PPP-Completeness with Connections to Cryptography

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Abstract

Polynomial Pigeonhole Principle (PPP) is an important subclass of TFNP with profound connections to the complexity of the fundamental cryptographic primitives: *collision-resistant hash functions* and *one-way permutations*. In contrast to most of the other subclasses of TFNP, no complete problem is known for PPP. Our work identifies the first PPP-complete problem without any circuit or Turing Machine given explicitly in the input: constrained-SIS, and thus we answer a longstanding open question from [Pap94].

constrained-SIS: a generalized version of the well-known Short Integer Solution problem (SIS) from lattice-based cryptography.

In order to give intuition behind our reduction for constrained-SIS, we identify another PPPcomplete problem with a circuit in the input but closely related to lattice problems: BLICHFELDT.

BLICHFELDT: the computational problem associated with Blichfeldt's fundamental theorem in the theory of lattices.

Building on the inherent connection of PPP with collision-resistant hash functions, we use our completeness result to construct the first natural hash function family that captures the hardness of all collision-resistant hash functions in a worst-case sense, i.e. it is natural and *universal in the worst-case*. The close resemblance of our hash function family with SIS, leads us to the first candidate collision-resistant hash function that is both natural and universal *in average-case sense*.

Finally, our results enrich our understanding of the connections between PPP, lattice problems and other concrete cryptographic assumptions, such as the discrete logarithm problem over general groups.

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1 Introduction

The fundamental task of *Computational Complexity* theory is to classify computational problems according to their inherent computational difficulty. This led to the definition of *complexity classes* such as NP which contains the *decision* problems with *efficiently* verifiable proofs in the "yes" instances. The *search* analog of the class NP, called FNP, is defined as the class of *search* problems whose decision version is in NP. The same definition extends to the class FP, as the search analog of P. The seminal works of [JPY88, Pap94] considered search problems in FNP that are *total*, i.e. their decision version is always affirmative and thus a solution must always exist. This totality property makes the definition of FNP inadequate to capture the intrinsic complexity of total problems in the appropriate way as it was first shown in [JPY88]. Moreover, there were evidences for the hardness of total search problems e.g. in [HPV89]. Megiddo and Papadimitriou [MP89] defined the class **TFNP** that contains the total search problems of FNP, and Papadimitriou [Pap94] proposed the following classification rule of problems in TFNP:

Total search problems should be classified in terms of the profound mathematical principles that are invoked to establish their totality.

Along these lines, many subclasses for TFNP have been defined. Johnson, Papadimitriou and Yannakakis [JPY88] defined the class **PLS**. A few years later, Papadimitriou [Pap94] defined the complexity classes **PPA**, **PPAD**, **PPADS** and **PPP**, each one associated with a profound mathematical principle in accordance with the above classification rule. More recently, the classes **CLS** and **PWPP** were defined in [DP11] and [Jer16], respectively. In Section 1.1 we give a high-level description of all these classes.

Finding complete problems for the above classes is important as it enhances our understanding of the underlying mathematical principles. In turn, such completeness results reveal equivalences between total search problems, that seemed impossible to discover without invoking the definition of these classes. Since the definition of these classes in [JPY88, Pap94] it was clear that the completeness results about problems that do not have explicitly a Turing machine or a circuit as a part of their input are of particular importance. For this reason it has been established to call such problems *natural* in the context of the complexity of total search problems (see [FRG18]).

Many natural complete problems are known for PLS and PPAD, and recently natural complete problems for PPA were identified too (see Section 1.1). However, no natural complete problems are known for the classes PPP, PWPP that have profound connections with the hardness of important cryptographic primitives, as we explain later in detail.

Our Contributions. Our main contribution is to provide the first natural complete problems for PPP and PWPP, and thus solve a longstanding open problem from [Pap94]. Beyond that, our PPP completeness results lead the way towards answering important questions in cryptography and lattice theory as we highlight below.

UNIVERSAL COLLISION-RESISTANT HASH FUNCTION. Building on the inherent connection of PWPP with *collision-resistant hash functions*, we construct a natural hash function family \mathcal{H}_{cSIS} with the following properties:

- Worst-Case Universality. No efficient algorithm can find a collision in every function of the family \mathcal{H}_{cSIS} , unless worst-case collision-resistant hash functions do not exist.

Moreover, if an (average-case hard) collision-resistant hash function family exists, then there exists an efficiently samplable distribution \mathcal{D} over \mathcal{H}_{cSIS} , such that $(\mathcal{D}, \mathcal{H}_{cSIS})$ is an (average-case hard) collision-resistant hash function family.

- Average-Case Hardness. No efficient algorithm can find a collision in a function chosen *uniformly at random* from \mathcal{H}_{cSIS} , unless we can efficiently find short lattice vectors in any (worst-case) lattice.

The first property of \mathcal{H}_{cSIS} is reminiscent of the existence of *worst-case* one-way functions from the assumption that $P \neq NP$ [Sel92]. The corresponding assumption for the existence of worst-case collision-resistance hash functions is assuming $FP \neq PWPP$, but our hash function family \mathcal{H}_{cSIS} is the first natural definition that does not involve circuits, and admits this strong completeness guarantee in the worst-case.

The construction and properties of \mathcal{H}_{cSIS} lead us to the first candidate of a *natural* and *uni-versal collision-resistant hash function family*. The idea of universal constructions of cryptographic primitives was initiated by Levin in [Lev87], who constructed the first universal one-way function and followed up by [Lev03, KN09]. Using the same ideas we can also construct collision a universal collision-resistant hash function family as we describe in Appendix C. The constructed hash function though invokes in the input an explicit description of a Turing machine and hence it fails to be *natural*, with the definition of naturality that we described before. In contrast, our candidate construction is natural, simple, and could have practical applications.

COMPLEXITY OF LATTICE PROBLEMS IN PPP. The hardness of lattice problems in NP \cap coNP [AR04] has served as the foundation for numerous cryptographic constructions in the past two decades. This line of work was initiated by the breakthrough work of Ajtai [Ajt96], and later developed in a long series of works (e.g. [AD97, MR07, Reg09, GPV08, Pei09, GVW13, BLP⁺13, BV14, GSW13, GVW15b, GKW17, WZ17, PRSD17]). This wide use of search (approximation) lattice problems further motivates their study.

We make progress in understanding this important research front by showing that:

- 1. the computational problem BLICHFELDT associated with Blichfeldt's theorem, which can be viewed as a generalization of Minkowski's theorem, is PPP-complete,
- 2. the cSIS problem, a constrained version of the Short Integer Solution (SIS), is PPP-complete,
- 3. we combine known results and techniques from lattice theory to show that most approximation lattice problems are reducible to BLICHFELDT and cSIS.

These results create a new path towards a better understanding of lattice problems in terms of complexity classes.

COMPLEXITY OF OTHER CRYPTOGRAPHIC ASSUMPTIONS. Besides lattice problems, we discuss the relationship of other well-studied cryptographic assumptions and PPP. Additionally, we formulate a white-box variation of the *generic group model* for the discrete logarithm problem [Sho97]; we observe that this problem is in PPP and is another natural candidate for being PPP-complete.

1.1 Related Work

In this section we discuss the previous work on the complexity of total search problems, that has drawn attention from the theoretical computer science community over the past decades. We start with a high-level description of the total complexity classes and then discuss the known results for each one of them.

- **PLS.** The class of problems whose totality is established using a potential function argument. *Every finite directed acyclic graph has a sink.*
- **PPA.** The class of problems whose totality is proved through a parity argument. *Any finite graph has an even number of odd-degree nodes.*
- **PPAD.** The class of problems whose totality is proved through a directed parity argument. *All directed graphs of degree two or less have an even number of degree one nodes.*
- **PPP.** The class of problems whose totality is proved through a pigeonhole principle argument. *Any map from a set S to itself either is onto or has a collision.*

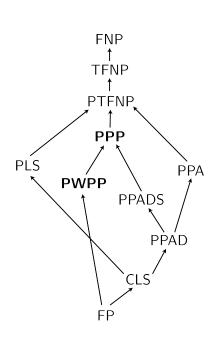
Using the same spirit two more classes were defined after [Pap94], in [DP11] and [Jer16].

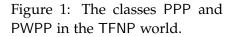
- **CLS.** The class of problems whose totality is established using both a potential function argument and a parity argument.
- **PWPP.** The class of problems whose totality is proved through a weak pigeonhole principle. *Any map from a set S to a strict subset of S has a collision.*

Recently, a syntactic analog PTFNP of the semantic class TFNP has been defined in [GP17], and a complete problem for this class has been identified. It has also been shown that all the classes we described above are subsets of PTFNP. Oracle separations between all these classes are known [BCE⁺98], with the only exception of whether PLS is contained in PPAD. **PLS-completeness.** The class PLS represents the complexity of local optimization problems. Some important problems that have been shown to be PLS-complete are: Local Max-Cut [SY91], Local Travelling Salesman Problem [Pap92], and Finding a Pure Nash Equilibrium [FPT04]. Recently, important results for the smoothed complexity of the Local Max-Cut problem were shown in [ER17, ABPW17].

PPAD-completeness. Arguably, the most celebrated application of the complexity of total search problems is the characterization of the computational complexity of finding a Nash equilibrium in terms of PPAD-completeness [DGP09, CDT09]. This problem lies in the heart of game theory and economics. The proof that Nash Equilibrium is PPAD-complete initiated

a long line of research in the intersection of computer science and game theory and revealed connections between the two scientific communities that were unknown before (e.g.





[EGG06, CDDT09, VY11, KPR⁺13, CDO15, Rub15, Rub16, CPY17, SSB17]).

PPA-completeness. PPA-complete problems usually arise as the undirected generalizations of their PPAD-complete analogs. For example, Papadimitriou [Pap94] showed that Sperner's Lemma in a 3-D cube is PPAD-complete and later Grigni [Gri01] showed that Sperner's Lemma in a 3-manifold consisting of the product of a Möbius strip and a line segment is PPA-complete. Since Möbius strip is non-orientable, this indeed is a non-directed version of the Sperner's

Lemma. Similarly, other problems have been showed to be PPA-complete, all involving some circuit as an input in their definition [ABB15, DEF⁺16, BIQ⁺17]. Recently, the first natural PPAcomplete problem, without a circuit as part of the input, has been identified in [FRG18]. This illustrates an interesting relation between PPA and complexity of social choice theory problems. CLS-completeness. The CLS class was defined in [DP11] to capture the complexity of problems such as P-matrix LCP, computing KKT-points, and finding Nash equilibria in congestion and network coordination games. Recently, it has been proved that the problem of finding a fixed point whose existence invokes Banach's Fixed Point Theorem, is CLS-complete [DTZ18, FGMS17]. TFNP and cryptography. The connection of TFNP and cryptography was illustrated by Papadimitriou in [Pap94], where he proved that if PPP = FP then one-way permutations cannot exist. In [BO06], a special case of integer factorization was shown to be in PPA \cap PPP. This was generalized in [Jer16] by proving that the problem of factoring integers is in PPA \cap PPP under randomized reductions. Recently, strong cryptographic assumptions were used to prove the average-case hardness of PPAD and CLS [BPR15, GPS16, HY17]. In [RSS17] it was shown that average-case PPAD hardness does not imply one-way function, whereas in [HNY17] it was shown that any hard on average problem in NP implies the average case hardness of TFNP. Finally, in [KNY17] it is proved that the existence of multi-collision resistant hash functions is equivalent with a variation of the total search problem RAMSEY, which is not known to belong to any of the above complexity classes. Interestingly, they prove that another variation of RAMSEY called *colorful*-Ramsey (C-RAMSEY) is PWPP-hard. Although this an important result, the problem C-RAMSEY still invokes a circuit in the input and in not known to be in PWPP, hence does not resolve the problem of identifying a natural complete problem for PWPP.

TFNP and lattices. In [BJP⁺15] it was shown that the computational analog of Minkowski's theorem (namely MINKOWSKI) is in PPP, was conjectured that it is also PPP-complete. The authors justified their conjecture by showing that EQUAL-SUMS, a problem from [Pap94] that is conjectured to be PPP-complete, reduces to MINKOWSKI. Additionally, they show that a number theoretic problem called DIRICHLET reduces to MINKOWSKI, and thus is in PPP. In [HRRY17] it is proven that the problem NUMBER-BALANCING is equivalent to a polynomial approximation of Minkowski's theorem in the ℓ_2 norm (via Cook reductions for both directions).

1.2 Roadmap of the paper

We start our exposition with a brief description of the results contained in this paper. First we briefly describe the PPP-completeness of BLICHFELDT that illustrates some of the basic ideas behind our main result that cSIS is PPP-complete. The complete proof of the PPP-completeness of BLICHFELDT can be found in Section 3. We suggest to readers that have experience with the fundamental concepts of the theory of lattices to skip the details Section 3.

Then, we present a brief description of our main theorem and its proof. The complete proof of the PPP-completeness of cSIS can be found in Section 4.

In Section 5 we describe the PPP-completeness of a weaker version of cSIS and its relation with the definition of the first natural universal collision resistant hash function family in the worst-case sense. This proof also provides the first candidate for a collision resistant hash function family that is both natural and universal in the average-case sense.

Finally in Section 6 we present, for completeness of our exposition, other lattice problems that are already known to belong to PPP and PWPP and in Section 7 we present more general other cryptographic assumptions that belong to PPP and PWPP.

1.3 Overview of the Results

Before we describe our results in more detail we define the class PPP more formally. The class PPP contains the set of problems that are reducible to the PIGEONHOLE CIRCUIT problem. The input to PIGEONHOLE CIRCUIT is a binary circuit $C : \{0,1\}^n \to \{0,1\}^n$ and its output is either an $\underline{x} \in \{0,1\}^n$ such that $C(\underline{x}) = \underline{0}$, or a pair $\underline{x}, \underline{y} \in \{0,1\}^n$ such that $\underline{x} \neq \underline{y}$ and $C(\underline{x}) = C(\underline{y})$.

Our first and technically most challenging result is to identify and prove the PPP-completeness of two problems, both of which share similarities with lattice problems. For our exposition, a lattice $\mathcal{L} \subseteq \mathbb{Z}^n$ can be viewed as a finitely generated additive subgroup of \mathbb{Z}^n . A lattice $\mathcal{L} = \mathcal{L}(\mathbf{B}) := \mathbf{B} \cdot \mathbb{Z}^n$ is generated by a full-rank matrix $\mathbf{B} \in \mathbb{Z}^{n \times n}$, called *basis*. In the rest of this section we also use the *fundamental parallelepiped* of \mathcal{L} defined as $\mathcal{P}(\mathcal{L}) := \mathbf{B} \cdot [0, 1)^n$.

1.3.1 BLICHFELDT is PPP-complete.

We define the BLICHFELDT problem as the computational analog of Blichfeldt's theorem (see Theorem 3.1). Its input is a basis for a lattice $\mathcal{L} \subseteq \mathbb{Z}^n$ and a set $S \subseteq \mathbb{Z}^n$ of cardinality greater or equal to the volume of $\mathcal{P}(\mathcal{L})$. Its output is either a point in *S* that belongs to \mathcal{L} , or for two (different) points in *S* such that their difference belongs to \mathcal{L} . In the overview below, we explain why such an output always exists. Notice that finding a solution to BLICHFELDT becomes trivial if the input representation of *S* has length proportional to its size, i.e. one can iterate over all element pairs of *S*. The problem becomes challenging when *S* is represented succinctly. We introduce a notion for a succinct representation of sets that we call *value function*. Informally, a value function for a set *S* is a small circuit that takes as input $\lceil \log(|S|) \rceil$ bits that describe an index $i \in \{0, ..., |S| - 1\}$, and outputs $\mathbf{s}_i \in S$.

We give a proof overview of our first main theorem, and highlight the obstacles that arise, along with our solutions.

Theorem 3.5. The BLICHFELDT problem is PPP-complete.

PPP MEMBERSHIP OF BLICHFELDT OVERVIEW. We denote with [n] the set $\{0, ..., n-1\}$. We define the map $\sigma : \mathbb{Z}^n \to \mathcal{P}(\mathcal{L}) \cap \mathbb{Z}^n$ that reduces any point in \mathbb{Z}^n modulo the parallelepiped to $\mathcal{P}(\mathcal{L}) \cap \mathbb{Z}^n$, i.e. (mod $\mathcal{P}(\mathcal{L})$). Using σ we can efficiently check the membership of any $\mathbf{v} \in \mathbb{Z}^n$ in \mathcal{L} , by checking if σ maps \mathbf{v} to the origin. Observe that if $\sigma(\mathbf{x}) = \sigma(\mathbf{y})$ then $\mathbf{x} - \mathbf{y} \in \mathcal{L}$.

We show in Lemma 3.2 that $\operatorname{vol}(\mathcal{P}(\mathcal{L})) = |\mathcal{P}(\mathcal{L}) \cap \mathbb{Z}^n|$, hence the input requirement for *S* is equivalent to $|S| \ge |\mathcal{P}(\mathcal{L}) \cap \mathbb{Z}^n|$. Notice that the points of *S* after applying the map σ , either have a collision in $\mathcal{P}(\mathcal{L}) \cap \mathbb{Z}^n$ or a preimage of the origin exists in *S*. It follows by a pigeonhole argument that a solution to BLICHFELDT always exists. For the rest of this part we assume that $|S| = |\mathcal{P}(\mathcal{L}) \cap \mathbb{Z}^n|$ and let $n = \lceil \log(|S|) \rceil$.

We construct a circuit $C : \{0,1\}^n \to \{0,1\}^n$ that on input an appropriate index *i*, evaluates the value function of *S* to obtain $\mathbf{s}_i \in S$, and computes $\sigma(\mathbf{s}_i)$. The most challenging part of the proof is to construct an efficient map from $\sigma(S)$ to $[|\mathcal{P}(\mathcal{L}) \cap \mathbb{Z}^n|]$ in the following way. We define an appropriate parallelepiped $D = [L_1] \times [L_2] \times \cdots \times [L_n]$ where the L_i are non-negative integers, and a bijection $\pi : \mathcal{P}(\mathcal{L}) \cap \mathbb{Z}^n \to D$. Because *D* is a cartesian product, a natural efficient indexing procedure exists as described in Lemma 2.1. This allows to map $\pi(\sigma(\mathbf{s}_i))$ to $j \in [|\mathcal{P}(\mathcal{L}) \cap \mathbb{Z}^n|]$. The circuit *C* outputs the binary decomposition of *j*. It follows that any \underline{x} such that $C(\underline{x}) = \underline{0}$ corresponds to a vector $\mathbf{x} \in S$ such that $\sigma(\mathbf{x}) = \mathbf{0}$. On the other hand, a collision $C(\underline{x}) = C(\underline{y})$ with $\underline{x} \neq \underline{y}$ corresponds to a collision $\sigma_S(\mathbf{x}) = \sigma_S(\mathbf{y})$, where σ_S is the restriction of σ on *S*, and hence $\mathbf{x} - \mathbf{y} \in \mathcal{L}$.

