Adv}$  receives  $\text{pp} := (\text{hk}_1, \dots, \text{hk}_\kappa)$  as input, interacts with the challenger  $\text{Chal}$  as per Definition 3.7 and outputs  $\text{id}^*$  as the challenge identity.

- **Hybrid  $H_1$ : Encrypt  $m = 0$  using P.** The ciphertext  $\text{ct}$  given to the adversary is formed as follows.
  1. For  $j \in [\kappa]$  sample  $\text{hk}_j \leftarrow \text{HGen}(1^\kappa, 0)$ .
  2.  $\text{id}^* \leftarrow \text{Adv}(\text{hk}_1, \dots, \text{hk}_\kappa)$ .
  3.  $\text{ct} \leftarrow \text{Obf}(P[0, \text{id}^*, \text{rt}, \text{hk}_1, \dots, \text{hk}_t, r])$ , where  $\text{rt}$  is the root of the tree,  $t$  is the depth of the tree, and  $r \leftarrow \{0, 1\}^*$ .
- **Hybrid  $H_2$ : Encrypt  $m = 0$  using  $P_1$ .** The ciphertext  $\text{ct}$  given to the adversary is formed as follows.
  1. For  $j \in [\kappa]$  sample  $\text{hk}_j \leftarrow \text{HGen}(1^\kappa, 0)$ .
  2.  $\text{id}^* \leftarrow \text{Adv}^{\text{Reg}_{\text{sel}}, \text{Reg}_{\text{smp}}}(\text{hk}_1, \dots, \text{hk}_\kappa)$ .
  3.  $\text{ct} \leftarrow \text{Obf}(P_1[0, \text{id}^*, \text{pth}^*, \text{rt}, \text{hk}_1, \dots, \text{hk}_t, r])$ , where  $\text{pth}^*$  is the path in the tree leading to the challenge node,  $\text{rt}$  is the root of  $\text{pth}^*$ ,  $t$  is the depth of the tree, and  $r \leftarrow \{0, 1\}^*$ .
- **Hybrid  $H_3$ : Encrypt  $m = 1$  using  $P_1$ .** The ciphertext  $\text{ct}$  given to the adversary is formed as follows.
  1. For  $j \in [\kappa]$  sample  $\text{hk}_j \leftarrow \text{HGen}(1^\kappa, 0)$ .
  2.  $\text{id}^* \leftarrow \text{Adv}(\text{hk}_1, \dots, \text{hk}_\kappa)$ .

3.  $\text{ct} \leftarrow \text{Obf}(P_1[1, \text{id}^*, \text{pth}^*, \text{rt}, \text{hk}_1, \dots, \text{hk}_t, r])$ , where  $\text{pth}^*$  is the path in the tree leading to the challenge node,  $\text{rt}$  is the root of  $\text{pth}^*$ ,  $t$  is the depth of the tree, and  $r \leftarrow \{0, 1\}^*$ .

• **Hybrid  $H_4$ : Encrypt  $m = 1$  using  $P$ .** The ciphertext  $\text{ct}$  given to the adversary is formed as follows.

1. For  $j \in [\kappa]$  sample  $\text{hk}_j \leftarrow \text{HGen}(1^\kappa, 0)$ .
2.  $\text{id}^* \leftarrow \text{Adv}(\text{hk}_1, \dots, \text{hk}_\kappa)$ .
3.  $\text{ct} \leftarrow \text{Obf}(P[1, \text{id}^*, \text{rt}, \text{hk}_1, \dots, \text{hk}_t, r])$ , where  $\text{rt}$  is the root of the underlying tree,  $t$  is the depth of the tree, and  $r \leftarrow \{0, 1\}^*$ .

**Notation.** We use  $\text{ct}\langle H_i \rangle$  to denote the value of the ciphertext  $\text{ct}$  in Hybrid  $H_i$ .

**Lemma 4.6.** *We have,*

1.  $\text{ct}\langle H_1 \rangle \stackrel{c}{\approx} \text{ct}\langle H_2 \rangle$ ,
2.  $\text{ct}\langle H_3 \rangle \stackrel{c}{\approx} \text{ct}\langle H_4 \rangle$ .

*Proof.* We will prove Part 1, and the proof for Part 2 will be exactly the same.