PPP HARDNESS OF BLICHFELDT OVERVIEW. We start with a circuit $C : \{0,1\}^n \to \{0,1\}^n$ that is an input to PIGEONHOLE CIRCUIT. We construct a set *S* and a lattice \mathcal{L} as input to BLICHFELDT in the following way. The set *S* contains the elements $\mathbf{s}_{\underline{x}} = \begin{bmatrix} \underline{x} \\ \mathcal{C}(\underline{x}) \end{bmatrix} \in \{0,1\}^{2n}$ and is represented succinctly with the value function that maps \underline{x} to $\mathbf{s}_{\underline{x}}$. Notice that $|S| = 2^n$. The lattice \mathcal{L} consists of all $\underline{v} \in \{0,1\}^{2n}$ that satisfy the equation $[\mathbf{0}_n \ \mathbf{I}_n] \cdot \underline{v} = \mathbf{0} \pmod{2}$. By Lemma 2.3, one can efficiently obtain a basis from this description of \mathcal{L} and in addition the volume of $\mathcal{P}(\mathcal{L})$ is at most 2^n . Thus, *S* and \mathcal{L} is a valid input for BLICHFELDT.

The output of BLICHFELDT is either an $\mathbf{s}_{\underline{x}} = \begin{bmatrix} \underline{x} \\ \mathcal{C}(\underline{x}) \end{bmatrix} \in S \cap \mathcal{L}$ that (by construction of \mathcal{L}) implies $\mathcal{C}(\underline{x}) = \mathbf{0}$, or two different elements of S, $\mathbf{s}_{\underline{x}} = \begin{bmatrix} \underline{x} \\ \mathcal{C}(\underline{x}) \end{bmatrix}$, $\mathbf{s}_{\underline{y}} = \begin{bmatrix} \underline{y} \\ \mathcal{C}(\underline{y}) \end{bmatrix}$ with $\mathbf{s}_{\underline{x}} - \mathbf{s}_{\underline{y}} \in \mathcal{L}$ that implies $\underline{x} \neq \underline{y}$ and (by construction of \mathcal{L}) $\mathcal{C}(\underline{x}) = \mathcal{C}(\underline{y})$.

1.3.2 cSIS is PPP-complete.

Part of the input to BLICHFELDT is represented with a value function which requires a small circuit. As we explained before this makes BLICHFELDT a non-natural problem with the respect to the definition of naturality in the context of the complexity of total search problems. We now introduce a natural problem that we call *constrained Short Integer Solution* (cSIS), and show that it is PPP-complete. The cSIS problem is a generalization of the well-known *Short Integer Solution* (SIS) problem, and discuss their connection in Section 5.2.

The input is $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, $\mathbf{G} \in \mathbb{Z}_q^{d \times m}$ and $\mathbf{b} \in \mathbb{Z}_q^d$, for some positive integer q and $m \ge (n+d)\lceil \log(q) \rceil$. The matrix \mathbf{G} has the property that for every \mathbf{b} we can efficiently find an $\mathbf{x} \in \{0,1\}^m$ such that $\mathbf{G}\mathbf{x} = \mathbf{b} \pmod{q}$. We define such matrices as *binary invertible*. The output is either a vector $\mathbf{x} \in \{0,1\}^m$ such that $\mathbf{A}\mathbf{x} = \mathbf{0} \pmod{q}$ and $\mathbf{G}\mathbf{x} = \mathbf{b} \pmod{q}$, or two different vectors $\mathbf{x}, \mathbf{y} \in \{0,1\}^m$ such that $\mathbf{A}(\mathbf{x} - \mathbf{y}) = \mathbf{0} \pmod{q}$ and $\mathbf{G}\mathbf{x} = \mathbf{G}\mathbf{y} = \mathbf{b} \pmod{q}$. We give a proof overview of the next theorem, and a full proof in Section 4.

Theorem 4.11. The cSIS problem is PPP-complete.

PPP MEMBERSHIP OF cSIS OVERVIEW. We show the membership of cSIS in PPP for the general class of binary invertible matrices **G** in Section 4. In order to simplify the exposition, we assume that $q = 2^{\ell}$ and **G** to be the "gadget" matrix concatenated with a random matrix **V**. That is, **G** has the form $[\mathbf{I}_d \otimes \gamma^T \mathbf{V}]$ where $\gamma^T = [1, 2, ..., 2^{\ell}]$.

Let $\mathbf{A} \in \mathbb{Z}_q^{h \times m}$, $\mathbf{G} \in \mathbb{Z}_q^{d \times m}$, and $\mathbf{b} \in \mathbb{Z}_q^d$ be the input to cSIS. We now explain why $m \ge (n+d)\ell$ suffices to always guarantee a solution to cSIS. First, observe that the first $\ell \cdot d$ columns of \mathbf{G} , corresponding to the gadget matrix $[\mathbf{I}_d \otimes \gamma^T]$, are enough to guarantee that for every $\mathbf{r}' \in \mathbb{Z}_q^{m-\ell d}$ there exists an \mathbf{r} such that $\mathbf{G}\begin{bmatrix}\mathbf{r}\\\mathbf{r}'\end{bmatrix} = \mathbf{b} \pmod{q}$. Hence, there are at least $q^{m-\ell d}$ solutions to the equation $\mathbf{G}\mathbf{x} = \mathbf{b} \pmod{q}$. Also, there are $2^{\ell n}$ possible values for $\mathbf{A}\mathbf{x} \pmod{q}$. By a pigeonhole argument a solution to cSIS always exists. To complete the membership proof, issues similar to BLICHFELDT with the circuit representation of the problem instance appear, but we overcome them using similar ideas.

PPP HARDNESS OF cSIS OVERVIEW. We start with a circuit $C : \{0,1\}^n \to \{0,1\}^n$ that is an input to PIGEONHOLE CIRCUIT. Since the input of PIGEONHOLE CIRCUIT is a circuit and the input

of cSIS is a pair of matrices and a vector, we need to represent this circuit in an algebraic way. In particular, we device a way to encode the circuit in a binary invertible matrix **G** and a vector **b**. To gain a better intuition of why this is possible, we note that a NAND gate $x \land y = z$ can be expressed as the linear modular equation $x + y + 2z - w = 2 \pmod{4}$, where $x, y, z, w \in \{0,1\}$. By a very careful construction, we can encode these linear modular equations in a binary invertible matrix **G**. For further details we defer to Section 4.

Since cSIS with q = 4 returns a vector such that $\mathbf{Ax} = \mathbf{0} \pmod{4}$ and PIGEONHOLE CIRCUIT asks for a binary vector such that $\underline{x} = \underline{\mathbf{0}}$, a natural idea is to let \mathbf{A} be of the form $[\mathbf{0} \ \mathbf{I}_n]$, where the identity matrix corresponds to the columns representing the output of circuit C in \mathbf{G} . Finally, we argue that a solution to cSIS with input \mathbf{A} , \mathbf{G} and \mathbf{b} as constructed above, gives either a collision or a preimage of zero for the circuit C as required.

It can be argued that this reduction shares common ideas with the reduction of 3-SAT to SUBSET-SUM; this shows the importance of the input conditions for cSIS and hints to the numerous complications that arise in the proof. Without these conditions, we could end up with a trivial reduction to an NP-hard problem! Fortunately, we are able to show that our construction satisfies the input conditions of cSIS.

1.3.3 Towards a Natural and Universal Collision-Resistant Rash Family.

PWPP is a subclass of PPP in which a collision always exists; it is not hard to show that variations of both BLICHFELDT and cSIS are PWPP-complete. We tweak the parameters of valid inputs in order to guarantee that a collision always exists. The PWPP-complete variation of cSIS, which we denote by weak-cSIS, gives a function family which is a universal collision-resistant hash function family in a worst-case sense: if there is a function family that contains at least one function for which it is hard to find collisions, then our function family also includes a function for which it is hard to find collisions.

We now describe the differences of cSIS and weak-cSIS. As before we assume that $q = 2^{\ell}$. The input to weak-cSIS is a matrix $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, and a binary invertible matrix $\mathbf{G} \in \mathbb{Z}_q^{d \times m}$. Notice that there is no vector **b** in the input, and the relation between n, m, d and ℓ is that m has to be *strictly* greater that $\ell(n + d)$. Namely, $m > \ell(n + d)$. This change in the relation of the parameters might seem insignificant, but is actually very important, as it allows us to replace **b** in cSIS by the zero vector. This transforms weak-cSIS into a pure lattice problem: on input matrices \mathbf{A}, \mathbf{G} with corresponding bases $\mathbf{B}_{\mathbf{A}}$ and $\mathbf{B}_{\mathbf{G}}$, where \mathbf{G} is binary invertible, find two vectors **x** and **y** such that $\mathbf{x}, \mathbf{y} \in \mathcal{L}(\mathbf{B}_{\mathbf{G}})$ and $\mathbf{x} - \mathbf{y} \in \mathcal{L}(\mathbf{B}_{\mathbf{A}})$.

The great resemblance of weak-cSIS with SIS and its completeness for PWPP lead us to the first candidate for a universal collision-resistant hash function $\mathcal{H}_{cSIS} = \{h_s : \{0,1\}^k \to \{0,1\}^{k'}\}$:

- The key \underline{s} is a pair of matrices (\mathbf{A}, \mathbf{G}) , where \mathbf{G} is binary invertible.
- Given a key $\underline{s} = (\mathbf{A}, \mathbf{G})$ and a binary vector $\underline{x} \in \{0, 1\}^k$, $h_s(\underline{x})$ is the binary decomposition

of **Au** (mod *q*), where
$$\mathbf{u} = \begin{bmatrix} \mathbf{r} \\ \underline{x} \end{bmatrix}$$
 such that $\mathbf{Gu} = \mathbf{0} \pmod{q}$.

Because lattice problems have worst-to-average case reductions and our hash family is based on a lattice problem, this gives hope for showing that our construction is universal in the averagecase sense.

1.3.4 Other Lattice Problems Known to be in PPP.

We show that the computational analog of Minkowski's theorem, namely MINKOWSKI, is in PPP via a Karp-reduction to BLICHFELDT. We note that a Karp-reduction showing MINKOWSKI \in PPP was shown in [BJP⁺15]. Based on these two problems and the known reductions between lattice problems, we conclude that a variety of lattice (approximation) problems belong to PPP; the most important among them are *n*-SVP, $\tilde{O}(n)$ -SIVP and $n^{2.5}$ -CVP (see Figure 2).

1.3.5 Other Cryptographic Assumptions in PPP.

By the definition, the class PWPP contains all cryptographic assumptions that imply collisionresistant hash functions. These include the factoring of Blum integers, the Discrete Logarithm problem over \mathbb{Z}_p^* and over elliptic curves, and the SIS lattice problem (a special case of weak-cSIS). Also, Jeřábek [Jer16] showed that the problem of factoring integers is in PWPP.

We extend the connection between PPP and cryptography by introducing a white-box model to describe *general groups*, which we define to be cyclic groups with a succinct representation of their elements and group operation (i.e. a small circuit). We show that the Discrete Logarithm over general groups is in PPP. An example of a general group is \mathbb{Z}_q^* . These connections are also summarized in Figure 2.

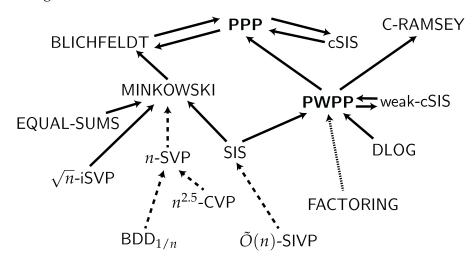


Figure 2: Solid arrows denote a Karp reduction, and dashed arrows denote a Cook reduction.

1.4 Open questions.

Numerous new questions arise from our work and the connections we draw between PPP, cryptography and lattices. We summarize here some of them.

Open Problem 1.1. *Is there a worst-to-average case reduction from* weak-cSIS *to itself?*

This result will provide the first *natural*, in the sense that does not invoke explicitly a Turing machine in the input, and *universal* collision resistant hash function family.

Open Problem 1.2. Is SIS or MINKOWSKI PPP-hard?

Open Problem 1.3. *Is* γ -SVP *in* PPP *for* $\gamma = o(n)$?

Open Problem 1.4. *Is* γ -CVP PPP*-hard for* $\gamma = \Omega(\sqrt{n})$?

Open Problem 1.5. Is the discrete logarithm problem in PPP for general elliptic curves?

2 Preliminaries

General Notation. Let [m] be the set $\{0, ..., m-1\}$, $\mathbb{N} = \{0, 1, 2, ...\}$ and $\mathbb{Z}_+ = \{1, 2, 3, ...\}$ We use small bold letters \mathbf{x} to refer to real vectors in finite dimension \mathbb{R}^d and capital bold letters \mathbf{A} to refer to matrices in $\mathbb{R}^{d \times \ell}$. For a matrix \mathbf{A} , we denote by \mathbf{a}_i^T its *i*-th row and by $a_{i,j}$ its (i, j)-th element. Let \mathbf{I}_n denote the *n*-dimensional identity matrix. We denote with $\mathbf{E}_{i,j}$ the matrix that has all zeros except that $e_{i,j} = 1$. A function negl(k) is *negligible* if negl(k) < $1/k^c$ for any constant c > 0 and sufficiently large k. All logarithms $\log(\cdot)$ are in base 2.

Vector Norms. We define the ℓ_p -norm of $\mathbf{x} \in \mathbb{R}^d$ to be $\|\mathbf{x}\|_p = (\sum_i x_i^p)^{1/p}$ and the ℓ_∞ -norm of \mathbf{x} to be $\|\mathbf{x}\|_{\infty} = \max_i |x_i|$. For simplicity we use $\|\cdot\|$ for the ℓ_2 -norm instead of $\|\cdot\|_2$. It is well known that $\|\mathbf{x}\|_p \le n^{1/p-1/q} \|\mathbf{x}\|_q$ for $p \le q$ and $\|\mathbf{x}\|_p \le \|\mathbf{x}\|_q$ for p > q.

2.1 Complexity Classes and Reductions

Binary Strings and Natural Numbers. We use bold and underlined small letters to refer to binary strings. Binary strings $\underline{x} \in \{0,1\}^k$ of length k can also be viewed as vectors in \mathbb{Z}^k . Every binary string $\underline{x} \in \{0,1\}^k$ can be mapped to a non negative integer number through the nonlinear map bc : $\{0,1\}^* \to \mathbb{Z}_+$ called *bit composition*, where $bc(\underline{x}) = \sum_{i=0}^{k-1} \underline{x}_{k-i} 2^i$. It is trivial to see that actually bc is a bijective mapping and hence we can define the inverse mapping bd : $\mathbb{Z}_+ \to \{0,1\}^*$ called *bit decomposition*, which is also trivial to compute for any given number $m \in \mathbb{Z}_+$.

The bit decomposition function bd is extended to integer vectors and the result is the concatenation of the bit decomposition of each coordinate of the vector. Similarly, this is also extended to integer matrices. Then of course the bit composition function bc is no longer well defined because its output can be either a number or a vector of numbers, but for simplicity we still use the notation bc and it will be made clear from the context whether the output is a number or a vector of numbers. When bd is applied to a set $\{m_1, \ldots, m_k\}$ the output is still a set with the bit decomposition of each element $\{bd(m_1), \ldots, bd(m_k)\}$.

Boolean Circuits. A *boolean circuit* C with n inputs and 1 output is represented as a labeled directed acyclic graph with in-degree at most 2, with exactly n source nodes and exactly 1 sink node. Each source node is an input of C and the sink node is the output of C. Each of the input nodes of C is labeled with a number in [n] denoting the ordering of the input variables. Each node with in-degree 1 is labeled with one of the 2 boolean functions with 1 variable $\{id, \neg\}$. Each node with in-degree 2 can be labeled with one of the 16 boolean function with two variables, but for our purposes we are going to use only the following five boolean functions: *nand*, *nor*, *xor*, *and*, *or*, with corresponding symbols $\{ \land, \lor , \oplus, \land, \lor \}$. Every boolean circuit defines a boolean function on n variables $C : \{0,1\}^n \to \{0,1\}$. Let \underline{x} be a binary string of length n, i.e. $\underline{x} \in \{0,1\}^n$. The value $C(\underline{x})$ of the circuit on input \underline{x} is computed by evaluating of C one by one in a topological sorting of C, starting from the input nodes. Then, $C(\underline{x})$ is the value of the output node. The size |C| of C is the number of nodes in the graph that represents C.

Circuits. We can now define a circuit C with n inputs and m outputs as an ordered tuple of m boolean circuits $C = (C_1, \ldots, C_m)$ which defines a function $C : \{0, 1\}^n \to \{0, 1\}^m$, where $C(\underline{x}) = (C_1(\underline{x}), \ldots, C_m(\underline{x}))$. The size |C| of C is equal to $|C_1| + \cdots + |C_2|$. It is known that $P \subseteq P/poly$ (see [AB09]), where P/poly is the class of polynomial-sized circuits. Thus, any polynomial time procedure we describe, implies an equivalent circuit of polynomial size.

Search Problems. A *search problem* in FNP is defined by a relation \mathcal{R} that on input *x* of size *n* and for every *y* of size poly(n), $\mathcal{R}(x, y)$ is polynomial-time computable on *n*. A solution to the search problem with input *x* is a *y* of size poly(n) such that $\mathcal{R}(x, y)$ holds.

A search problem is *total* if for every input *x* of size *n*, there exists a *y* of size poly(n) such that $\mathcal{R}(x, y)$ holds. The class of total search problems in FNP is called TFNP.

Karp Reductions Between Search Problems. A search problem \mathcal{P}_1 is *Karp-reducible* to a search problem \mathcal{P}_2 if there exist polynomial-time (in the input size of \mathcal{P}_1) computable functions f and g such that if x is an input of \mathcal{P}_1 , then f(x) is an input of \mathcal{P}_2 and if y is any solution of \mathcal{P}_2 with input f(x) then g(x, f(x), y) is a solution of \mathcal{P}_1 .