Recall that in hybrid  $H_1$  we encrypt  $m = 0$  by obfuscating  $P$  and that in hybrid  $H_2$  we encrypt  $m = 0$  by obfuscating  $P_1$ . Let  $t$  be the depth of the tree at the time of encryption.

We will define intermediate hybrids  $P_{2,i}$  for  $i \in [2t+1]$ , and we will show  $P \equiv P_{2,1}$ ,  $P_1 \equiv P_{2,2t+1}$  and for all  $i \in [2t]$ ,  $\text{Obf}[P_{2,i}] \stackrel{c}{\approx} \text{Obf}[P_{2,i+1}]$ . These circuit programs are given in Figure 6.

Informally, the program  $P_{2,i}$  works as follows: it checks whether its given path is “correct” and whether, in addition, the last  $i$  elements of the path are in accordance with the challenge path  $\text{pth}^*$  that was hardwired into the program. For example, if  $i = 5$ , then the root of the path and the two levels below it (five nodes in total) should match the corresponding nodes in the challenge path  $\text{pth}^*$ . If both these conditions hold, then  $P_{2,i}$  will encrypt the hardwired plaintext bit ( $m = 0$ ) using the public key provided in the corresponding leaf of the path.

We will now define a Hybrid  $H_{2,i}$  below, which uses program  $P_{2,i}$ .

• **Hybrid  $H_{2,i}$ : Encrypt  $m = 0$  using  $P_{2,i}$ .** The given ciphertext  $\text{ct}$  is as:

1. For  $j \in [\kappa]$  sample  $\text{hk}_j \leftarrow \text{HGen}(1^\kappa, 0)$ .
2.  $\text{id}^* \leftarrow \text{Adv}(\text{hk}_1, \dots, \text{hk}_\kappa)$ .
3.  $\text{ct} \leftarrow \text{Obf}(P_{2,i}[0, \text{id}^*, \text{pth}^*, \text{rt}, \text{hk}_1, \dots, \text{hk}_t, r])$ , where  $\text{pth}^*$  is the challenge path in the system,  $\text{rt}$  is the root of  $\text{pth}^*$ ,  $t$  is the depth of the tree, and  $r \leftarrow \{0, 1\}^*$ .

First, by inspection we can see that  $\text{ct}\langle H_1 \rangle \stackrel{c}{\approx} \text{ct}\langle H_{2,1} \rangle$  and  $\text{ct}\langle H_2 \rangle \stackrel{c}{\approx} \text{ct}\langle H_{2,2t+1} \rangle$ . This is because the underlying two circuits  $P$  and  $P_{2,1}$  are functionally equivalent. Same holds for  $P_1$  and  $P_{2,2t+1}$ .

Thus, for any fixed  $w \in [2t]$  we just need to prove

$$\text{ct}\langle H_{2,w} \rangle = \text{ct}\langle H_{2,w+1} \rangle. \quad (8)$$

Below, we fix  $w \in [2t]$ . To prove Equation 8, we introduce two hybrids  $H'_{2,w}, H'_{2,w+1}$  and show

$$\text{ct}\langle H_{2,w} \rangle \stackrel{c}{\approx} \text{ct}\langle H'_{2,w} \rangle \stackrel{c}{\approx} \text{ct}\langle H'_{2,w+1} \rangle \stackrel{c}{\approx} \text{ct}\langle H_{2,w+1} \rangle. \quad (9)$$

This will establish Equation 8.

Informally, the hybrids  $H'_{2,w}$  and  $H'_{2,w+1}$  are defined similarly to  $H_{2,w}$  and  $H_{2,w+1}$ , except that one of the many hash keys is now sampled in a different way, in order to make some binding property happen.

For  $z \in \{w, w+1\}$ , the hybrid  $H'_{2,z}$  is defined as follows.

• **Hybrid  $H'_{2,z}$  for  $z \in \{w, w+1\}$ .** The given ciphertext  $\text{ct}$  is formed as follows.