Cook Reductions Between Search Problems. A search problem \mathcal{P}_1 is *Cook-reducible* to a search problem \mathcal{P}_2 if there exists a polynomial-time (in the input size of \mathcal{P}_1) oracle Turing machine \mathcal{T} such that if x is an input of \mathcal{P}_1 , \mathcal{T} computes a y such that y is a solution of \mathcal{P}_1 whenever all the oracle answers are solutions of \mathcal{P}_2 . The set of all search problems that are Cook-reducible to problem \mathcal{P} is denoted by $\mathsf{FP}^{\mathcal{P}}$.

The PPP **Complexity Class.** The class PPP is a subclass of TFNP and consists of all search problems Karp-reducible to the following problem called PIGEONHOLE CIRCUIT.

PIGEONHOLE CIRCUIT Problem.

INPUT: A circuit C with n inputs and n outputs.

OUTPUT: One of the following:

- 1. a binary vector \underline{x} such that $C(\underline{x}) = \underline{0}$, or
- 2. two binary vectors $\underline{x} \neq y$, such that $C(\underline{x}) = C(y)$.

The weak PPP **Complexity Class.** The class PWPP is the set of all search problems Karpreducible to the following problem called COLLISION.

COLLISION Problem.

INPUT: A circuit C with n inputs and m outputs with m < n. OUTPUT: Two binary vectors $\underline{x} \neq \underline{y}$, such that $C(\underline{x}) = C(\underline{y})$.

2.2 Set Description Using Circuits

Let $S \subseteq \mathbb{N}^n$ and let $bd(S) \subseteq \{0,1\}^k$, i.e. the elements of *S* can be represented using *k* bits. As we will see later there is an inherent connection between proofs of both the inclusion and the hardness of PPP and the succinct representation of subsets *S* using circuits. We define here three such representations: the *characteristic function*, the *value function* and the *index function*.

Characteristic Function. We say that a circuit CH_S with *k* binary inputs and one output is a characteristic function representation of *S* if $CH_S(\underline{x}) = 1$ if and only if $\underline{x} \in bd(S)$.

Value Function. Let (s, \mathcal{V}_S) be a tuple where \mathcal{V}_S is a circuit with $\lceil \log(s) \rceil$ binary inputs and k outputs and $s \in \mathbb{Z}_+$. Let $f_{(s,\mathcal{V}_S)} : [s] \to \{0,1\}^k$ be a function such that $f_{(s,\mathcal{V}_S)}(bc(\underline{x})) = \mathcal{V}_S(\underline{x})$ for all \underline{x} with $bc(\underline{x}) < s$. Then, (s, \mathcal{V}_S) is a value function representation of S if and only if $f_{(s,\mathcal{V}_S)}$ is a bijective map between [s] and bd(S). The value $\mathcal{V}_S(\underline{x})$ can be arbitrary when $bc(\underline{x}) \geq s$.

Index Function. Let (s, \mathcal{I}_S) be a tuple where \mathcal{I}_S is a circuit with k binary inputs and $\lceil \log(s) \rceil$ outputs and $s \in \mathbb{Z}_+$. Let $f_{(s,\mathcal{I}_S)} : bd(S) \rightarrow [s]$ be a function such that $f_{(s,\mathcal{I}_S)}(\underline{x}) = bc(\mathcal{I}_S(\underline{x}))$ for all $\underline{x} \in bd(S)$. Then, (s,\mathcal{I}_S) is an index function representation of S if and only if $f_{(s,\mathcal{I}_S)}$ is a bijective map between bd(S) and [s]. The value $\mathcal{I}_S(\underline{x})$ can be arbitrary when $\underline{x} \notin bd(S)$.

Some remarks about the above definitions are in order. First, given a succinct representation of *S* it is computationally expensive to compute |S|, thus we provide it explicitly using *s*. Second, even though the input and the output of each circuit have to be binary vectors, we abuse notation and let the input of the index function and the characteristic function, and the output of the value function to be a element in \mathbb{N}^n . Formally, according to the above definitions the output of \mathcal{V}_S is the bit decomposition of an element in *S*, namely $bc(\mathcal{V}_S(\underline{x})) \in S$. In the rest of the paper, we abuse notation and drop bc to denote by $\mathcal{V}_S(\underline{x})$ the vector in *S*. Similarly, we drop bc and bd for the characteristic and the index functions.

To illustrate the definitions of succinct representations of sets, we explain how to define them in the simple case of the set $([0, L_1] \times \cdots \times [0, L_n]) \cap \mathbb{Z}^n$. Although this is a simple example, it is an ingredient that we need when we show the connection of lattice problems with the class PPP.

Lemma 2.1. Let $L_1, \ldots, L_n > 0$ and $S = ([0, L_1] \times \cdots \times [0, L_n]) \cap \mathbb{Z}^n$. Then, the following exist:

- 1. *a characteristic function representation* CH_S *of* S*, where* $|CH_S| = O(n \max_i \log L_i)$ *,*
- 2. a value function representation (\mathcal{V}_S, s) of S, where $|\mathcal{V}_S| = O(n^2 \max_i \log L_i)$ and
- 3. an index function representation (\mathcal{I}_S, s) of S, where $|\mathcal{I}_S| = O(n^2 \max_i \log L_i)$.

Proof. Let $\ell_1 = \lfloor L_1 \rfloor + 1, \ldots, \ell_n = \lfloor L_n \rfloor + 1$.

- 1. Given the bit decomposition of a vector $\mathbf{x} \in \mathbb{Z}^n$, we can easily test if \mathbf{x} belongs to *S* by checking whether $x_i \leq \ell_i$ for all *i*. Such a comparison needs $O(\log L_i)$ boolean gates and hence the size of $C\mathcal{H}_S$ is $O(n \max_i \log L_i)$.
- 2. Let *k* be the input number to our value function, the vector that we assing to *k* is the *k*-th vector in the lexicographical ordering of the elements in bd(S). We compute this vector $\mathbf{x} \in \mathbb{Z}^n$ coordinate by coordinate. We start from x_1 . Observe for any $t \in \mathbb{Z}_+$, the number of vectors in *S* with $x_1 = t$ is equal to $\prod_{i=2}^n \ell_i$ and hence the number of vectors with $x_1 \leq t$ is equal to $(t+1) \cdot \prod_{i=2}^n \ell_i$. Therefore, $x_1 = \lfloor k / \prod_{i=2}^n \ell_i \rfloor$. Then, the dimension of the problem reduces by one and therefore we can repeat the same procedure to compute x_2 as the first coordinate of the $(k (\prod_{i=2}^n \ell_i) \cdot \lfloor k / \prod_{i=2}^n \ell_i \rfloor)$ -th vector in the set $S' = ([0, L_2] \times \cdots \times [0, L_n]) \cap \mathbb{Z}^{n-1}$. Then we apply the procedure recursively. Moreover, this whole task can be made into an iterative procedure, with a circuit of size $O(n^2 \max_i \log L_i)$.
- 3. Let $\mathbf{x} \in S$ be the vector whose index we want to compute. The index that we assign to this vector is its position in the lexicographical ordering of the vectors in bd(S). The number of vectors $\mathbf{y} \in S$ with $y_1 < x_1$ that are before \mathbf{x} in the lexicographical ordering is equal to $(x_1 1) \cdot \prod_{i=2}^{n} \ell_i$. Therefore, using the recursion of the construction of the circuit for the value function of *S* above, we see that the lexicographical index can be computed by a circuit of size $O(n^2 \max_i \log L_i)$.

In the next sections, the constructions of value and index functions for sets lie at the heart of our proofs for showing membership in PPP, and we make frequent use of the above Lemma 2.1. Specifically, in Section 3, we see that the set of integer cosets of a lattice admits an index function that can be implemented with a polynomial sized circuit, and in Section 7 we see that any cyclic group admits a value function that can be implemented with a polynomial-size circuit. In the latter case, we demonstrate that an efficient implementation of an index function of a group shows that the Discrete Logarithm problem for this group belongs to the class PPP.

2.3 Lattices and Lattice Problems

Lattice. A *n*-dimensional *lattice* $\mathcal{L} \subset \mathbb{R}^n$ is the set of all integer linear combinations of *d* linearly independent vectors $\mathbf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_d)$ in \mathbb{R}^n , $\mathcal{L} = \mathcal{L}(\mathbf{B}) = \left\{ \sum_{i=1}^d a_i \mathbf{b}_i : a_i \in \mathbb{Z} \right\}$. The integer *d* is the rank of the lattice, and if it is equal to the dimension *n* we refer to \mathcal{L} as *full-rank*. The matrix **B** is called the *basis* of lattice $\mathcal{L}(\mathbf{B})$. Unless explicitly stated, the lattices on all definitions, statements and proofs are assumed to be full-rank integer lattices. Our results can be easily extended to *d*-rank lattices, but for ease of exposition we present only the full-rank case. A useful lemma we will use is the following:

Lemma 2.2. Let $\mathbf{B} \in \mathbb{Z}^{n \times n}$, $\mathbf{U} \in \mathbb{Z}^{n \times n}$ be a unimodular matrix, i.e. $\det(\mathbf{U}) = \pm 1$, then $\mathcal{L}(\mathbf{B}) = \mathcal{L}(\mathbf{B}\mathbf{U})$.

Smith Normal Form. A matrix $\mathbf{D} \in \mathbb{Z}^{n \times n}$ is in *Smith Normal Form* (*SNF*) if it is diagonal and $d_{i+1,i+1}$ divides $d_{i,i}$ for $1 \le i < n$. Moreover, any non-singular matrix $\mathbf{A} \in \mathbb{Z}^{n \times n}$ can be written as $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}$, where $\mathbf{D} \in \mathbb{Z}^{n \times n}$ is a unique matrix in SNF, and $\mathbf{U}, \mathbf{V} \in \mathbb{Z}^{n \times n}$ are unimodular matrices. Also, the matrices $\mathbf{U}, \mathbf{D}, \mathbf{V}$ can be computed in polynomial-time in *n* [KV05].

Dual Lattice. The *dual lattice* \mathcal{L}^* is defined as the set of all vectors in the span of **B** that have integer inner product with \mathcal{L} . That is, $\mathcal{L}^* = \{ \mathbf{y} \in \text{span}(\mathbf{B}) : \forall \mathbf{x} \in \mathcal{L}, \langle \mathbf{y}, \mathbf{x} \rangle \in \mathbb{Z} \}.$

q-ary Lattice. A lattice $\mathcal{L} \subseteq \mathbb{Z}^n$ is called a *q*-ary lattice if $(q\mathbb{Z})^n \subseteq \mathcal{L}$. Let $\mathbf{A} \in \mathbb{Z}^{m \times n}$, we define the following two types of *q*-ary lattices

$$\Lambda_q(\mathbf{A}) = \left\{ \mathbf{x} \in \mathbb{Z}^n \mid \mathbf{x}^T = \mathbf{y}^T \mathbf{A} \pmod{q} \text{ where } \mathbf{y} \in \mathbb{Z}^m \right\},$$
(2.1)

$$\Lambda_q^{\perp}(\mathbf{A}) = \{ \mathbf{x} \in \mathbb{Z}^n \mid \mathbf{A}\mathbf{x} = \mathbf{0} \; (\bmod \; q) \} \,. \tag{2.2}$$

The following lemma for *q*-ary lattices is useful in our proofs. For a simple proof of this Lemma 2.3 we refer to Sections 2.3 and 2.4 of [AP11].

Lemma 2.3. Let $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$, then:

- 1. $\Lambda_q^{\perp}(\mathbf{A}) = q \Lambda_q^*(\mathbf{A})$ and $\det\left(\Lambda_q^{\perp}(\mathbf{A})\right) \leq q^n$,
- 2. there exists a polynomial-size circuit \mathcal{BS} that on input $bd(\mathbf{A})$ it outputs $bd(\mathbf{B})$ such that $\mathbf{B} \in \mathbb{Z}^{n \times m}$ and $\Lambda_a^{\perp}(\mathbf{A}) = \mathcal{L}(\mathbf{B})$.

Fundamental Parallelepiped. The fundamental parallelepiped of $\mathcal{L}(\mathbf{B})$ is defined as the set $\mathcal{P}(\mathbf{B}) = \{\sum_{i=1}^{n} t_i \mathbf{b}_i : 0 \le t_i < 1\}$. Given a full-rank lattice $\mathcal{L}(\mathbf{B})$, we can define the operator (mod $\mathcal{P}(\mathbf{B})$) on vectors in \mathbb{R}^n such that $\mathbf{y} = \mathbf{x} \pmod{\mathcal{P}(\mathbf{B})}$ if $\mathbf{y} = \mathbf{B} (\mathbf{B}^{-1}\mathbf{x} - \lfloor \mathbf{B}^{-1}\mathbf{x} \rfloor)$.

Determinant. The *determinant* of a lattice is the volume of the fundamental parallelepiped, $det(\mathcal{L}) = \sqrt{det(\mathbf{B}^T \mathbf{B})}$ or simply $|det(\mathbf{B})|$ for full-rank lattices.

Lattice Cosets. For every $\mathbf{c} \in \mathbb{R}^n$ we define the lattice coset as $\mathcal{L} + \mathbf{c} = {\mathbf{x} + \mathbf{c} : \mathbf{x} \in \mathcal{L}}$. Let $\mathbf{B} \in \mathbb{Z}^{n \times n}$ be a set of *n* linearly independent integer vectors. We define the set of *integer cosets* of $\mathcal{L}(\mathbf{B})$ as follows $\operatorname{Co}(\mathcal{L}(\mathbf{B})) = {\mathcal{L}(\mathbf{B}) + \mathbf{c} \mid \mathbf{c} \in \mathbb{Z}^n}$.

We now state a fundamental relation between $Co(\mathcal{L}(\mathbf{B}))$, $det(\mathcal{L}(\mathbf{B}))$ and $\mathcal{P}(\mathcal{L}(\mathbf{B}))$.

Proposition 2.4. Let $\mathbf{B} \in \mathbb{Z}^{n \times n}$ be a set of *n*-dimensional linearly independent vectors. It holds that $\det(\mathcal{L}(\mathbf{B})) = |\operatorname{Co}(\mathcal{L}(\mathbf{B}))| = |\mathcal{P}(\mathcal{L}(\mathbf{B})) \cap \mathbb{Z}^d|.$

For a proof of Proposition 2.4 see the proof of Lemma 3.2.

3 BLICHFELDT is PPP-Complete

The concept of lattices was introduced by Hermann Minkowski in his influential book *Geometrie der Zahlen* [Min10], first published at 1896. In his book, Minkowski developed the theory of the geometry of numbers and resolved many difficult problems in number theory. His fundamental theorem, known as *Minkowski's Convex Body Theorem*, was the main tool of these proofs.

Despite the excitement created by Minkowski's groundbreaking work, it was only after 15 years that a new principle in geometry of numbers was discovered. The credit for this discovery goes to Hans Frederik Blichfeldt, who in 1914 published a paper [Bli14] with his new theorem and some very important applications in number theory ¹.

In this section, we characterize the computational complexity of Blichfeldt's existence theorem and in later sections we discuss its applications (see Section 6). We recall the statement of Blichfeldt's theorem below, introduce its computational search version, and prove that it is PPP-complete.

Theorem 3.1 (Blichfeldt's Theorem [Bli14]). Let $\mathbf{B} \in \mathbb{Z}^{n \times n}$ be a set of *n*-dimensional linearly independent integer vectors and a measurable set $S \subseteq \mathbb{R}^n$. If $\operatorname{vol}(S) > \det(\mathcal{L}(\mathbf{B}))$, then there exist $x, y \in S$ with $x \neq y$ and $x - y \in \mathcal{L}(\mathbf{B})$.

A proof of Theorem 3.1 can be found in Chapter 9 of [OLD01]. We now define the computational discrete version of Blichfeldt's theorem.

BLICHFELDT Problem.

INPUT: An *n*-dimensional basis $\mathbf{B} \in \mathbb{Z}^{n \times n}$ and a set $S \subseteq \mathbb{Z}^n$ described by the value function representation (s, \mathcal{V}_S) .

OUTPUT: If $s < \det(\mathcal{L}(\mathbf{B}))$, then the vector **0**. Otherwise, one of the following:

- 0. a number $z \in [s]$ such that $\mathcal{V}_S(z) \notin S$ or two numbers $z, w \in [s]$ such that $\mathcal{V}_S(z) = \mathcal{V}_S(w)$,
- 1. a vector **x** such that $\mathbf{x} \in S \cap \mathcal{L}$,
- 2. two vectors $\mathbf{x} \neq \mathbf{y}$, such that $\mathbf{x}, \mathbf{y} \in S$ and $\mathbf{x} \mathbf{y} \in \mathcal{L}$.

Lemma 3.2. BLICHFELDT is in PPP.

Proof. We show a Karp reduction from BLICHFELDT to the PIGEONHOLE CIRCUIT problem.

Let **B** and (s, \mathcal{V}_S) be the inputs of BLICHFELDT. We define the lattice $\mathcal{L} = \mathcal{L}(\mathbf{B})$ and $R = \mathcal{P}(\mathbf{B}) \cap \mathbb{Z}^n$ to be the set of all the integer vectors in the fundamental parallelepiped $\mathcal{P}(\mathbf{B})$. Let $\ell = \lceil \log(\det(\mathcal{L})) \rceil$ and $m = \lceil \log(s) \rceil$. We remark that $\ell = \lceil \log(|R|) \rceil$ (see Proposition 2.4) and m is equal to the number of binary inputs of \mathcal{V}_S . Our goal is to construct a circuit \mathcal{C} with ℓ inputs and ℓ outputs such that given a solution of the PIGEONHOLE CIRCUIT problem with input \mathcal{C} , we efficiently find a solution of BLICHFELDT. If $s < \det(\mathcal{L})$, then we output **0** without invoking PIGEONHOLE CIRCUIT. Therefore, for the rest of the proof we focus on the case $s \ge \det(\mathcal{L})$ which implies $m \ge \ell$. We can also assume without loss of generality that $s = \det(\mathcal{L})$. If this is not the case, then we can equivalently work with the set S' which is described by the value function representation $(\det(\mathcal{L}), \mathcal{V}'_S)$, where \mathcal{V}'_S has ℓ inputs and outputs the value of \mathcal{V}_S after padding the input with $m - \ell$ inputs in the most significant bits. We do this so that the input conditions of PIGEONHOLE CIRCUIT are satisfied. For simplicity, we assume S = S' for the rest of the proof.

¹These introductory paragraphs were inspired from Chapter 9 of [OLD01].

Before defining C, we note that numbers that satisfy case "0." in the output of BLICHFELDT exist if and only if (s, V_S) is not a valid value function of S. Therefore, an output of case "0." certifies that the input of BLICHFELDT is not a valid input.

First, let us assume that $det(\mathcal{L}) = 2^{\ell}$. Intuitively, we want \mathcal{C} to be a mapping from any vector $\mathbf{x} \in S$ to the integer coset of \mathbf{x} in $Co(\mathcal{L})$. A useful observation is that if two vectors in S belong to the same coset, then their difference belongs to \mathcal{L} . The first step to implement this idea is to find a set of representatives for $Co(\mathcal{L})$. The set R of integer vectors in $\mathcal{P}(\mathbf{B})$ is such a set. Finally, to finish the construction of \mathcal{C} we need an index function representation of R, so that the output of \mathcal{C} is in $\{0,1\}^{\ell}$. When $det(\mathcal{L}) \neq 2^{\ell}$ we need to make some simple changes in the reduction, we decribe these changes at the end of the proof.

The main difficulty that we encounter in formalizing the above intuition hides in the construction of a polynomial-size circuit \mathcal{I}_R for the index function of the set R. Let r = |R|. We define the circuit $\mathcal{I}_R : bd(R) \to \{0,1\}^\ell$ such that \mathcal{I}_R defines a bijective map between bd(R) and bd([r]). The circuit \mathcal{I}_R first computes the Smith Normal Form of the basis $\mathbf{B} = \mathbf{UDV}$, which can be done by a circuit that has size polynomial in $|bd(\mathbf{B})|$ [KV05], then uses the index function of the set $\mathcal{P}(\mathbf{D}) \cap \mathbb{Z}^n$, as defined in Lemma 2.1, to map each element of R to a number in [r].

Claim 3.3. There exists a bijection π between R and $\mathcal{P}(\mathbf{D}) \cap \mathbb{Z}^n$, which can be implemented by a polynomial-size circuit.

Proof of Claim 3.3. In the Smith Normal Form of **B**, we have that $\mathbf{U} \in \mathbb{Z}^{n \times n}$, $\mathbf{V} \in \mathbb{Z}^{n \times n}$ are unimodular matrices and $\mathbf{D} \in \mathbb{Z}^{n \times n}$ is a diagonal matrix. Since **V** is unimodular by Lemma 2.2, $\mathcal{L}(\mathbf{DV}) = \mathcal{L}(\mathbf{D})$ and, hence, **D** is a basis of $\mathcal{L}(\mathbf{DV})$. This implies that the function $\boldsymbol{\phi}(\mathbf{x}) = \mathbf{x} \pmod{\mathcal{P}(\mathbf{D})}$ is a bijection between $\mathcal{P}(\mathbf{DV}) \cap \mathbb{Z}^n$ and $\mathcal{P}(\mathbf{D}) \cap \mathbb{Z}^n$, with inverse map $\boldsymbol{\phi}^{-1}(\mathbf{y}) = \mathbf{y} \pmod{\mathcal{P}(\mathbf{DV})}$. Observe that both $\boldsymbol{\phi}$ and $\boldsymbol{\phi}^{-1}$ can be implemented with a polynomial-size circuit. By the unimodularity of **U**, the map $h(\mathbf{x}) = \mathbf{U}^{-1}\mathbf{x}$ defines a bijection between $\mathcal{P}(\mathbf{B}) \cap \mathbb{Z}^n$ and $\mathcal{P}(\mathbf{DV}) \cap \mathbb{Z}^n$, with $h^{-1}(\mathbf{y}) = \mathbf{Uy}$. As above, both h and h^{-1} can be implemented by polynomial-size circuits. Hence, the function $\pi(\mathbf{x}) = h(\boldsymbol{\phi}(\mathbf{x}))$ is a bijection between $R = \mathcal{P}(\mathbf{B}) \cap \mathbb{Z}^n$ and $\mathcal{P}(\mathbf{D}) \cap \mathbb{Z}^n$ with inverse map $\pi^{-1}(\mathbf{y}) = \boldsymbol{\phi}^{-1}(h^{-1}(\mathbf{y}))^2$. The size of the circuits that implement π and π^{-1} is also polynomial.

Proof of Proposition 2.4. By the unimodularity of **U** and **V** we have that $\det(\mathbf{D}) = \det(\mathbf{B})$. Since **D** is diagonal, the number of integer points in $\mathcal{P}(\mathbf{D})$ is equal to $\det(\mathcal{L}(\mathbf{D}))$. Hence, $|\mathcal{P}(\mathbf{B}) \cap \mathbb{Z}^n| = |\mathcal{P}(\mathbf{D}) \cap \mathbb{Z}^n| = \det(\mathcal{L}(\mathbf{D})) = \det(\mathcal{L}(\mathbf{B}))$.

Now we are ready to describe \mathcal{I}_R . Let **D** = diag(d_1, \ldots, d_n), observe that

$$R_{\mathbf{D}} = {\mathbf{x} \in \mathbb{Z}^n \text{ such that } \mathbf{x} \in \mathcal{P}(\mathbf{D})} = ([0, d_1] \times \cdots \times [0, d_n]) \cap \mathbb{Z}^n.$$

We use Lemma 2.1 to construct an index function $\mathcal{I}_{R_{\mathbf{D}}}$. Finally, we define the index function of *R* as $\mathcal{I}_{R}(\mathbf{x}) = \mathcal{I}_{R_{\mathbf{D}}}(\boldsymbol{\pi}(\mathbf{x}))$.

Let $\sigma : \mathbb{Z}^n \to R$ be the function that computes the modulo $\mathcal{P}(\mathbf{B})$, i.e. $\sigma(\mathbf{x}) = \mathbf{x} \pmod{\mathcal{P}(\mathbf{B})}$. Now, we have all the components to define our circuit \mathcal{C} to reduce BLICHFELDT to PPP. The circuit \mathcal{C} takes as input a boolean vector $\underline{x} \in \{0,1\}^{\ell}$ and computes $\mathcal{I}_R (\sigma (\mathcal{V}_S(\underline{x})))$, where \mathcal{I}_R is as defined above. Namely, the circuit \mathcal{C} first computes the vector $\mathbf{x} = \mathcal{V}_S(\underline{x})$, which belongs in S, then it computes a vector $\mathbf{c} \in \mathcal{P}(\mathbf{B})$ such that $\mathbf{x} \in \mathbf{c} + \mathcal{L}$, maps it to a vector in $\mathcal{P}(\mathbf{D})$ and lastly maps

²The fact that π is a bijection shows that $|\mathcal{P}(\mathbf{D}) \cap \mathbb{Z}^n| = |\mathcal{P}(\mathbf{B}) \cap \mathbb{Z}^n|$.

this vector to a boolean vector using the index function of R_D . Since we assume $|R| = 2^{\ell}$ we have that C has ℓ inputs and ℓ outputs and hence it is a valid input to PIGEONHOLE CIRCUIT. Any solution of this PIGEONHOLE CIRCUIT instance gives a solution to our BLICHFELDT instance. The PIGEONHOLE CIRCUIT with input C returns one of the following:

1. a boolean vector $\underline{x} \in \{0,1\}^{\ell}$ such that $\mathcal{C}(\underline{x}) = \underline{0}$.

If $\mathcal{V}_{S}(\mathsf{bc}(\underline{x})) \notin S$, then $\mathsf{bc}(\underline{x})$ is a solution to our initial BLICHFELDT instance. Otherwise, let $\underline{y} = \sigma(\mathcal{V}_{S}(\underline{x}))$ and $\underline{y} = \mathsf{bd}(\underline{y})$. In this case, we have that $\mathcal{I}_{R}(\underline{y}) = \underline{0}$, which implies that $\mathcal{I}_{R_{D}}(\pi(\underline{y})) = \underline{0}$. From the definition of $\mathcal{I}_{R_{D}}$ in Lemma 2.1, we get that $\pi(\underline{y}) = \mathbf{0}$ and hence $\underline{y} = \mathbf{0}$. Finally, $\sigma(\mathcal{V}_{S}(\underline{x})) = \mathbf{0}$ implies $\mathcal{V}_{S}(\underline{x}) \in \mathbf{0} + \mathcal{L}$ and so $\mathbf{x} = \mathcal{V}_{S}(\underline{x})$ is a solution to our initial BLICHFELDT instance.

2. two boolean vectors $\underline{x}, y \in \{0, 1\}^{\ell}$, such that $\mathcal{C}(\underline{x}) = \mathcal{C}(y)$.

Using the same reasoning as in the previous case we conclude that either $bc(\underline{x}), bc(\underline{y})$ is a solution to our initial BLICHFELDT instance or there exists a $\mathbf{c} \in R$ such that $\mathcal{V}_S(\underline{x}), \mathcal{V}_S(\underline{y}) \in \mathbf{c} + \mathcal{L}(\mathbf{B})$ and hence if $\mathbf{x} = \mathcal{V}_S(\underline{x}), \mathbf{y} = \mathcal{V}_S(\underline{y})$ then we have that $\mathbf{x} - \mathbf{y} \in \mathcal{L}$ and so \mathbf{x}, \mathbf{y} is a solution to our initial BLICHFELDT instance.

The only thing left to finish the proof is the case in which $det(\mathcal{L}(\mathbf{B})) < 2^{\ell}$. In this case the circuit \mathcal{C} is defined as

$$\mathcal{C}(\underline{x}) = egin{cases} \underline{x} & ext{ if } \mathsf{bc}(\underline{x}) \geq \mathsf{det}(\mathcal{L}(\mathbf{B})) \ \mathcal{I}_R\left(\sigma\left(\mathcal{V}_S(\underline{x})
ight)
ight) & ext{ if } \mathsf{bc}(\underline{x}) < \mathsf{det}(\mathcal{L}(\mathbf{B})) \end{cases}.$$

When C is as above, there are no solutions \underline{x} and \underline{y} to the PIGEONHOLE CIRCUIT problem with input C such that $bc(\underline{x}), bc(\underline{y}) \ge det(\mathcal{L}(\mathbf{B}))$. Hence, all solutions of PIGEONHOLE CIRCUIT imply a solution to our initial BLICHFELDT instance as described before.

We now proceed to show the PPP-hardness of BLICHFELDT.

Lemma 3.4. BLICHFELDT is PPP-hard.

Proof. We prove that PIGEONHOLE CIRCUIT is reducible to the BLICHFELDT problem.

Let $C : \{0,1\}^n \to \{0,1\}^n$ be an arbitrary instance of PIGEONHOLE CIRCUIT. We construct an instance of BLICHFELDT based on *q*-ary lattices as follows. Fix q = 2 and let $\mathbf{A} = [\mathbf{0} \ \mathbf{I}_n] \in \mathbb{Z}_2^{n \times 2n}$, where \mathbf{I}_n is the *n*-dimensional identity matrix. We define the lattice $\Lambda_q^{\perp}(\mathbf{A})$ and using Lemma 2.3 we compute $\mathbf{B} = \mathcal{BS}(\mathbf{A})$ such that $\mathcal{L}(\mathbf{B}) = \Lambda_q^{\perp}(\mathbf{A})$. Next, we define the set $S = \left\{ \begin{bmatrix} \mathbf{x} \\ \mathcal{C}(\mathbf{x}) \end{bmatrix}$ such that $\mathbf{x} \in \{0,1\}^n \right\} \subseteq \mathbb{Z}_2^{2n}$. Accordingly, the circuit \mathcal{V}_S has C hardcoded, and on input $\mathbf{x} \in \{0,1\}^n$ it outputs $\mathcal{V}_S(\mathbf{x}) = \begin{bmatrix} \mathbf{x} \\ \mathcal{C}(\mathbf{x}) \end{bmatrix}$, where $\begin{bmatrix} \mathbf{x} \\ \mathcal{C}(\mathbf{x}) \end{bmatrix}$ is viewed as a vector in \mathbb{Z}_2^{2n} and not as a binary string. Notice that $|S| = 2^n$ since for every $\mathbf{x} \in \{0,1\}^n$ there is one element in S with its *n*-bit prefix equal to \mathbf{x} . Thus, the BLICHFELDT instance is defined by \mathbf{B} and $(2^n, \mathcal{V}_S)$. Notice that for $\mathbf{y} = \begin{bmatrix} \mathbf{x} \\ \mathcal{C}(\mathbf{x}) \end{bmatrix} \in S$ we have

$$\mathbf{A}\mathbf{y} = \mathcal{C}(\underline{x}) \pmod{2}. \tag{3.1}$$

We will see that any solution to the above BLICHFELDT instance gives a solution for the PIGEONHOLE CIRCUIT problem with input C. The problem BLICHFELDT returns one of the following:

1. a single vector $\mathbf{y} = \begin{bmatrix} \mathbf{x} \\ \mathcal{C}(\mathbf{x}) \end{bmatrix} \in S \cap \mathcal{L}(\mathbf{B}).$

Then, by (3.1) it holds that $C(\underline{x}) = \mathbf{0} \pmod{2}$ which means $C(\underline{x}) = \underline{\mathbf{0}}$ and hence \underline{x} is a solution to PIGEONHOLE CIRCUIT.

2. two vectors $\mathbf{x}, \mathbf{y} \in S$, such that $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{x} - \mathbf{y} \in \mathcal{L}(\mathbf{B})$.

In this case, we have $\mathbf{x} = \begin{bmatrix} \mathbf{x} \\ C(\mathbf{x}) \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} \mathbf{x} \\ C(\mathbf{x}) \end{bmatrix}$ such that $\mathbf{x} - \mathbf{y} \in \Lambda_q^{\perp}(\mathbf{A})$. Thus, $\mathbf{A}(\mathbf{x} - \mathbf{y}) = \mathbf{0} \pmod{2}$ which by (3.1) implies $C(\mathbf{x}) = C(\mathbf{y})$, and because $\mathbf{x} \neq \mathbf{y}$ it has to be that $\mathbf{x} \neq \mathbf{y}$. So, the solution to PIGEONHOLE CIRCUIT is the pair \mathbf{x}, \mathbf{y} .

Finally, we argue that $2^n \ge \det(\mathcal{L}(\mathbf{B}))$ so that the BLICHFELDT problem does not output **0** trivially. This follows directly from Lemma 2.3 since q = 2.

Combining Lemma 3.2 and Lemma 3.4, we prove the following theorem.

Theorem 3.5. BLICHFELDT is PPP-complete.

4 Constrained Short Integer Solution is PPP-Complete

In this section we define the first PPP-complete problem that is *natural*, i.e. does not explicitly invoke any circuit as part of its input in contrast to the BLICHFELDT problem. We call this problem the *constrained Short Integer Solution* (cSIS) problem because it shares a similar structure with the well-known and well-studied *Short Integer Solution* problem that was defined in the seminal work of Ajtai [Ajt96], and later studied in [Mic04, MR07, GPV08, MP12].

To expand on the complexity theoretic importance and potential of cSIS, we note the SIS problem is contained in PPP by its collision-resistance nature but it is unknown if it is also PPP-hard. This poses a fascinating open question, since it implies a unique characterization of a concrete cryptographic assumption using a complexity class and vice versa. We view our result in this section (as well as in the next section) as a first step towards this direction, by showing that cSIS is PPP-complete.

Similar to the presentation of the previous section; we first define cSIS, then prove its PPP membership and finally show its PPP-hardness. In all these steps, a matrix with a special structure, that we call *binary invertible* plays an important role, and thus we define it below and prove a key property that we use.

Definition 4.1 (BINARY INVERTIBLE MATRIX). Let $\ell \in \mathbb{Z}_+$, $q \leq 2^{\ell}$ and $d, k \in \mathbb{N}$. First, we define the ℓ -th gadget vector γ_{ℓ} to be the vector $\gamma_{\ell} = \begin{bmatrix} 1 & 2 & 4 & \dots & 2^{\ell-1} \end{bmatrix}^T \in \mathbb{Z}_q^{\ell}$. Second, let $\mathbf{U} \in \mathbb{Z}_q^{d \times (d \cdot \ell)}$ be a matrix with non-zero elements only above the $(\ell + 1)$ -diagonal and $\mathbf{V} \in \mathbb{Z}_q^{d \times k}$ be an arbitrary matrix. We define the matrix $\mathbf{G} = \begin{bmatrix} (\mathbf{I}_d \otimes \gamma_{\ell}^T + \mathbf{U}) & \mathbf{V} \end{bmatrix} \in \mathbb{Z}_q^{d \times (d \cdot \ell + k)}$ to be a *binary invertible* matrix.

To illustrate the form of a binary invertible matrix we give an example below with $\ell = 3$, i.e. in

 \mathbb{Z}_8 , where the symbol \star represents any element in \mathbb{Z}_8 .

G =	1 0 0	2 0 0	4 0 0	* 1 0	* 2 0	* 4 0	* * 1	* * 2	* * 4	 	* * *	* * *	* * *	* * *	* * *	 	* * *	* * *	$\left. \right\rangle_{d \text{ rows}}$
_	0	: 0			:			:		٠.		:				:			
	_					d·ℓ	colu	mns					- 、		k col	umns	_		

It is evident from the definition of a binary invertible matrix, that it is not a fixed matrix but rather a collection of matrices. That is, after we fix *q*, the exact values of **G** depend on the choice of **U** and **V**. For example, a special case of a binary invertible matrix is the *gadget matrix* $\mathbf{G} = \mathbf{I}_d \otimes \gamma_\ell^T$ with $\mathbf{U} = \mathbf{0}^{d \times (d \cdot \ell)}$ and k = 0, that was defined in [MP12] and used in many cryptographic constructions (e.g. [GSW13, BGG⁺14, GVW15b, GVW15a, MW16, BP16, BKM17, BTVW17, PS18]).

Next, we formalize the main property of binary invertible matrices that is in the core of our proof for the inclusion of cSIS in PPP, and also explains the name "binary invertible".

Proposition 4.2. Let $\mathbf{G} = [(\mathbf{I}_d \otimes \boldsymbol{\gamma}_{\ell}^T + \mathbf{U}) \ \mathbf{V}] \in \mathbb{Z}_q^{d \times (d \cdot \ell + k)}$ be a binary invertible matrix and \mathbf{r}' be an arbitrary vector in \mathbb{Z}_q^k . Then, for every $\mathbf{b} \in \mathbb{Z}_q^d$, there exists a vector $\mathbf{r} \in \{0,1\}^{d \cdot \ell}$ such that $\mathbf{G}\begin{bmatrix}\mathbf{r}\\\mathbf{r}'\end{bmatrix} = \mathbf{b} \pmod{q}$. Additionally, the vector \mathbf{r} is computable by a polynomial-size circuit and it is guaranteed to be unique when $q = 2^{\ell}$.