1. Let  $q := t - \lfloor \frac{w}{2} \rfloor - 1$ . Intuitively,  $q$  denotes the level index in the tree for which we want to use a different hash key. For all  $i \in [\kappa] \setminus \{q\}$ : sample  $\text{hk}'_i \leftarrow \text{HGen}(1^\kappa, 0)$ . Sample

$$\text{hk}'_q \leftarrow \text{HGen}(1^\kappa, v), \text{ where } v := (w+1) \bmod 2.$$

2.  $\text{id}_1^* \leftarrow \text{Adv}(\text{hk}'_1, \dots, \text{hk}'_\kappa)$ .
3.  $\text{ct} \leftarrow \text{Obf}(\text{P}_{2,i}[0, \text{id}_1^*, \text{pth}_1^*, \text{rt}_1, \text{hk}'_1, \dots, \text{hk}'_t, r])$ , where  $\text{pth}_1^*$  is the challenge path in the system,  $\text{rt}_1$  is the root of  $\text{pth}_1^*$  and  $r \leftarrow \{0, 1\}^*$ .

Toward proving Equation 9, first note that by the index hiding property of  $(\text{HGen}, \text{Hash})$  we have  $\text{ct}\langle H_{2,w} \rangle \stackrel{c}{\approx} \text{ct}\langle H'_{2,w} \rangle$  and  $\text{ct}\langle H_{2,w+1} \rangle \stackrel{c}{\approx} \text{ct}\langle H'_{2,w+1} \rangle$ . Thus, it remains to prove

$$\text{ct}\langle H'_{2,w} \rangle \stackrel{c}{\approx} \text{ct}\langle H'_{2,w+1} \rangle. \quad (10)$$

To prove Equation 10, we claim that the underlying two programs are equivalent,

$$\text{P}_{2,w}[0, \text{id}_1^*, \text{pth}_1^*, \text{rt}_1, \text{hk}'_1, \dots, \text{hk}'_t, r] = \text{P}_{2,w+1}[0, \text{id}_1^*, \text{pth}_1^*, \text{rt}_1, \text{hk}'_1, \dots, \text{hk}'_t, r]. \quad (11)$$

By IO security, Equation 11 implies Equation 10, and thus we just need to prove Equation 11. To prove equivalence of the two circuits in Equation 11, assume to the contrary that there exists an input  $\text{pth}$  for which we have  $\text{P}_{2,w}(\text{pth}) \neq \text{P}_{2,w+1}(\text{pth})$ . (Here for better readability we dropped the hardwired values.) By simple inspection we can see that we have  $\text{P}_{2,w}(\text{pth}) \neq \text{P}_{2,w+1}(\text{pth})$  iff all the following conditions hold:

1.  $\text{Valid}(\text{hk}'_1, \dots, \text{hk}'_t, \text{pth}) = \top$ ; and
2.  $\text{Last}(\text{pth}, w) \subseteq \text{pth}_1^*$ ; and
3.  $\text{Last}(\text{pth}, w+1) \not\subseteq \text{pth}_1^*$ .

This, however, is a contradiction because by the somewhere statistical binding property of  $(\text{KGen}, \text{Hash})$  and by the way in which we have sampled  $\text{hk}'_q$ , Conditions 1 and 2 contradict Condition 3.  $\square$

**Description of Circuit  $\text{P}_{2,i}$ .**

**Hardwired:**  $m \in \{0, 1\}$ ,  $\text{id}_1^*$ ,  $\text{pth}_1^*$ ,  $\text{rt}_1$ ,  $\text{hk}_1, \dots, \text{hk}_t$  and randomness  $r$

**Input:**  $\text{pth} := [(\text{id}, \text{pk}), (\text{h}_1^0, \text{h}_1^1, b_1), \dots, (\text{h}_{t-1}^0, \text{h}_{t-1}^1, b_t), \text{rt}']$

1. If  $\text{id} \neq \text{id}^*$  or  $\text{rt} \neq \text{rt}'$  or  $\text{Valid}(\text{hk}_1, \dots, \text{hk}_t, \text{pth}) \neq \top$ , then output  $\perp$  and end.
2. If  $\text{Last}(\text{pth}, i) \subseteq \text{pth}^*$ , then output  $E(\text{pk}, \text{m}; r)$  and end.
3. Otherwise, output  $\perp$  and end.