Proof. For a simple illustration of the proposition, for a moment assume that $\mathbf{G} = \begin{bmatrix} \mathbf{W} & \mathbf{V} \end{bmatrix} \in \mathbb{Z}_q^{d \times (d+k)}$, where *q* is prime, $\mathbf{W} \in \mathbb{Z}_q^{d \times d}$ is an upper triangular matrix, and $\mathbf{V} \in \mathbb{Z}_q^{k \times d}$ is arbitrary. Then, for every $\mathbf{b} \in \mathbb{Z}_q^d$, using backwards substitution we can efficiently compute a vector $\mathbf{x} \in \mathbb{Z}_q^d$ such that $\mathbf{G} \begin{bmatrix} \mathbf{x} \\ \mathbf{r}' \end{bmatrix} = \mathbf{b} \pmod{q}$.

For the general case where $\mathbf{G} = [(\mathbf{I}_d \otimes \boldsymbol{\gamma}_{\ell}^T + \mathbf{U}) \ \mathbf{V}]$, we use again backward substitution. But because we require \mathbf{r} to be binary, we make \mathbf{r} to be the the binary decomposition of the corresponding solution in \mathbb{Z}_q . More precisely, we divide \mathbf{r} into d parts of ℓ coordinates each, such that $\mathbf{r} = [\mathbf{r}_1 \dots \mathbf{r}_d]^T$. We define \mathbf{g}_i^T to be the *i*-th row of the matrix \mathbf{G} . Then, the *d*-th part of \mathbf{r} is equal to $\mathbf{r}_d = \operatorname{bd} \left(b_d - \mathbf{g}_d^T \begin{bmatrix} \mathbf{0} \\ \mathbf{r}' \end{bmatrix} \pmod{q} \right)$. Next, we recursively compute the *t*-th part \mathbf{r}_t of \mathbf{r} , assuming we have already computed the parts $\mathbf{r}_{t+1}, \dots, \mathbf{r}_d$. The recursive relation for \mathbf{r}_t is

$$\mathbf{r}_{t} = \operatorname{bd} \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{r}_{t+1} \\ \mathbf{r}_{t+2} \\ \cdots \\ \mathbf{r}_{d} \\ \mathbf{r}' \end{bmatrix} \pmod{q} , \qquad (4.1)$$

where $\mathbf{b} = \begin{bmatrix} b_1 & b_2 & \dots & b_d \end{bmatrix}^T$. The fact that **r** can be computed by a polynomial sized circuit, follows easily from (4.1).

In the special case of $q = 2^{\ell}$, it is easy to see that for every $x \in [q]$ there exists a unique $\mathbf{r}_t \in \{0,1\}^\ell$ such that $\gamma_\ell^T \mathbf{r}_t = x \pmod{q}$. Additionally, (when $q = 2^\ell$) for any $\mathbf{r}_t \in \{0,1\}^\ell$, it holds that $\gamma_{\ell}^T \mathbf{r}_t < q$, and thus $\gamma_{\ell}^T \mathbf{r}_t = x$ over \mathbb{Z} . But in this case, \mathbf{r}_t is the binary decomposition of x, and it is unique. Because every \mathbf{r}_t is unique, we get that \mathbf{r} is also unique.

Remark 4.3. The property of Proposition 4.2 is the only property of binary invertible matrices that we need for our proofs. We could potentially define binary invertible matrices in a more general way. For example, a binary invertible matrix could be a permutation of the columns of a matrix of Definition 4.1. Our results follow immediately for the more general class of matrices that satisfy the properties of Proposition 4.2. Let us denote by S this set of matrices. We focus on the more restrictive case of Definition 4.1, not only for ease of the exposition, but also because given a matrix **A** there is no known efficient procedure to check whether $\mathbf{A} \in \mathcal{S}$. In fact, this problem is NP-complete, since we can encode a SUBSET-SUM instance in A and reduce SUBSET-SUM to checking whether $A \in S$. Alternatively, we could define a promise version of cSIS where G, is promised to be in S. However, this would deprive us from a syntactic definition of cSIS.

We now define the Constrained Short Integer Solution problem.

cSIS Problem.

INPUT: A matrix $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, a binary invertible matrix $\mathbf{G} \in \mathbb{Z}_q^{d \times m}$ and a vector $\mathbf{b} \in \mathbb{Z}_q^d$ where $\ell \in \mathbb{Z}_+, q \leq 2^{\ell} \text{ and } m \geq (n+d) \cdot \ell.$

OUTPUT: One of the following:

1. a vector $\mathbf{x} \in \{0,1\}^m$ such that $\mathbf{x} \in \Lambda_q^{\perp}(\mathbf{A})$ and $\mathbf{G}\mathbf{x} = \mathbf{b} \pmod{q}$,

2. two vectors $\mathbf{x}, \mathbf{y} \in \{0, 1\}^m$ such that $\mathbf{x} \neq \mathbf{y}$ with $\mathbf{x} - \mathbf{y} \in \Lambda_q^{\perp}(\mathbf{A})$ and $\mathbf{G}\mathbf{x} = \mathbf{G}\mathbf{y} = \mathbf{b} \pmod{q}$.

Lemma 4.4. cSIS is in PPP.

Proof. We show a Karp reduction from cSIS to the PIGEONHOLE CIRCUIT problem that works for any positive integer $q \ge 2$.

Let $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, $\mathbf{G} \in \mathbb{Z}_q^{d \times m}$, $\mathbf{b} \in \mathbb{Z}_q^d$ be the inputs to the cSIS problem and define $k = m - d \cdot \ell$. From the definition of cSIS, we have that $k \ge n \cdot \ell$. We define the circuit \mathcal{C} that takes as input a vector $\underline{x} \in \{0,1\}^{n \cdot \ell}$ and outputs a vector $\underline{z} \in \{0,1\}^{n \cdot \ell}$. For any input $\underline{x} \in \{0,1\}^{n \cdot \ell}$, we define the vector $\mathbf{r}' = \begin{vmatrix} \mathbf{x} \\ \mathbf{0}^{k-n \cdot \ell} \end{vmatrix} \in \mathbb{Z}_{q'}^k$ and by Proposition 4.2 we compute a vector $\mathbf{r} \in \{0,1\}^{d \cdot \ell}$ such that $\mathbf{G}\begin{bmatrix}\mathbf{r}\\\mathbf{r}'\end{bmatrix} = \boldsymbol{b} \pmod{q}. \text{ Let } \mathcal{C}_1 : \{0,1\}^{n \cdot \ell} \to \{0,1\}^{d \cdot \ell} \text{ be the circuit that on input } \boldsymbol{x} \text{ computes } \mathbf{r}.$

The circuit \mathcal{C} on input \underline{x} , first use \mathcal{C}_1 to compute $\mathbf{r} = \mathcal{C}_1(\underline{x})$, and then outputs the binary decomposition of the vector $\mathbf{A}\begin{bmatrix} \mathbf{r} \\ \mathbf{r}' \end{bmatrix} \pmod{q}$, where $\mathbf{r}' = \begin{bmatrix} \underline{x} \\ \mathbf{0}^{k-n \cdot \ell} \end{bmatrix}$. Overall, the description of \mathcal{C} is

$$\mathcal{C}(\underline{x}) = \mathsf{bd} \left(\mathbf{A} \begin{bmatrix} \mathcal{C}_1(\underline{x}) \\ \underline{x} \\ \mathbf{0}^{k-n \cdot \ell} \end{bmatrix} \pmod{q} \right).$$

Clearly, this is a polynomial-time computation, and thus the circuit \mathcal{C} is of polynomial size. To complete the proof, we show that a solution to PIGEONHOLE CIRCUIT with input C, gives a solution for the cSIS instance. The output of PIGEONHOLE CIRCUIT with input C is one of the following:

1. a vector $\underline{x} \in \{0, 1\}^{n \cdot \ell}$ such that $C(\underline{x}) = \mathbf{0}^{n \cdot \ell}$. In this case, for the vector $\mathbf{x} = \begin{bmatrix} C_1(\underline{x}) \\ \underline{x} \\ \mathbf{0} \end{bmatrix}$, we get $C(\underline{x}) = bd (\mathbf{Ax} \pmod{q}) = \underline{\mathbf{0}}$. Because the

binary decomposition bd defines a bijective map, this implies that $Ax = 0 \pmod{q}$. Also, by the definition of C_1 , we get that $Gx = b \pmod{q}$. Hence, x is a solution of cSIS with input (A, G, b).

2. two vectors $\underline{x}, \underline{y} \in \{0, 1\}^{n \cdot \ell}$, such that $\underline{x} \neq \underline{y}$ and $C(\underline{x}) = C(\underline{y})$. In this case, we define the vectors $\mathbf{x} = \begin{bmatrix} C_1(\underline{x}) \\ \underline{x} \\ \mathbf{0} \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} C_1(\underline{y}) \\ \underline{y} \\ \mathbf{0} \end{bmatrix}$ such that

$$\mathsf{bd}\left(\mathbf{A}\mathbf{x}\;(\bmod\;q)\right) = \mathcal{C}(\underline{\mathbf{x}}) = \mathcal{C}(\underline{\mathbf{y}}) = \mathsf{bd}\left(\mathbf{A}\mathbf{y}\;(\bmod\;q)\right).$$

Because the binary decomposition bd defines a bijective map, this implies that $\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{y} \pmod{q}$ and hence $\mathbf{A}(\mathbf{x} - \mathbf{y}) = \mathbf{0} \pmod{q}$. Therefore, $\mathbf{x} - \mathbf{y} \in \Lambda_q^{\perp}(\mathbf{A})$. Also, it has to be the case that $\mathbf{x} \neq \mathbf{y}$, because $\mathbf{x} \neq \mathbf{y}$ and by the definition of C_1 we get $\mathbf{G}\mathbf{x} = \mathbf{G}\mathbf{y} = \mathbf{b} \pmod{q}$. Therefore, \mathbf{x} , \mathbf{y} form a valid solution for the cSIS problem with input (\mathbf{A} , \mathbf{G} , \mathbf{b}).

Lemma 4.5. cSIS is PPP-hard.

Proof. We show a Karp reduction from PIGEONHOLE CIRCUIT to the cSIS problem.

Let $C = (C_1, \ldots, C_n)$ be the input circuit to the PIGEONHOLE CIRCUIT problem with n inputs and n outputs. Also, let d = |C| be the size of C. As we explained in Section 2.1, we may assume without loss of generality, that C consists of gates in the set $\{ \Lambda, \nabla, \oplus, \wedge, \vee \}^3$. The circuit C is represented as n directed acyclic graphs. We first describe how to construct from the i-th circuit C_i part of a cSIS instance and then we combine these parts to form a cSIS instance. Let $d_i = |C_i|$ be the size of C_i and $\mathcal{G}^{(i)} = (V^{(i)}, E^{(i)})$ be its directed acyclic graph, where $V^{(i)}$ is the set of nodes of C_i . Let $(v_1^{(i)}, v_2^{(i)}, \ldots, v_{d_i}^{(i)})$ be a topological ordering of the graph $\mathcal{G}^{(i)}$, where the first n nodes are the source nodes of $\mathcal{G}^{(i)}$ and the last node is the unique sink of $\mathcal{G}^{(i)}$. As we have already explained in Section 2.1, the source nodes of $\mathcal{G}^{(i)}$ correspond to the inputs of C_i and the sink of $\mathcal{G}^{(i)}$ corresponds to the output of C_i .

We denote by $\mathbf{G}^{(i)}$ and $\mathbf{b}^{(i)}$ the part of the final cSIS instance that corresponds to circuit C_i . We prove our hardness result for $\ell = 2$ and $q = 4^4$. We introduce two variables for each node of $\mathcal{G}^{(i)}$. The set of the first variable in each pair represents the value of the corresponding node in the evaluation of C_i and we call it the set of *value variables* and the set of the second variable in each pair is the set of *auxiliary variables*. Let us remind that in the topological ordering of $\mathcal{G}^{(i)}$ we start with the *n* input nodes v_1, \ldots, v_n , for which we use x_1, \ldots, x_n to represent their corresponding value variables. The last node of the topological ordering is the output node, and

³In fact, it is well known that only the NAND (Λ) gates suffice, but we discuss here the implementation of all these five gates, because we are going to use them in Section 5.

⁴At the end of the proof, we also show how to generalize the result for any $\ell \in \mathbb{Z}_+$ and $q = 2^{\ell}$

since it is the *i*-th output of the circuit C, we denote its value variable by y_i . We denote the remaining value variables by $z_{n+1}^{(i)}, \ldots, z_{d_i-1}^{(i)}$. Additionally, we denote by w_1, \ldots, w_n the auxiliary variables that correspond to the input nodes, by t_i the auxiliary variable of the output node and by $r_{n+1}^{(i)}, \ldots, r_{d_i-1}^{(i)}$ the remaining auxiliary variables. We summarize the notation for the variables in the Table 1. We observe that we may use $z_j^{(i)} = x_j$ for $j \le n$ and $z_{d_i}^{(i)} = y_i$ and the same holds for the auxiliary variables. Each column of $\mathbf{G}^{(i)}$ corresponds to one of these variables.

nodes	$v_1^{(i)}$	 $v_n^{(i)}$	$v_{n+1}^{(i)}$	 $v_{d_i-1}^{(i)}$	$v_{d_i}^{(i)}$
value variables	<i>x</i> ₁	 x_n	$z_{n+1}^{(i)}$	 $z_{d_i-1}^{(i)}$	y_i
auxiliary variables	w_1	 w_n	$r_{n+1}^{(i)}$	 $r_{d_i-1}^{(i)}$	t_i

Table 1: The value and auxiliary variables that correspond to every node of the graph $\mathcal{G}^{(i)}$.

Since we focus on a fixed graph $\mathcal{G}^{(i)}$, we occasionally drop the superscript (i) for simplicity. We reintroduce the superscripts when we combine all matrices $\mathbf{G}^{(i)}$ to a matrix \mathbf{G} . Our goal is to define a $\mathbf{G}^{(i)}$ and a $\mathbf{b}^{(i)}$ such that every binary solution of $\mathbf{G}^{(i)}\mathbf{s} = \mathbf{b}^{(i)} \pmod{4}$ corresponds to a valid evaluation of the circuit \mathcal{C}_i .

As explained in Section 2.1, it suffices to assume that the in-degree of every non-input node is two. Let $p_1(j)$ be the index of the first, in the topological ordering, predecessor of the node v_j and $p_2(j)$ be the index of the second. Since nodes are indexed in topological ordering we have that $p_1(j) < p_2(j) < j$. Every row of $\mathbf{G}^{(i)}$ corresponds to a node v_j , with j > n, of $\mathcal{G}^{(i)}$, and contains the coefficients of the variables in the modular equation of a form that appears in Table 2, depending on the label of v_j . We prove the correctness of these equations later in the text but it becomes also clear from the following Claim 4.6. The proof of Claim 4.6 goes through a simple enumeration of the different values for the boolean variables and can be found in Appendix A.1. The equation of node v_{d_i-j} defines also the *j*-th element of $\mathbf{b}^{(i)}$ according to Table 2.

,	equation of v_j	b_j
$\bar{\wedge}$	$\begin{aligned} r_j + 2z_j - z_{p_1(j)} - z_{p_2(j)} &= 2 \pmod{4} \\ r_j + 2z_j - z_{p_1(j)} - z_{p_2(j)} &= 3 \pmod{4} \\ z_j + 2r_j - z_{p_1(j)} - z_{p_2(j)} &= 0 \pmod{4} \\ r_j + 2z_j - z_{p_1(j)} - z_{p_2(j)} &= 0 \pmod{4} \\ r_j + 2z_j + z_{p_1(j)} + z_{p_2(j)} &= 0 \pmod{4} \end{aligned}$	2
$\bar{\vee}$	$r_j + 2z_j - z_{p_1(j)} - z_{p_2(j)} = 3 \pmod{4}$	3
\oplus	$z_j + 2r_j - z_{p_1(j)} - z_{p_2(j)} = 0 \pmod{4}$	0
\wedge	$r_j + 2z_j - z_{p_1(j)} - z_{p_2(j)} = 0 \pmod{4}$	0
\vee	$r_j + 2z_j + z_{p_1(j)} + z_{p_2(j)} = 0 \pmod{4}$	0

Table 2: Forms of equation of a non-input node $v_i^{(i)}$ of the graph $\mathcal{G}^{(i)}$, depending on its label.

Claim 4.6. Let $x, y, z, w \in \{0, 1\}$, then the following equivalences holds

1. $w + 2z - x - y = 2 \pmod{4} \Leftrightarrow x \land y = z, w = x \oplus y$ 2. $w + 2z - x - y = 3 \pmod{4} \Leftrightarrow x \lor y = z, w = \neg (x \oplus y)$ 3. $z + 2w - x - y = 0 \pmod{4} \Leftrightarrow x \oplus y = z, w = x \land y$ 4. $w + 2z - x - y = 0 \pmod{4} \Leftrightarrow x \land y = z, w = x \oplus y$ 5. $w + 2z + x + y = 0 \pmod{4} \Leftrightarrow x \lor y = z, w = x \oplus y$.