Figure 6: Circuit  $P_{2,i}$  for  $i \in [\ell]$

**Lemma 4.7.**  $\text{ct}\langle H_2 \rangle \stackrel{c}{\approx} \text{ct}\langle H_3 \rangle$ .

*Proof.* The proof is similar to the proof of Lemma 4.4. □

## 5 Basing Weakly-Efficient RBE on Standard Assumptions

In this section, we describe our construction of RBE based on *hash garbling* and is inspired by our IO based construction from previous section. This notion and its construction has been implicit in prior works [CDG<sup>+</sup>17, DG17], and it was shown [DG17, DGHM18, BLSV18] that hash garbling can be realized based on CDH, Factoring or LWE assumptions. Specifically, implicit in these prior works are constructions of hash garbling based on hash encryption and garbled circuits. Below, we abstract out this notion and use it in our work directly. This abstract primitive significantly simplifies exposition.

**Definition 5.1** (Hash garbling). A *hash garbling* scheme consists of four PPT algorithms HGen, Hash, HG, and HInp, defined as follows.

- $\text{HGen}(1^\kappa, 1^\ell) \rightarrow \text{hk}$ . This algorithm takes the security parameter  $\kappa$  and an output length parameter  $1^\ell$  for  $\ell \leq \text{poly}(\kappa)$ , and outputs a hash key  $\text{hk}$ . (HGen runs in  $\text{poly}(\kappa)$  time.)
- $\text{Hash}(\text{hk}, x) = y$ . This takes  $\text{hk}$  and  $x \in \{0, 1\}^\ell$  and outputs  $y \in \{0, 1\}^\kappa$ .
- $\text{HG}(\text{hk}, C, \text{stt}) \rightarrow \tilde{C}$ . This algorithm takes a hash key  $\text{hk}$ , a circuit  $C$ , and a secret state  $\text{stt} \in \{0, 1\}^\kappa$  as input and outputs a circuit  $\tilde{C}$ .
- $\text{HInp}(\text{hk}, y, \text{stt}) \rightarrow \tilde{y}$ . This algorithm takes a hash key  $\text{hk}$ , a value  $y \in \{0, 1\}^\kappa$ , and a secret state  $\text{stt}$  as input and outputs  $\tilde{y}$ .

We require the following properties for a hash garbling scheme:

- **Correctness.** For all  $\kappa, \ell$ ,  $\text{hk} \leftarrow \text{HGen}(1^\kappa, 1^\ell)$ , circuit  $C$ , input  $x \in \{0, 1\}^\ell$ ,  $\text{stt} \in \{0, 1\}^\kappa$ ,  $\tilde{C} \leftarrow \text{HG}(\text{hk}, C, \text{stt})$  and  $\tilde{y} \leftarrow \text{HInp}(\text{hk}, \text{Hash}(\text{hk}, x), \text{stt})$ , then  $\tilde{C}(\tilde{y}, x) = C(x)$ .
- **Security.** There exists a PPT simulator  $\text{Sim}$  such that for all  $\kappa, \ell$  (recall that  $\ell$  is polynomial in  $\kappa$ ) and PPT (in  $\kappa$ )  $\mathcal{A}$  we have that

$$(\text{hk}, x, \tilde{C}, \tilde{y}) \stackrel{c}{\approx} (\text{hk}, x, \text{Sim}(\text{hk}, x, 1^{|\tilde{C}|}, C(x))),$$

where  $\text{hk} \leftarrow \text{HGen}(1^\kappa, 1^\ell)$ ,  $(C, x) \leftarrow \mathcal{A}(\text{hk})$ ,  $\text{stt} \leftarrow \{0, 1\}^\kappa$ ,  $\tilde{C} \leftarrow \text{HG}(\text{hk}, C, \text{stt})$  and  $\tilde{y} \leftarrow \text{HInp}(\text{hk}, \text{Hash}(\text{hk}, x), \text{stt})$ .

**Notation on binary trees.** Just like the IO construction, in our construction below,  $\text{Tree}$  is a *full* binary tree where the label of each node in  $\text{Tree}$  is calculated as the hash of its left and right children and, now additionally, with an extra identity. Looking ahead, this identity will be the largest identity among the users registered in the left child. (Such information is useful if one wants to a binary search of an identity over this tree.) Just as in the IO-based construction, we define the size of a tree  $\text{Tree}$  as the number of its *leaves*, denoted by  $\text{size}(\text{Tree})$ , and we denote the root of  $\text{Tree}$  as  $\text{rt}(\text{Tree})$ , and use  $\text{d}(\text{Tree})$  to refer to the depth of  $\text{Tree}$ . Again, when  $\text{Tree}$  is clear from the context, we use  $\text{rt}$  and  $\text{d}$  to denote the root and the depth of  $\text{Tree}$ .

Before describing the construction, recall that without loss of generality, we can assume that public keys, secret keys, and identities, are all of length security parameter  $\kappa$ .

**Comparison with Construction 4.1 using signs (=) and (\*\*).** To help the reader familiar with Construction 4.1, we have denoted the steps that are identical to Construction 4.1 by (=) and the steps that are significantly *different* by (\*\*). Other steps are close but not identical.

**Construction 5.2** (Construction of RBE from hash garbling). We will use a hash garbling scheme ( $\text{HGen}, \text{Hash}, \text{HG}, \text{HIInp}$ ) and a public key encryption scheme ( $\text{G}, \text{E}, \text{D}$ ). Using them we show how to implement the subroutines of RBE according to Definition 3.1.

- $\text{Stp}(1^\kappa) \rightarrow (\text{pp}_0)$ , where  $\text{pp}_0 = \text{hk}$  is sampled from  $\text{HGen}(1^\kappa, 1^{3\kappa})$ .
- $\text{Reg}^{\text{[aux]}}(\text{pp}_n, \text{id}, \text{pk}) \rightarrow \text{pp}_{n+1}$ . This algorithm works as follows:
  1. (=) Parse  $\text{aux}_n := (\{\text{Tree}_1, \dots, \text{Tree}_\eta\}, (\text{id}_1, \dots, \text{id}_n))$  where the trees have corresponding depths  $\text{d}_1 > \text{d}_2 \dots > \text{d}_\eta$ , and  $(\text{id}_1, \dots, \text{id}_n)$  is the order the identities registered.<sup>8</sup>
  2. Parse  $\text{pp}_n$  as a sequence  $(\text{hk}, (\text{rt}_1, \text{d}_1), \dots, (\text{rt}_\eta, \text{d}_\eta))$  where  $\text{rt}_i \in \{0, 1\}^\kappa$  represents the root of tree  $\text{Tree}_i$  and  $\text{d}_i$  represents the depth of  $\text{Tree}_i$ .
  3. Create a new tree  $\text{Tree}_{\eta+1}$  with leaves  $\text{id}, \text{pk}$  and set its root as  $\text{rt}_{\eta+1} \leftarrow \text{Hash}(\text{hk}, \text{id} \parallel \text{pk} \parallel 0^\kappa)$  and thus its depth would be  $\text{d}_{\eta+1} = 1$ .
  4. (=) Let  $\mathcal{T} = \{\text{Tree}_1, \dots, \text{Tree}_{\eta+1}\}$ . (We will keep changing  $\mathcal{T}$  in step below.)
  5. While there are two different trees  $\text{Tree}_L, \text{Tree}_R \in \mathcal{T}$  of the same depth  $\text{d}$  and size  $s = 2^{\text{d}}$  (recall that our trees are always full binary trees).
    - (a) Obtain new Tree of depth  $\text{d} + 1$  by merging the two trees  $\text{Tree}_L$  and  $\text{Tree}_R$  as follows.
    - (b) (\*\*) Let  $\text{id}_1 \dots \text{id}_{n'}$  and  $\text{pk}_1 \dots \text{pk}_{n'}$  be the identities and public keys of  $n'$  users in both trees  $\text{Tree}_L$  and  $\text{Tree}_R$  combined in *sorted order* according to identities.
    - (c) For each  $i \in [n']$ , let  $h_{0,i} := \text{Hash}(\text{hk}, \text{id}_i \parallel \text{pk}_i \parallel 0^\kappa)$ .
    - (d) (\*\*) Next for each  $j \in \{1, \dots, \log n'\}$  and  $k \in \{0, \dots, (n'/2^j) - 1\}$ , let

$$h_{j,k} = \text{Hash}(\text{hk}, h_{j-1,2k} \parallel h_{j-1,2k+1} \parallel \text{id}[j, k])$$

where  $\text{id}[j, k]$  is the largest identity in the left child (which is the node with label  $h_{j-1,2k}$ ); namely  $\text{id}[j, k] = \text{id}_{(2^{k+1}) \cdot 2^{j-1}}$ . This completes the description of  $\text{Tree}$ .