So as we said, each column of $\mathbf{G}^{(i)}$ corresponds to a variable, each row of $\mathbf{G}^{(i)}$ corresponds to an equation as described above and $\mathbf{b}^{(i)}$ is defined based on the label of each node of $\mathcal{G}^{(i)}$ according to Table 2. The order of both the rows and the columns is specified by the topological sorting of $\mathcal{G}^{(i)}$. Specifically, the first row of $\mathbf{G}^{(i)}$ describes the equation corresponding to node $v_{d_i}^{(i)}$ (the output node of $\mathcal{G}^{(i)}$), the second row of $\mathbf{G}^{(i)}$ describes the equation corresponding to node $v_{d_i-1}^{(i)}$. In general, the *k*-th row of $\mathbf{G}^{(i)}$ describes the equation corresponding to node $v_{d_i-k}^{(i)}$. We emphasize that, since there are no equations for the input nodes of $\mathcal{G}^{(i)}$, we have $d_i - n$ equations in total. The order of columns follows a similar rule, i.e. it corresponds to the reverse order of the topological ordering of $\mathcal{G}^{(i)}$. The first two columns correspond to variables z_{d_i} , r_{d_i} of node v_{d_i} followed by pairs of rows corresponding to the variables of all non-input nodes of $\mathcal{G}^{(i)}$. Among the two columns of each node, the first corresponds to the auxiliary variable and the second to the value variable, unless the label of the node is " \oplus ". In the " \oplus " case, the first corresponds to the value variable and the second to the auxiliary variable. Finally, $\mathbf{G}^{(i)}$ has *n* columns at the end for the variables that correspond to the input nodes of $\mathcal{G}^{(i)}$. For the last 2*n* columns, all the columns of the value variables precede the columns of the auxiliary variables. These rules completely define matrix $\mathbf{G}^{(i)}$ (see Table 3 for an illustration).

	r_{d_i}	z_{d_i}	r_{d_i-1}	z_{d_i-1}		r_{n+1}	z_{n+1}	x_n		x_1	w_n		w_1
eq. of v_{d_i}	1	2	*	*	•••	*	*	¦ *		*	0	•••	0
eq. of v_{d_i-1}	0	0	1	2	•••	*	*	*	•••	*	0		0
:													
eq. of v_{n+1}	0	0	0	0	•••	1	2		•••	*	0		0

Table 3: Illustration of the matrix $\mathbf{G}^{(i)}$ (assuming that \mathcal{C}_i has no " \oplus " gates).

Before defining the final matrix \mathbf{G} , we state and prove some basic properties of $\mathbf{G}^{(i)}$.

Claim 4.7. The matrix $\mathbf{G}^{(i)}$ is binary invertible.

Proof of Claim 4.7. We remind that the dimensions of $\mathbf{G}^{(i)}$ are $(d_i - n) \times (2d_i)$. Because of the order of rows and columns of $\mathbf{G}^{(i)}$, and by the form of the equations of Table 2, we have that the 1 × 2 vectors that appear in the diagonal of $\mathbf{G}^{(i)}$ are equal to $\gamma_2^T = \begin{bmatrix} 1 & 2 \end{bmatrix}$. The only other non-zero elements of the *k*-th row of $\mathbf{G}^{(i)}$ appear in the columns corresponding to $v_{p_1(d_i-k)}^{(i)}$ and $v_{p_2(d_i-k)}^{(i)}$. But, by construction $v_{p_1(d_i-k)}^{(i)}$ and $v_{p_2(d_i-k)}^{(i)}$ are always before $v_{d_i-k}^{(i)}$ in the topological ordering of $\mathcal{G}^{(i)}$. Therefore, their corresponding columns are after the columns of $v_{d_i-k}^{(i)}$. Hence, the only non-zero elements of $\mathbf{G}^{(i)}$ are above its 3rd diagonal. This implies that $\mathbf{G}^{(i)}$ has the form $\left[\left(\mathbf{I}_{d_i-n} \otimes \gamma_2^T + \mathbf{U}^{(d_i-n) \times (2(d_i-n))}\right) \quad \mathbf{V}^{n \times (2n)}\right]$ and as shown $\mathbf{U}^{(d_i-n) \times (2(d_i-n))}$ has non-zero elements only above the 3rd diagonal. This concludes the claim that $\mathbf{G}^{(i)}$ is binary invertible.

Claim 4.8. Let $\mathbf{s} \in \{0,1\}^{2d_i}$ be a solution to the modular linear equation $\mathbf{G}^{(i)}\mathbf{s} = \mathbf{b}^{(i)} \pmod{4}$. Let $\underline{\mathbf{x}}$ be a binary string consisting of the value variables of the input nodes, i.e. $\underline{\mathbf{x}} = (s_{2d_i-2n+1}, s_{2d_i-2n+2}, \dots, s_{2d_i-n+1})$. Then, the second coordinate s_2 of \mathbf{s} is equal to $s_2 = C_i(\underline{\mathbf{x}})$.

Proof of Claim 4.8. We inductively prove that the value of the coordinate s_{2d_i-2j+2} of **s** is equal to the value of the non-input node v_j in $\mathcal{G}^{(i)}$ in the evaluation of $\mathcal{C}_i(\mathbf{x})$.

INDUCTION BASE. By the definition of \underline{x} we have that the coordinates $(s_{2d_i-2n+1}, \ldots, s_{2d_i-n+1})$ are equal to the input values.

INDUCTIVE HYPOTHESIS. Assume that for any k such that n < k < j the value of the coordinate s_{2d_i-2k+2} of **s** is equal to the value of the non-input node v_k in $\mathcal{G}^{(i)}$ in the evaluation of $\mathcal{C}_k(\underline{x})$.

INDUCTIVE STEP. The vector **s** has to satisfy the $(d_i - n - j + 1)$ -th modular equation of the system $\mathbf{G}^{(i)}\mathbf{s} = \mathbf{b}^{(i)} \pmod{4}$. Without loss of generality, we assume that the label of v_j is " $\bar{\wedge}$ ". This equation then suggests that $s_{2d_i-2j+1} + 2s_{2d_i-2j+2} - s_{2d_i-2p_1(j)+2} - s_{2d_i-2p_2(j)+2} = 2 \pmod{4}$ and by Claim 4.6 we get that

$$s_{2d_i-2j+2} = \left(s_{2d_i-2p_1(j)+2}\right) \land \left(s_{2d_i-2p_2(j)+2}\right).$$
(4.2)

But, from inductive hypothesis we know that $s_{2d_i-2p_1(j)+2}$ and $s_{2d_i-2p_2(j)+2}$ take the correct values of $v_{p_1(j)}$ and $v_{p_2(j)}$ in the evaluation of $C_i(\underline{x})$. Hence, from Equation (4.2) we immediately get that s_{2d_i-2j+2} takes the value of node v_j . Similarly, we can show the inductive step for all other possible labels of v_j .

For $j = d_i$ the statement that we just proved through Induction implies that $s_2 = C_i(\underline{x})$.

We are finally ready to describe our matrix **G** and vector **b**. We remind that $d_i = |C_i|$ and we define $d'_i = d_i - n$. Let $d = \sum_{i=1}^n d'_i$. The matrix **G** is of dimension $d \times 2(d+n)$. We remind that from Claim 4.7 matrices **G**⁽ⁱ⁾ are binary invertible and hence let **U**⁽ⁱ⁾ and **V**⁽ⁱ⁾ the matrices that satisfy the equation

$$\mathbf{G}^{(i)} = \begin{bmatrix} \left(\mathbf{I}_{d'_i} \otimes \boldsymbol{\gamma}_2^T + \mathbf{U}^{(i)} \right) & \mathbf{V}^{(i)} \end{bmatrix}.$$
(4.3)

We define **G** to be equal to

$$\mathbf{G} = \begin{bmatrix} \left(\mathbf{I}_{d_{1}^{\prime}} \otimes \gamma_{2}^{T} + \mathbf{U}^{(1)} \right) & \mathbf{0} & \dots & \mathbf{0} & \mathbf{V}^{(1)} \\ \mathbf{0} & \left(\mathbf{I}_{d_{2}^{\prime}} \otimes \gamma_{2}^{T} + \mathbf{U}^{(2)} \right) & \dots & \mathbf{0} & \mathbf{V}^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \left(\mathbf{I}_{d_{n}^{\prime}} \otimes \gamma_{2}^{T} + \mathbf{U}^{(n)} \right) & \mathbf{V}^{(n)} \end{bmatrix}, \quad (4.4)$$

and the vector **b** to be

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}^{(1)} \\ \mathbf{b}^{(2)} \\ \vdots \\ \mathbf{b}^{(n)} \end{bmatrix}.$$
 (4.5)

From the definition of **G** and Claim 4.7 it is immediate that **G** is binary invertible.

Claim 4.9. The matrix **G** defined in Equation (4.4) is binary invertible.

Additionally, let $k_i = 2 + \sum_{j=1}^{i-1} d'_j$, then the following claim is a simple corollary of Equations (4.4) and (4.5) and Claim 4.8.

Claim 4.10. Let $\mathbf{s} \in \{0,1\}^{2(d+n)}$ be a solution to the modular linear equation $\mathbf{Gs} = \mathbf{b} \pmod{4}$. Let also \underline{x} the binary string that is equal to the value of the value variables of the input nodes, i.e. $\underline{x} = (s_{2d+1}, s_{2d+2}, \dots, s_{2d+n})$. Then, the binary string $\underline{z} = (s_{k_1}, s_{k_2}, \dots, s_{k_n})$ is equal to $\underline{z} = C(\underline{x})$.

To complete the description of the cSIS instance to which we reduce PIGEONHOLE CIRCUIT, we have to describe also the matrix **A**. The dimensions of **A** are $n \times 2(d + n)$. We describe as a concatenation of three matrices $\mathbf{A}_1 \in \mathbb{Z}_q^{n \times 2d}$, $\mathbf{A}_2 \in \mathbb{Z}_q^{n \times n}$, $\mathbf{A}_3 \in \mathbb{Z}_q^{n \times n}$, such that $\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \end{bmatrix}$. Each row of \mathbf{A}_1 corresponds to an output of C and has a single 1 in the position (i, k_i) . More precisely $\mathbf{A}_1 = \sum_{i=1}^n \mathbf{E}_{i,k_i}$, where $\mathbf{E}_{i,j}$ is the matrix with all zeros except in position (i, j). We also set $\mathbf{A}_2 = \mathbf{0}$ and $\mathbf{A}_3 = 2\mathbf{I}_n$ which implies $\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} & 2\mathbf{I}_n \end{bmatrix}$ (see Table 4 for an illustration).

	$r_{d_1}^{(1)}$	$z_{d_1}^{(1)}$		$r_{d_2}^{(2)}$	$z_{d_2}^{(2)}$		$r_{d_n}^{(n)}$	$z_{d_n}^{(n)}$	x_n		x_1	w_n		w_1
y_1	0	1	•••	0	0		0	0	0	•••	0	2	•••	0
y_2	0	0		0	1		0	0	0		0	0		0
÷						÷				÷		 	÷	
y_n	0	0	•••	0	0		0	1	0		0	0		2

Table 4: Illustration of the matrix **A** (assuming that C has no " \oplus " gates).

At a high level, the first 2*s* columns of **A** and **G** are meant for the description of circuit C. In particular, there are two columns for each gate of C, the first corresponds to an auxiliary variable and the second to the output of the gate ⁵. Then, there are *n* columns corresponding to the input of C and *n* auxiliary columns, one for each output of C. Similarly to Lemma 3.4, **A** has non-zero elements in the columns corresponding to the *n* outputs of C and **G** and **b** play the role of the set *S*, namely they guarantee that the output of cSIS will encode \underline{x} and $C(\underline{x})$.

The output of cSIS on input $(\mathbf{A}, \mathbf{G}, \mathbf{b})$ is one of the following:

1. a vector $\mathbf{s} \in \{0,1\}^{2(n+s)}$ such that $\mathbf{s} \in \Lambda_4^{\perp}(\mathbf{A})$ and $\mathbf{Gs} \equiv \mathbf{b} \pmod{4}$.

Let $\underline{x} \in \{0,1\}^n$ and $\underline{y} \in \{0,1\}^n$ be the *n* input and *n* output coordinates of **s** respectively, i.e. $\underline{x} = (s_{2d+1}, \ldots, s_{2d+n})$ and $\underline{y} = (s_{k_1}, \ldots, s_{k_n})$, as defined above. Let $\underline{w} \in \{0,1\}^n$ be the *n* last coordinates of **s**. Then, since each row of **A** has exactly one coordinate equal to 1, corresponding to a value in \underline{y} , and one coordinate equal to 2, corresponding to a coordinate in \underline{w} , $\mathbf{Ax} = \mathbf{0} \pmod{4}$ implies both that $\underline{y} = \mathbf{0}$ and $\underline{w} = \mathbf{0}$. We can now use Claim 4.10, and get that $C(\underline{x}) = \underline{y}$, which in turn implies that $C(\underline{x}) = \mathbf{0}$. Hence, \underline{x} is a valid solution to PIGEONHOLE CIRCUIT with input C.

2. two vectors $\mathbf{s}, \mathbf{t} \in \{0, 1\}^{2(n+s)}$, such that $\mathbf{s} \neq \mathbf{t}, \mathbf{s} - \mathbf{t} \in \Lambda_4^{\perp}(\mathbf{A})$ and $\mathbf{Gs} \equiv \mathbf{b} \pmod{4}$, $\mathbf{Gt} \equiv \mathbf{b} \pmod{4}$.

Let $\underline{x}_1, \underline{x}_2\{0,1\}^n$ and $\underline{y}_1, \underline{y}_2\{0,1\}^n$ be the *n* input and *n* output coordinates of **s** and **t** respectively, i.e. $\underline{x}_1 = (s_{2d+1}, \ldots, s_{2d+n}), \underline{y}_1 = (s_{k_1}, \ldots, s_{k_n})$ and $\underline{x}_2 = (t_{2d+1}, \ldots, t_{2d+n}), \underline{y}_2 = (t_{k_1}, \ldots, t_{k_n})$. Let also $\underline{w}_1 \in \{0,1\}^n$ and $\underline{w}_2 \in \{0,1\}^n$ be the *n* last coordinates of **s** and **t** respectively. Then, similarly to the previous case, $\mathbf{A}(\mathbf{s} - \mathbf{t}) = \mathbf{0} \pmod{4}$ implies $\underline{y}_1 = \underline{y}_2$ and $\underline{w}_1 = \underline{w}_2$. From $\underline{w}_1 = \underline{w}_2$, $\mathbf{s} \neq \mathbf{t}$ and the uniqueness guaranteed by Proposition 4.2 we can easily conclude that $\underline{x}_1 \neq \underline{x}_2$. Also, using Claim 4.10 we get that $\mathcal{C}(\underline{x}_1) = \underline{y}_1$, $\mathcal{C}(\underline{x}_2) = \underline{y}_2$ and since $\underline{y}_1 = \underline{y}_2$ we get $\mathcal{C}(\underline{x}_1) = \mathcal{C}(\underline{x}_2)$ with $\underline{x}_1 \neq \underline{x}_2$. Therefore, the pair $\underline{x}_1, \underline{x}_2$ is a valid solution to PIGEONHOLE CIRCUIT with input \mathcal{C} .

This completes the hardness proof for q = 4.

⁵This is ice versa in the case of \oplus gate, but we can assume without loos of generality that our circuit consists of only $\overline{\wedge}$ gates and then this holds.

For the $q = 2^{\ell}$ case, we need to augment **A** and **G**. This is done by introducing ℓ variables for every node of C. One of the is still the value variable and the $\ell - 1$ rest are auxiliary variables. We also have to concatenate a zero matrix of size $s \times (\ell - 1)(n + s)$ on the right of **G**. For the matrix $\mathbf{A} \in \mathbb{Z}^{n \times \ell(n+s)}$, we concatanate $\ell - 1$ matrices of the form $2^i \mathbf{I}$ for $i \in \{2, 3, 4, \dots, \ell\}$ on the right. The vector **b** remains the same. The new tuple is still a valid input for cSIS, since the parameters are appropriately set and **G** remains binary invertible. Since only zero entries on the right have been added to the matrix **G**, it decribes the circuit C as we argued above. Finally, let $\mathbf{x} \in \{0,1\}^{\ell(n+s)}$ such that $\mathbf{A}\mathbf{x} = 0 \pmod{4}$, then the last $(\ell - 1)n$ coordinates of \mathbf{x} must be 0 and the rest of the proof is as above.

Combining Lemma 4.4 and Lemma 4.5, we prove the main theorem of this section.

Theorem 4.11. The cSIS problem is PPP-complete.

We provide an example of our reduction for very simple circuit C in Figure 3.

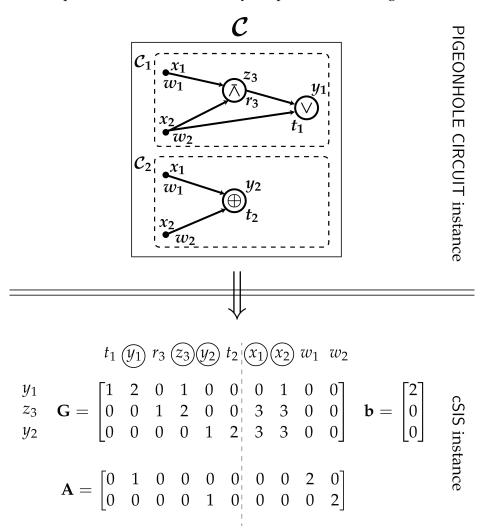


Figure 3: A simple example of the construction of Lemma 4.5.

5 Complete Collision Resistant Hash Function

The similarities between the cSIS problem and SIS raise the question of whether cSIS has cryptographic applications. In this section, we propose a candidate family of *collision-resistant* hash (CRH) functions based on the *average-case hardness* of the cSIS problem, and also discuss its worstcase hardness. The computational problem weak-cSIS associated with our collision resistant hash function family is a variant of cSIS presented in Section 4. In this case, the modular constraint are *homogeneous*, i.e. $\mathbf{b} = \mathbf{0}$, and the inequality constraints on the dimension of the matrices are strict. This change in the relation of the parameters might seem insignificant, but it is actually very important since it transforms or problem into a purely lattice problem: On input matrices \mathbf{A} , \mathbf{G} with corresponding bases $\mathbf{B}_{\mathbf{A}}$ and $\mathbf{B}_{\mathbf{G}}$, where \mathbf{G} is binary invertible, find two vectors \mathbf{x} and \mathbf{y} such that $\mathbf{x}, \mathbf{y} \in \mathcal{L}(\mathbf{B}_{\mathbf{G}})$ and $\mathbf{x} - \mathbf{y} \in \mathcal{L}(\mathbf{B}_{\mathbf{A}})$. Our proof that weak-cSIS is PWPP-complete, increases our hope that SIS is PWPP-complete too, since it overcomes one important difficulty towards reducing weak-cSIS to SIS. We start with a formal definition of collision-resistant hash function.

Collision-Resistant Hash Functions: Let *k* be the security parameter. A family of functions

$$\mathcal{H} = \left\{ \mathbf{H}_{\underline{s}} : \{0,1\}^k \to \{0,1\}^{p(k)} \right\}_{\underline{s} \in \{0,1\}^{p'(k)}}$$

where p(k) and p'(k) are polynomials, is *collision-resistant* if:

(SHRINKING) The output of H_s is smaller than its input. Namely, p(k) < k.

(EFFICIENT SAMPLING) There exists a probabilistic polynomial-time algorithm Gen that on input 1^k samples a uniform key <u>s</u>.

(Efficient Evaluation) There exists a deterministic polynomial-time algorithm that on input a key \underline{s} and an $\underline{x} \in \{0,1\}^k$ outputs $H_s(\underline{x}) \in \{0,1\}^{p(k)}$.

(Collision-Resistance) For every probabilistic polynomial-time adversary A, there exists a negligible function $\nu(k) = \operatorname{negl}(k)$, such that for any $k \in \mathbb{Z}_+$:

$$\Pr_{\underline{s} \leftarrow \mathsf{Gen}(1^k)} \left[(\underline{x}_1, \underline{x}_2) \leftarrow \mathcal{A}(1^k, \underline{s}) \text{ s.t. } \underline{x}_1 \neq \underline{x}_2 \text{ and } \mathsf{H}_{\underline{s}}(\underline{x}_1) = \mathsf{H}_{\underline{s}}(\underline{x}_2) \right] \leq \nu(k).$$

Next, we define our new CRH function family. Let *k* be the security parameter. Let $\ell \in \mathbb{Z}_+$, $q = 2^{\ell}$ and $d \in \mathbb{Z}_+$ be parameters. Let r = poly(k) such that $r\ell < k$. The family of hash functions,

$$\mathcal{H}_{\mathsf{cSIS}} = \left\{ \mathrm{H}_{\underline{s}} : \{0,1\}^k \to \{0,1\}^{r\ell} \right\}_{\underline{s} \in \{0,1\}^{p(k)}}$$

is defined as follows.

- Gen_{cSIS}(1^{*k*}) samples a uniform $\underline{s} \in \{0,1\}^{p(k)}$ and interprets it as a uniform matrix $\mathbf{A} \in \mathbb{Z}_q^{r \times (k+\ell d)}$ and a uniformly chosen binary invertible matrix $\mathbf{G} \in \mathbb{Z}_q^{d \times (k+\ell d)}$. This algorithm runs in polynomial-time in *k*.

-
$$H_{(\mathbf{A},\mathbf{G})}(\underline{x})$$
: On input $\underline{x} \in \{0,1\}^k$, compute the unique $\underline{u} \in \{0,1\}^{\ell d}$ such that $\mathbf{G}\begin{bmatrix}\underline{u}\\\underline{x}\end{bmatrix} = \mathbf{0} \pmod{q}$ as in Proposition 4.2, and output $\mathsf{bd}\left(\mathbf{A}\begin{bmatrix}\underline{u}\\\underline{x}\end{bmatrix} \pmod{q}\right)$.

Remark 5.1. We note that if SIS ⁶ is hard, then \mathcal{H}_{cSIS} is collision-resistant. In fact, a more general statement holds: if cSIS with $\mathbf{b} = \mathbf{0}$ is hard on average, then \mathcal{H}_{cSIS} is collision-resistant. Also, based on the above description of $H_{(\mathbf{A},\mathbf{G})}$, the vector $\mathbf{z} = \begin{bmatrix} \mathbf{u} \\ \mathbf{x} \end{bmatrix}$ is a solution to cSIS with input $(\mathbf{A},\mathbf{G},\mathbf{0})$. Finally, note that for $\ell = 1$ finding collisions is trivial and hence we assume that $\ell \geq 2$.

In the rest of this section, we analyze the hardness of finding collision in the \mathcal{H}_{cSIS} family. We call this problem *weak*-cSIS (weak-cSIS) because of its similar nature to cSIS. First, we consider its *worst-case* hardness and draw connections between the family \mathcal{H}_{cSIS} , a restricted version of the cSIS problem, and the complexity class PWPP. Then, we move on to the average-case hardness of weak-cSIS and argue that it defines a canditate for a *universal* CRH function family.

5.1 Worst-case Hardness of \mathcal{H}_{cSIS}

The class Polynomial Weak Pigeon Principle PWPP (a subclass of PPP) is particularly interesting for cryptography because it contains all collision-resistant hash functions. In this part, we show that a generalized version of the SIS problem we define below, namely weak-cSIS, is complete for the class PWPP.

weak-cSIS Problem.

INPUT: Any key $\underline{s} = (\mathbf{A}, \mathbf{G}) \in \mathbb{Z}_q^{r \times (\ell d + k)} \times \mathbb{Z}_q^{d \times (\ell d + k)}$ that indexes a function $\mathbf{H}_{\underline{s}} \in \mathcal{H}_{cSIS}$. OUTPUT: Two boolean vectors $\underline{x}_1 \neq \underline{x}_2$, such that $\mathbf{H}_{\underline{s}}(\underline{x}_1) = \mathbf{H}_{\underline{s}}(\underline{x}_2)$.

The membership of weak-cSIS in PWPP is straight-forward. The challenging part of the proof is in showing its PWPP-hardness, which shares some common ideas with the proof of Lemma 4.5.

Lemma 5.2. weak-cSIS is in PWPP.

Proof. We show a Karp-reduction from weak-cSIS to COLLISION. Let \underline{s} be an input to weak-cSIS and let C be the poly($|\underline{s}|$) size circuit that on input \underline{x} outputs $H_{\underline{s}}(\underline{x})$. Because $r\ell < k$, the C is a valid input for COLLISION. Let $\underline{x}_1, \underline{x}_2$ be the two boolean vectors that COLLISION outputs. Thus, $C(\underline{x}_1) = C(\underline{x}_2)$, which directly implies that $(\underline{x}_1, \underline{x}_2)$ is a solution of weak-cSIS with input \underline{s} . \Box

Now, we move to the more challenging hardness proof. Even though some of the proof techniques are reminiscent of the ones in Section 4, we need new ideas, mainly because of the homogeneity of the constraints in weak-cSIS. Before presenting the proof, we define a special version of COLLISION problem. Then, we state and prove an easy lemma. This lemma has appeared in various previous works (see Lemma 2.2 in [Jer16]), but it is useful for us to state and prove it using our notation.

COLLISION_{$p(\kappa)$} Problem.

INPUT: A circuit C with κ inputs and $p(\kappa)$ outputs. OUTPUT: Two boolean vectors $\underline{x}_1 \neq \underline{x}_2$, such that $C(\underline{x}_1) = C(\underline{x}_2)$.

Then, the following lemma is easy to check:

Lemma 5.3. *The* COLLISION *problem is Karp-reducible to* COLLISION_{κ -2}*.*

⁶For a definition of SIS, see Section 5.2.

Proof. First, we claim that it suffices to show that $\text{COLLISION}_{\kappa-1}$ is Karp-reducible to $\text{COLLISION}_{\kappa-2}$. This is because every circuit C with κ inputs and m outputs can be transformed into a circuit C' with κ inputs and $\kappa - 1$ outputs by padding the output with zeros. We note that every collision of C' is a collision of C.

Now, we show that $\text{COLLISION}_{\kappa-1}$ is Karp-reducible to $\text{COLLISION}_{\kappa-2}$. Let C be a circuit with n inputs and n-1 outputs, we create a new circuit C' with n+1 inputs and n-1 outputs such that $C'(\underline{x}, b) = C(C(\underline{x}), b)$, where $\underline{x} \in \{0, 1\}^n$ and $b \in \{0, 1\}$. Then, $\text{COLLISION}_{\kappa-2}$ with input C' outputs $\underline{y}_1 = (\underline{x}_1, b_1)$ and $\underline{y}_2 = (\underline{x}_2, b_2)$ such that $\underline{y}_1 \neq \underline{y}_2$ and $C'(\underline{y}_1) = C'(\underline{y}_2)$. We consider the following possible cases:

- $b_1 \neq b_2$, then \underline{y}_1 and \underline{y}_2 form a collision for C.
- $b_1 = b_2$, then $\mathbf{x}_1^{-1} \neq \mathbf{x}_2^{-2}$ and one of the following holds: $\mathcal{C}(\mathbf{x}_1) \neq \mathcal{C}(\mathbf{x}_2)$ or $\mathcal{C}(\mathbf{x}_1) = \mathcal{C}(\mathbf{x}_2)$. In the first case, \mathbf{y}_1 and \mathbf{y}_2 form a collision for \mathcal{C} . Otherwise, \mathbf{x}_1 and \mathbf{x}_2 form a collision for \mathcal{C} .

The above lemma naturally generalizes to any polynomial shrinkage p(k) of the input by repeating the same construction. For our purposes shrinking the input by two bits is enough.

Lemma 5.4. weak-cSIS is PWPP-hard.

Proof. From Lemma 5.3, it suffices to show a Karp-reduction from $\text{COLLISION}_{\kappa-2}$ to weak-cSIS. Let C be an input of $\text{COLLISION}_{\kappa-2}$ with n inputs, r = n - 2 outputs and d gates. Also, let q = 4. We can generalize to any $q = 2^{\ell}$ for $\ell > 1$ similarly to the proof of Lemma 4.5.

If we set $\underline{u} \in \{0,1\}^{2d}$ to be the unique binary vector such that $\mathbf{G} \begin{bmatrix} \underline{u} \\ \underline{x} \end{bmatrix} = 2 \cdot \mathbf{1} \pmod{4}$ in the construction of $\mathrm{H}_{\underline{s}}(\underline{x})$, then the proof follows exactly the same steps as the one of Lemma 4.5. However, in this case we can prove a stronger statement, where $\begin{bmatrix} \underline{u} \\ \underline{x} \end{bmatrix}$ belongs to the lattice $\Lambda^{\perp}(\mathbf{G})$. First, let us restate the part of the Claim 4.6 that we need for our proof.

Claim 5.5. The following equivalences hold:

• $x \oplus y = z \iff \exists w \in \{0,1\} \ s.t. \ -x - y + w + 2z \equiv 0 \pmod{4}$ • $x \lor y = z \iff \exists w \in \{0,1\} \ s.t. \ x + y + w + 2z \equiv 0 \pmod{4}$

If we could implement a circuit using $\{\oplus, \lor\}$, then combining Claim 5.5 and the proof techniques of Lemma 4.5, we would have the desired result. Even though this is not possible, we note that we can implement a circuit using $\{\oplus, \lor, 1\}$, since the implementation of ∇ suffices and $(x \lor y) \oplus 1 = x \nabla y$. Inspired by the above observations, our approach in showing the hardness of weak-cSIS is to construct a valid input $\underline{s} = (\mathbf{A}, \mathbf{G})$ for weak-cSIS such that \mathbf{G} encodes a circuit consisting of only XOR and OR gates and every solution of this instance of weak-cSIS gives a solution for COLLISION_{$\kappa-2$} with input C.

First, we observe that the fact that r = n - 2 guarantees the existence of at least $6\lfloor k/4 \rfloor$ pairs of \underline{x}_1 and \underline{x}_2 such that $\underline{x}_1 \neq \underline{x}_2$ and $H_{\underline{s}}(\underline{x}_1) = H_{\underline{s}}(\underline{x}_2)$. In particular, there exist collision pairs such that $\underline{x}_1 \neq \underline{0}$ and $\underline{x}_2 \neq \underline{0}$.

As we explained, without loss of generality, we assume that C consists of only $\{\lor, \oplus, 1\}$ gates. Before starting with the reduction we change our circuit C to a circuit C' such that any collision of C' corresponds to a collision of C. We construct a circuit C' such that it first computes the OR of all the inputs, $z = x_1 \lor x_2 \lor \cdots \lor x_n$. Then, in the circuit's graph we substitute the outgoing edges of nodes with label "1" with outgoing edges from the node *z*, that we introduced. Hence, C' only contains $\{\lor, \oplus\}$ gates. The next claim follows by construction of C'.

Claim 5.6. For every $\underline{x} \in \{0,1\}^k \setminus \underline{0}$, it holds that $C'(\underline{x}) = C(\underline{x})$ and $C'(\underline{0}) = \underline{0}$.

Proof. Let $\underline{x} \in \{0,1\}^k \setminus \underline{0}$, then for this input the value of the node *z* that we introduced is certainly equal to 1. Since we replaced all the uses of the constant gate 1 with the output of *z* and *z* outputs one, the results of the circuit will be exactly the same. Also, it is easy to observe that any circuit with only $\{\vee, \oplus\}$ gates, on input $\underline{0}$ outputs always $\underline{0}$, and hence $C'(0) = \underline{0}$.

Let $\mathbf{G} \in \mathbb{Z}_q^{d \times 2(n+r)}$ be the binary invertible matrix that encodes \mathcal{C}' (as described in Lemma 4.5) and $\mathbf{A} \in \mathbb{Z}_q^{r \times 2(n+r)}$ be as in Lemma 4.5. Observe that since we have only used the gates $\{\vee, \oplus\}$ in \mathcal{C}' the vector **b** in the reduction described in Lemma 4.5 is always equal to **0**. Thus, (\mathbf{A}, \mathbf{G}) is a valid input for the problem weak-cSIS.

Let $(\underline{x}_1, \underline{x}_2)$ be a solution of weak-cSIS with $\underline{x}_1 \neq \underline{0}$ and $\underline{x}_2 \neq \underline{0}$, then the pair \underline{x}_1 and \underline{x}_2 is a collision for C. But, we cannot guarantee that $\underline{x}_1 \neq \underline{0}$ and $\underline{x}_2 \neq \underline{0}$ when we use C' as input for weak-cSIS. Thus, we need a modification of C' that guarantees that weak-cSIS will not return the $\underline{0}$ -vector as part of a solution. To achieve this, we construct a new circuit C'' that is exactly the same as C' with one more output variable set to be equal to $z = x_1 \lor x_2 \lor \ldots \lor x_n$. This last output of C'' is equal to 1 if and only if $\underline{x} \neq \underline{0}$. Observe that the output of C'' consists of n - 1 bits, and hence C'' still compresses its domain by one bit. Let $(\mathbf{A}', \mathbf{G}')$ be the matrices corresponding to C'' according to the construction of proof of Lemma 4.5, then $\underline{s}' = (\mathbf{A}', \mathbf{G}')$ is a valid input for weak-cSIS.

We conclude the proof by observing that every output of weak-cSIS with input \underline{s}' gives a collision $\mathcal{C}''(\underline{x}) = \mathcal{C}''(\underline{y})$, with $\underline{x} \neq \underline{y}$. If $\underline{x} \neq \underline{0}$ and $\underline{y} \neq \underline{0}$, then it follows from Claim 5.6 and the construction of \mathcal{C}'' that $\mathcal{C}(\underline{x}) = \mathcal{C}(\underline{y})$, and hence the pair $\underline{x}, \underline{y}$ is a solution to our initial COLLISION instance. Additionally, there is no collision of the form $\underline{0}$ and \underline{x} . Assume that there was, then since the last bit of $\mathcal{C}''(\underline{0})$ is 0, it must be that $\mathcal{C}''(\underline{x})$ is also 0. However, by construction of $\mathcal{C}'', \underline{0}$ is the unique binary vector for which the last bit of the output is 0, so it must be that $\underline{x} = 0$ which is a contradiction. Therefore, $\underline{x} \neq \underline{0}$ and $\underline{y} \neq \underline{0}$, and the lemma follows.

By combining Lemma 5.2 and Lemma 5.4, we get the following theorem.

Theorem 5.7. *The* weak-cSIS *problem is* PWPP*-complete.*

5.2 Average-case Hardness of \mathcal{H}_{cSIS}

In the previous section, we showed that weak-cSIS defines a *worst-case* collision resistant hash function, similar to the result of [Sel92] for one-way functions. However, the definition of collision-resistance in cryptography requires a stronger property. More specifically, it says that given a random chosen function in the family, it is hard to find a collision for this function. In this section, we investigate the average-case hardness of \mathcal{H}_{cSIS} in the search of a construction for a candidate of a both *natural* and *universal collision-resistant hash function*.

We have briefly mentioned the connection between \mathcal{H}_{cSIS} and SIS. Now, we describe it in more detail. We start by defining the SIS problem.

$SIS_{q,n,m,\beta,p}$ Problem.

INPUT: A uniformly random matrix $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, where $m > 2n \log(q)$. OUTPUT: A vector $\mathbf{r} \in \mathbb{Z}_q^m$ such that $\|\mathbf{r}\|_p \leq \beta$ and $\mathbf{Ar} = \mathbf{0} \pmod{q}$.

Whenever the parameters are clear from the context, we drop them from the subscripts for ease of notation.

It is easy to see that if SIS is hard on the average, then \mathcal{H}_{cSIS} is collision-resistant. Let $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$ be uniformly random and \underline{x}_1 and \underline{x}_2 be a collision of $H_{(\mathbf{A},\mathbf{0})}$. Then, $\underline{x}_1 - \underline{x}_2$ is a solution of SIS with input \mathbf{A} . This remark, combined with the known reduction from $\tilde{O}(n)$ -SIVP to $SIS_{\tilde{O}(n),n,\Omega(n \operatorname{polylog}(n)),\sqrt{n},2}$ [MR07], directly implies a reduction from $\tilde{O}(n)$ -SIVP to weak-cSIS.

Corollary 5.8. $\tilde{O}(n)$ -SIVP is reducible to weak-cSIS, and thus is contained in PWPP.

Finally we note that, since weak-cSIS is PWPP-complete and PWPP contains all collision-resistant hash functions, the following statement is also true.

Corollary 5.9. If there exists a family of collision-resistant hash functions \mathcal{H} , then there exists a distribution over keys $\underline{s} \in \{0,1\}^{p(k)}$ for \mathcal{H}_{cSIS} , where Gen draws key from this distribution and \mathcal{H}_{cSIS} is collision-resistant.

6 Lattice Problems and PPP

The lattice nature of the cSIS problem raises also the next question:

What is the connection between the class PPP and lattices?

In this section we present the known result for the lattice problems that are contained in PPP and we also mention how some of these results are implied by the completeness results that we presented in the previous sections. We start with the formal definition of the lattice problems that we are interested in.

Lattice Problems. We recall some of the most important lattice problems: the Shortest Vector Problem (γ -SVP), the Shortest Independent Vectors Problem (γ -SIVP) and the Closest Vector Problem (γ -CVP). We start by defining some important lattice quantities. For a lattice \mathcal{L} ,

$$\operatorname{dist}(\mathbf{t}, \mathcal{L}) = \min_{\mathbf{x} \in \mathcal{L}} \|\mathbf{x} - \mathbf{t}\|$$
(6.1)

$$\lambda_i(\mathcal{L}(\mathbf{B})) = \min_{\mathbf{x} \in \mathcal{L} \setminus \operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}), \mathbf{x} \neq \mathbf{0}} \|\mathbf{x}\|, \quad \text{for } i = 1, \dots, n$$
(6.2)

where $\|\mathbf{v}_i\| = \lambda_i(\mathcal{L}(\mathbf{B}))$.