- (e) (=) Remove both of  $\text{Tree}_L, \text{Tree}_R$  from  $\mathcal{T}$  and add  $\text{Tree}$  to  $\mathcal{T}$  instead.

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<sup>8</sup>Keeping this list is not necessary, but simplifies the presentation of the updates.

6. Let  $\mathcal{T} = \{\text{Tree}_1, \dots, \text{Tree}_\zeta\}$  where  $d'_1 > \dots > d'_\zeta$  is their corresponding depth and  $rt'_1, \dots, rt'_\zeta$  is their corresponding roots. Set  $\text{pp}_{n+1}, \text{aux}_{n+1}$  as

$$\text{aux}_{n+1} = (\mathcal{T}, (\text{id}_1, \dots, \text{id}_n, \text{id}_{n+1} = \text{id})), \text{pp}_{n+1} = (\text{hk}, (rt'_1, d'_1), \dots, (rt'_\zeta, d'_\zeta)).$$

- $\text{Enc}(\text{pp}, \text{id}, \text{m}) \rightarrow \text{ct}$ :

1. Parse  $\text{pp} := (\text{hk}, (rt_1, d_1), \dots, (rt_\eta, d_\eta))$ .
2. For each  $i \in \{1, \dots, \eta\}$  and  $j \in \{1, \dots, d_i\}$ , sample  $\text{stt}_{i,j} \leftarrow \{0, 1\}^\kappa$  and generate  $\tilde{P}_{i,j} \leftarrow \text{HG}(\text{hk}, P_{i,j}, \text{stt}_{i,j})$ , where  $P_{i,j}$  is explained below.
3. For each  $i \in [\eta]$  obtain  $\tilde{y}_{i,1} \leftarrow \text{HInp}(\text{hk}, rt_i, \text{stt}_{i,1})$ .
4. Output the ciphertext  $\text{ct} = (\text{pp}, \{\tilde{P}_{i,j}\}_{i,j}, \{\tilde{y}_{i,1}\}_i)$ .

The program  $P_{i,j}$  works as follows:

**Hardwired values:**  $rt_i, d_i, \text{hk}, \text{m}, \text{id}, r, \text{stt}_{i,j+1}$  (where  $\text{stt}_{i,d_i+1} = \perp$ )

**Input:**  $a||b||\text{id}^*$

1. If  $\text{id}^* = 0^\kappa$ <sup>9</sup> and  $a = \text{id}$  then output  $E(b, \text{m}; r)$ .
  2. If  $\text{id}^* = 0^\kappa$  and  $a \neq \text{id}$  then output  $\perp$ .
  3. If  $\text{id} > \text{id}^*$  then output  $\text{HInp}(\text{hk}, b, \text{stt}_{i,j+1})$ , else output  $\text{HInp}(\text{hk}, a, \text{stt}_{i,j+1})$ .
- $\text{Upd}^{\text{aux}}(\text{pp}, \text{id}) \rightarrow \text{u}$ : If  $\text{id}$  is a leaf in a tree of  $\text{aux}$ , say  $\text{Tree}$ , return the *whole* Merkle opening pth of leaf  $\text{id}$  and its sibling  $\text{pk}$  to the root  $\text{rt}(\text{Tree})$ . Otherwise, return  $\perp$ .
  - $\text{Dec}(\text{sk}, \text{u}, \text{ct}) \rightarrow \text{m}$ : Parse  $\text{ct} = (\text{pp}, \{\tilde{P}_{i,j}\}_{i,j}, \{\tilde{y}_{i,1}\}_i)$  and  $\text{u} := (z_1 \dots z_{d_{i^*}})$ . Let  $i^*$  be the index of the tree that holds the corresponding identity.<sup>10</sup> Decryption proceeds as follows:
    1. For  $j = \{1 \dots d_{i^*} - 1\}$  do
      - $\tilde{y}_{i^*,j+1} = \tilde{P}_{i^*,j}(\tilde{y}_{i^*,j}, z_j)$ .
    2. Let  $\text{ct} := \tilde{P}_{i^*,d_{i^*}}(\tilde{y}_{i^*,d_{i^*}}, z_{d_{i^*}})$ .
    3. Output  $D(\text{sk}, \text{ct})$ .

**Theorem 5.3.** *The RBE of Construction 5.2 satisfies the compactness, completeness (Definition 3.3), and security (Definition 3.7) properties.*

In the rest of this section, we prove Theorem 5.3. The completeness and compactness properties are proved similar to those of Construction 4.1. We can again verify that over the course of the system's execution, the tree that holds a user  $\text{id}$ , will not be merged with other trees more than  $\log n$  times. (Each merge increases the depth of the tree by one, and the depth cannot bypass  $\log n$ .) We may use this fact to conclude all the efficiency features for the RBE scheme.

In the rest of this section, we focus on proving security.

<sup>9</sup>Without loss of generality we assume that no user is assigned the identity  $0^\kappa$ .

<sup>10</sup>Alternatively, we may perform this with respect to all  $i^*$ , which is up to the number of trees in the system.

## 5.1 Proof of Security

Similar to our presentation of the proof of Construction 4.1, here also we first start by giving the proof for the case in which only *one* user has registered. We will then present the general proof.

<p><b>Hardwired:</b> <math>rt, hk, m \in \{0, 1\}, id', r</math> and <math>stt</math></p> <p><b>Input:</b> <math>(id, pk, id^*)</math></p> <ol style="list-style-type: none"> <li>1. If <math>id^* \neq 0^\kappa</math> or <math>id \neq id'</math>, then output <math>\perp</math> and end.</li> <li>2. Output <math>E(pk, m; r)</math> and end.</li> </ol>
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Figure 7: Circuit P used for encryption of  $m$  to identity  $id'$

**Theorem 5.4 (Security).** *For any identity  $id'$  we have*

$$(HG(hk, P_0, stt), HInp(hk, rt, stt)) \stackrel{c}{\approx} (HG(hk, P_1, stt), HInp(hk, rt, stt)) \quad (12)$$

where  $hk \leftarrow HGen(1^\kappa, 1^{3\kappa})$ ,  $stt \leftarrow \{0, 1\}^\kappa$ ,  $(pk, sk) \leftarrow G(1^\kappa)$ ,  $rt := Hash(hk, (id', pk, 0^\kappa))$  and for  $m \in \{0, 1\}$  the circuit program  $P_m$  is defined as

$$P_m := P[rt, hk, m, id', r, stt]. \quad (13)$$

*Proof.* For  $m \in \{0, 1\}$  let  $ct_m$  denote the challenge ciphertext, namely

$$ct_m := (HG(hk, P_0, stt), HInp(hk, rt, stt)), \quad (14)$$

where all the variables are sampled as in the theorem. We need to show  $ct_0 \stackrel{c}{\approx} ct_1$ . By simulation security of the hash garbling scheme, for both  $m \in \{0, 1\}$  we have

$$ct_m \stackrel{c}{\approx} Sim(hk, (id', pk, 0^\kappa), 1^{|P_m|}, E(pk, m; r)). \quad (15)$$

By semantic security of the underlying public-key encryption scheme we have

$$Sim(hk, (id', pk, 0^\kappa), 1^{|P_0|}, E(pk, 0; r)) \stackrel{c}{\approx} Sim(hk, (id', pk, 0^\kappa), 1^{|P_1|}, E(pk, 1; r)), \quad (16)$$

and so we obtain  $ct_0 \stackrel{c}{\approx} ct_1$ . □

**Proof for the general case.** As in the proof in Section 4.2.2 we may assume that at the time of encryption we have only one tree. The proof for the case of multiple trees is the same.

*Proof.* Suppose at the time of encryption the underlying tree with root  $rt$  has depth  $d$ . In the sequel we shall write  $P_j$  for  $j \in [d]$  to refer to the circuit program  $P_{1,j}$  described in our RBE construction. That is,

$$P_1 \equiv P_{1,1}[rt, d, hk, m, id, r, stt_{1,2}], \quad (17)$$

and for  $j > 1$

$$P_j \equiv P_{1,j}[rt, d, hk, m, id, r, stt_{1,j+1}], \quad (18)$$

where all the variables above are as in the encryption of the construction.

For  $j \in [d]$  we define  $\text{rt}_j$  to be the node in the  $j$ th level of the tree (where we consider the root as level one), whose sub-tree contains the leaf with label  $\text{id}$ .<sup>11</sup> For example, if the path leading to  $\text{id}$  is

$$[(\text{id}, \text{pk}, 0^\kappa), (a_1, b_1, \text{id}_1, \text{left}), \dots, (a_{d-1}, b_{d-1}, \text{id}_{d-1}, \text{right}), \text{rt}],$$

then  $\text{rt}_3 = b_{d-1}$ . For  $j > 1$  we define

$$\tilde{y}_j := \text{HInp}(\text{hk}, \text{rt}_j, \text{stt}_{1,j}). \quad (19)$$

We also define  $X_j$  for  $j \in [t+1]$  to be the concatenate result of the node values in level  $j$  of the path leading to  $\text{id}$ . For instance, in the example above we have  $X_1 = (a_{d-1}, b_{d-1}, \text{id}_{d-1})$ .

Let  $\text{stt}_i := \text{stt}_{1,i}$ . Recall that  $P_i$  has  $\text{stt}_{i+1}$  hardwired, which is the state used to hash-garble  $P_{i+1}$ . Via a sequence of hybrids, we show how to replace garbled versions of  $P_i$ 's, starting with  $i = 1$ , so that in the  $i$ th hybrid the values of  $\text{stt}_1, \dots, \text{stt}_i$  are never used.

- **Hybrid 0 (true encryption):** The ciphertext is  $\text{ct}_0 := (\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_d, \tilde{y}_1)$ , where all of the values are sampled as in the construction.
- **Hybrid 1:** The ciphertext is  $\text{ct}_1 := (\tilde{P}_{1,\text{sim}}, \tilde{P}_2, \dots, \tilde{P}_d, \tilde{y}_{1,\text{sim}})$ , where  $\tilde{P}_2, \dots, \tilde{P}_d$  are sampled as in the construction, and where  $\tilde{P}_{1,\text{sim}}$  and  $\tilde{y}_{1,\text{sim}}$  are sampled as follows:

$$(\tilde{P}_{1,\text{sim}}, \tilde{y}_{1,\text{sim}}) \leftarrow \text{Sim}(\text{hk}, X_1, 1^{|\tilde{P}_1|}, \tilde{y}_2). \quad (20)$$

- **Hybird  $i \in [d-1]$ :**

$$\text{ct}_i := (\tilde{P}_{1,\text{sim}}, \dots, \tilde{P}_{i,\text{sim}}, \tilde{P}_{i+1}, \dots, \tilde{P}_d, \tilde{y}_{1,\text{sim}}),$$

where for  $j \in [i]$ :

$$(\tilde{P}_{j,\text{sim}}, \tilde{y}_{j,\text{sim}}) \leftarrow \text{Sim}(\text{hk}, X_{j+1}, 1^{|\tilde{P}_j|}, \tilde{y}_{j+1}) \quad (21)$$

- **Hybrid  $d$ :**

$$\text{ct}_d := (\tilde{P}_{1,\text{sim}}, \dots, \tilde{P}_{d,\text{sim}}, \tilde{y}_{1,\text{sim}}),$$

where for  $j \in [d-1]$ :

$$(\tilde{P}_{j,\text{sim}}, \tilde{y}_{j,\text{sim}}) \leftarrow \text{Sim}(\text{hk}, X_{j+1}, 1^{|\tilde{P}_j|}, \tilde{y}_{j+1}), \quad (22)$$

and

$$(\tilde{P}_{d,\text{sim}}, \tilde{y}_{d,\text{sim}}) \leftarrow \text{Sim}(\text{hk}, (\text{id}, \text{pk}, 0^\kappa), 1^{|\tilde{P}_d|}, \text{E}(\text{pk}, \mathbf{m}; r)). \quad (23)$$

Now exactly as in the proof of Theorem 5.4, using the simulation security of the underlying HO scheme, we can show the indistinguishability of each two adjacent hybrids. Moreover, in the last hybrid, again using simulation security and as in the proof of Theorem 5.4, we may switch the underlying bit value of  $\mathbf{m}$ . The proof is now complete.  $\square$

<sup>11</sup>Recall that by Definition 3.7 the challenge identity  $\text{id}$  must have been registered before, and exactly once.

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