γ -SVP Problem.

INPUT: A *n*-dimensional basis $\mathbf{B} \in \mathbb{Z}^{n \times n}$. OUTPUT: A lattice vector $\mathbf{v} \in \mathcal{L}$ such that $\|\mathbf{v}\| \leq \gamma(n) \cdot \lambda_1(\mathcal{L}(\mathbf{B}))$.

γ -SIVP Problem.

INPUT: A *n*-dimensional basis $\mathbf{B} \in \mathbb{Z}^{n \times n}$.

OUTPUT: A set of *n* linearly independent lattice vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ such that $\max_i ||\mathbf{v}_i|| \le \gamma(n) \cdot \lambda_n(\mathcal{L}(\mathbf{B}))$.

γ -CVP Problem.

INPUT: A *n*-dimensional basis $\mathbf{B} \in \mathbb{Z}^{n \times n}$ and a target vector $\mathbf{t} \in \mathbb{Q}^n$. OUTPUT: A lattice vector \mathbf{v} such that $\|\mathbf{v} - \mathbf{t}\| \leq \gamma(n) \cdot \operatorname{dist}(\mathbf{v}, \mathcal{L}(\mathbf{B}))$.

$\frac{1}{\gamma}$ -BDD Problem.

INPUT: A *n*-dimensional basis $\mathbf{B} \in \mathbb{Z}^{n \times n}$ and a target vector $\mathbf{t} \in \mathbb{Q}^n$, with $\mathbf{t} \leq \frac{\lambda_1(\mathcal{L})}{2\gamma(n)}$. OUTPUT: A lattice vector \mathbf{v} such that $\|\mathbf{v} - \mathbf{t}\| = \text{dist}(\mathbf{t}, \mathcal{L}(\mathbf{B}))$.

Where $\gamma(n) \ge 1$ is a non-decreasing function in the lattice dimension *n*. For $\gamma = 1$ we get the exact version of the problem.

We now show that well-studied (approximation) lattice problems are contained in PPP. First, we define the MINKOWSKI problem and show a reduction to BLICHFELDT. Second, using known reductions, we conclude the membership of other lattice problems to PPP.

MINKOWSKI_p Problem.

INPUT: A *n*-dimensional basis $\mathbf{B} \in \mathbb{Z}^{n \times n}$ for a lattice $\mathcal{L} = \mathcal{L}(\mathbf{B})$. OUTPUT: A vector $\mathbf{x} \in \mathcal{L}$ such that $\|\mathbf{x}\|_p \leq n^{1/p} \det(\mathcal{L})^{1/n}$.

The authors in [BJP⁺15] give a reduction from MINKOWSKI to PIGEONHOLE CIRCUIT. We follow a different approach by showing a reduction from MINKOWSKI_p to BLICHFELDT. We emphasize that, even though a proof of Minkowski's theorem uses Blichfedlt's theorem, our reduction differs from this proof technique. We restrict to subsets of integer points, and thus the inherently continuous techniques used in the original proof of Minkowski's theorem via Blichfeldt's theorem to applied in our case.

Lemma 6.1. For $p \ge 1$ and $p = \infty$, MINKOWSKI_p is in PPP.

Proof. For any $p \ge 1$ it holds that $\|\mathbf{x}\|_{\infty} \le n^{1/p} \|\mathbf{x}\|_p$. This implies a Karp-reduction from MINKOWSKI_{*p*} to MINKOWSKI_∞. Hence, it suffices to show a Karp-reduction from MINKOWSKI_∞ to BLICHFELDT.

Let $\mathbf{B} \in \mathbb{Z}^{n \times n}$ be an input to MINKOWSKI_{*p*}, i.e. a basis for $\mathcal{L} = \mathcal{L}(\mathbf{B})$. Let $\ell = \lfloor \det(\mathcal{L})^{1/n} \rfloor$, $S = ([0, \ell]^n \cap \mathbb{Z}^n) \setminus \{\mathbf{0}\}$ and $s = |S| = (\ell + 1)^n - 1$. Define the value function representation for S to be $(s, \mathcal{V}_S(x)) = (s, \mathcal{V}_{[0,\ell]^n}(x+1))$, where the circuit $\mathcal{V}_{[0,\ell]^n}$ is constructed as in Lemma 2.1.

To show that **B** and *S* are a valid input for BLICHFELDT, we need to show $|S| \ge \det(\mathcal{L}) \Rightarrow$ $(|\det(\mathcal{L})^{1/n}| + 1)^n \ge \det(\mathcal{L})^{1/n} + 1$. This follows from the next claim for $x = \det(\mathcal{L})$.

Claim 6.2. *For any* $x \in \mathbb{Z}$ *and* $n \ge 1$, $(\lfloor x^{1/n} \rfloor + 1)^n \ge x + 1$.

Proof. If $x = k^n$ for some $k \in \mathbb{Z}$, then $\lfloor x^{1/n} \rfloor = k$. Hence, $(k+1)^n \ge k^n + 1 \Rightarrow (\lfloor x^{1/n} \rfloor + 1)^n \ge x + 1$. Otherwise, let $k \in \mathbb{Z}$ be the smallest integer such that $x < k^n$. Then, $\lfloor x^{1/n} \rfloor = k - 1$ and $x + 1 \le k^n$. Hence, $(\lfloor x^{1/n} \rfloor + 1)^n = k^n \ge x + 1$.

Finally, BLICHFELDT on input **B** and *S* will output one of following:

- 1. a vector **x** such that $\mathbf{x} \in S \cap \mathcal{L}$. Since $\mathbf{x} \in S$, we get $||\mathbf{x}||_{\infty} \leq \det(\mathcal{L})^{1/n}$. Hence, \mathbf{x} is a solution to MINKOWSKI_{∞}.
- 2. two vectors $\mathbf{x} \neq \mathbf{y}$, such that $\mathbf{x}, \mathbf{y} \in S$ and $\mathbf{x} \mathbf{y} \in \mathcal{L}$. Since $\mathbf{x}, \mathbf{y} \in S$, we get $\mathbf{x} \mathbf{y} \in [-\ell, \ell]^n$. Hence, $\|\mathbf{x} - \mathbf{y}\|_{\infty} \leq \det(\mathcal{L})^{1/n}$ and $\mathbf{x} - \mathbf{y}$ is a solution to MINKOWSKI_{∞}.

A direct corollary of the above lemma is that the most common version of the MINKOWSKI problem in the ℓ_2 -norm, is in PPP.

Corollary 6.3. MINKOWSKI₂ *is in* PPP.

Furthermore, there are known connections between γ -SVP with polynomial approximation factor and the class PPP. Specifically, it is known that *n*-SVP is Cook-reducible to MINKOWSKI_{∞} (see [Rot16, Theorem 1.23]).

This, along with the reduction from γ -SVP to $\sqrt{n\gamma^2}$ -CVP in [SD15] (for $\gamma = n$) and Lemma 6.1, implies the following.

Corollary 6.4. *n*-SVP and $n^{2.5}$ -CVP are Cook-reducible to BLICHFELDT.

A special type of lattices, that have gained a lot of attention due to their efficiency in cryptographic applications, are *ideal lattices*. The definition and cryptographic applications of ideal lattices are outside the scope of this work and can be found in [LPR13]. But, we include the following lemma, which needs only the basic fact that $\lambda_1(\mathcal{L}) = \lambda_2(\mathcal{L}) = \cdots = \lambda_n(\mathcal{L})$ in ideal lattices, where λ_i is the length of the *i*-th linearly independent vector. Let us denote by γ -iSVP the shortest vector problem on ideal lattices.

Lemma 6.5. \sqrt{n} -iSVP *is in* PPP.

Proof. For ideal lattices, it holds that $\lambda_1(\mathcal{L}) = \lambda_2(\mathcal{L}) = \cdots = \lambda_n(\mathcal{L})$. Minkowski's second Theorem states that

 $\lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n \geq \det(\mathcal{L}).$

Hence, on ideal lattices $\lambda_1 \ge \det(\mathcal{L})^{1/n}$. Combining this with the first Minkowski's theorem, that states that $\lambda_1 \le \sqrt{n} \det(\mathcal{L})^{1/n}$, we get that MINKOWSKI₂ with input \mathcal{L} solves \sqrt{n} -SVP of \mathcal{L} . \Box

7 Cryptographic Assumptions in PPP

The fact that the class PPP exhibits strong connections to cryptography was already known since its introduction [Pap94]. Papadimitriou [Pap94] showed that if PPP = FP, then one-way permutations do not exist. Since there are constructions of one-way permutations based on the discrete logarithm problem on \mathbb{Z}_p^* for a prime p, the problem of factoring Blum integers⁷ [Rab79], the RSA assumption [RSA78] and a special type of elliptic curves [Kal91], all these cryptographic assumptions become insecure if PPP = FP.

Moreover, if PWPP = FP, then collision-resistant hash functions do not exist; this follows directly from the definition of the class PWPP. Collision-resistant hash function families can be constructed based, for example, on the hardness assumption of the discrete logarithm problem over \mathbb{Z}_p^* for a prime *p* [CvP92], and from the SIS problem [MR07] problem, which implies that these assumptions become insecure if PPP = FP or even if PWPP = FP.

Additionally, recently it was shown that integer factorization is in PPP [Jer16]. In fact, Jeřábek [Jer16] showed that integer factorization lies in PWPP \cap PPA, which gives strong evidence that factoring is not PPP-complete.

We extend the connections between cryptographic assumptions and PPP by presenting a more general formulation of discrete logarithm and showing its membership to PPP. We call this formulation *discrete logarithm over general groups*.

⁷A Blum integer is the product of two distinct primes p, q such that $p \equiv 3 \pmod{4}$ and $q \equiv 3 \pmod{4}$.

Definition 7.1. A general group (\mathbb{G}, \star) is a cyclic group for which there exists a circuit describing a bijection $\mathcal{I}_{\mathbb{G}} : \mathbb{G} \to [\operatorname{ord}(\mathbb{G})]$, which we call the *index function of* \mathbb{G} in analogy to the definition of Section 2.2, and a circuit $f : [\operatorname{ord}(\mathbb{G})] \times [\operatorname{ord}(\mathbb{G})] \to [\operatorname{ord}(\mathbb{G})]$, which we call group operation function, such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{G}$, $\mathcal{I}_{\mathbb{G}}(\mathbf{x} \star \mathbf{y}) = f(\mathcal{I}_{\mathbb{G}}(\mathbf{x}), \mathcal{I}_{\mathbb{G}}(\mathbf{y}))$. The group (\mathbb{G}, \star) is defined by

- 1. a number *g* in [ord(G)] such that $\mathcal{I}_G(\mathbf{g}) = g$, where **g** is the generator of G,
- 2. a number *id* in $[ord(\mathbb{G})]$ such that $\mathcal{I}_{\mathbb{G}}(id) = id$, where *id* is the identity element of \mathbb{G} ,
- 3. the polynomial-size circuit for the group operation function f.

In order to illustrate the above definition we show that the multiplicative group \mathbb{Z}_p^* for prime p is a general group. First, note that $\operatorname{ord}(\mathbb{Z}_p^*) = p - 1$. Let $c \in \mathbb{Z}_p^*$ be a generator of \mathbb{Z}_p^* , then we set $g = c - 1 \in [p - 1]$, id = 0 and we define $f : [p - 1] \times [p - 1] \rightarrow [p - 1]$ as follows: let $x, y \in \mathbb{Z}_p^*$, then $f(x, y) = (x + 1) \cdot (y + 1) - 1 \pmod{q}$. Now, we are ready to define the discrete logarithm problem over a general group.

DLOG Problem.

INPUT: An alleged general group (G, \star) represented by $(g, id, f_G, \mathcal{I}_G)^a$ and a target $y \in [s]$. OUTPUT: One of the following:

- 1. an $x \in [s]$ such that $\mathcal{I}_{\mathbb{G}}(\mathbf{g}^x) = y$,
- 2. two $x, y \in [s]$ such that $x \neq y$ and $\mathcal{I}_{G}(\mathbf{g}^{x}) = \mathcal{I}_{G}(\mathbf{g}^{y})$.

^{*a*}Observe that in this definition g and *id* are just elements of G and f_G , \mathcal{I}_G are given in the form of circuits.

Remark 7.2. We aim at defining only syntantic problems, so even though we could define the promise version of the problem where (\mathbb{G} , \star) is always a general group, we choose to define it as above. In the first case, the output is the discrete logarithm of the target element, whereas in the second case, the output is a proof that the tuple (g, id, f_G) does not represent a general group.

Also, we observe that given an element $x \in [\operatorname{ord}(\mathbb{G})]$ and the representation $(g, id, f_{\mathbb{G}})$ of a general group, there exists an efficient procedure for computing $\mathcal{I}_{\mathbb{G}}(\mathbf{g}^x)$. For instance, using repeating squaring starting with $f_{\mathbb{G}}(g, g)$.

Theorem 7.3. DLOG is in PPP.

Proof. Let (g, id, f_G, y) be an input for DLOG and $n = \lceil \log(s) \rceil$, then we construct a circuit C : $\{0,1\}^n \to \{0,1\}^n$ as follows: For a binary vector $\underline{x} \in \{0,1\}^n$, $C(\underline{x}) = |\mathcal{I}_G(\underline{g}^{bc(\underline{x})}) - y|$. As we mentioned in the remark above, we can compute $C(\underline{x})$ efficiently using only g, id and f_G .

Then, the output of PIGEONHOLE CIRCUIT with input C is one of the following:

- 1. a binary vector \underline{x} such that $C(\underline{x}) = \underline{0}$.
 - In this case, $x = bc(\underline{x})$ is the discrete logarithm of y. Namely, $\mathcal{I}_{\mathbb{G}}(\mathbf{g}^x) = y$.
- 2. two binary vectors \underline{x} , y such that $\underline{x} \neq y$ and $C(\underline{x}) = C(y)$.

In this case, $\mathcal{I}_{G}(\mathbf{g}^{\mathbf{bc}(\underline{x})}) = \mathcal{I}_{G}(\mathbf{g}^{\mathbf{bc}(\underline{y})})$ implies one of the following: (a) f_{G} does not implement a valid group operation over G, (b) \mathcal{I}_{G} is not a valid index circuit of the set G, or (c) $\mathbf{g}^{\mathbf{bc}(\underline{x})} = \mathbf{g}^{\mathbf{bc}(\underline{y})}$. In either case this is a certificate that the input $(g, id, f_{G}, \mathcal{I}_{G})$ does not represent a valid general group.

Open Problem 7.4. Is DLOG PPP-hard?

We note that it is not known if elliptic curves are general groups, and hence discrete logarithm over general elliptic curves is not known to belong in PPP (see Open Problem 1.5).

Acknowledgements

We thank the anonymous FOCS reviewers for their helpful comments. We thank an anonymous reviewer and Nico Döttling for bringing in our attention a universal CRH following Levin's paradigm. We thank Vinod Vaikuntanathan, Daniel Wichs and Constantinos Daskalakis for helpful and enlightening discussions. We thank Christos-Alexandros Psomas and his coauthors for sharing their unpublished manuscript [BJP⁺15]. MZ also thanks Christos-Alexandros Psomas and Christos Papadimitriou for many fruitful discussions during his visit to Simons Institute at Berkeley at Fall 2015.

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A Missing Proofs

A.1 Proof of Claim 4.6

Proof of Claim 4.6. The proof of this claim follows easily from the next Table 5, from which we conclude various relationships between boolean functions with two inputs and modular equations (mod 4).

<i>x</i>	y	z	w	w+2z-x-y	z+2w-x-y	w + 2z + x + y
0	0	0	0	0	0	0
0	0	0	1	1	2	1
0	0	1	0	2	1	2
0	0	1	1	-1	-1	-1
0	1	0	0	-1	-1	1
0	1	0	1	0	1	2
0	1	1	0	1	0	-1
0	1	1	1	2	2	0
1	0	0	0	-1	-1	1
1	0	0	1	0	1	2
1	0	1	0	1	0	-1
1	0	1	1	2	2	0
1	1	0	0	2	2	2
1	1	0	1	-1	0	-1
1	1	1	0	0	-1	0
1	1	1	1	1	1	1

Table 5: The values of specific modular expressions (mod 4) for all different binary values of the variables.

B Parameters of cSIS and weak-cSIS

We give a summary of the notation that we use for the parameters of cSIS and weak-cSIS (or \mathcal{H}_{cSIS}).

Notation	Parameter	Problem
п	circuit input size	both
k	\mathcal{H}_{cSIS} input size	weak-cSIS
d	rows of G / gates of ${\cal C}$	both
т	columns of \mathbf{A} and \mathbf{G}	both
rℓ	output of \mathcal{H}_{cSIS}	weak-cSIS

Also, the known relations between these quantities for a valid cSIS or weak-cSIS input are:

 $k = \ell \cdot n, \quad m \ge d\ell + k = d\ell + n\ell, \quad k > r\ell.$

C Universal Collision Resistant Hash Function Family

We sketch the construction of a universal (average-case hard) hash function, following Levin's paradigm [Lev87] for a universal one-way function. Let *h* be a hash function family that takes two inputs a key *k* and a vector $x \in \{0,1\}^n$, and compresses the input *x* by one bit, i.e. $h(k,x) \in \{0,1\}^{n-1}$. Let $p(\cdot)$ be a polynomial that bounds the running time of $h(k, \cdot)$. First, using padding on the input we argue the existence of a hash function family *h'* that is defined as

$$h'(k, x \circ y) = h(k, x) \circ y$$

such that $|x \circ y| = p(|x|)$. This implies that $h'(k, \cdot)$ runs in quadratic time in $|x \circ y|$ (see [Gol06, §2.4.1]). Second, using standard domain extension we argue the existence of a hash function family h'' that compresses the input for more than one bits, i.e. $h'' : \{0,1\}^{n'} \rightarrow \{0,1\}^{n-1}$ (we exclude the key k from the description of the domain). Specifically, we require that the compressing ratio is enough so that the concatenation of m copies of $h''(k, \cdot)$ is smaller than the input, meaning that $n' > m \cdot (p(n) - 1)$. Let $p''(\cdot)$ be a polynomial that bounds the running time of h''. The hash function $h'(k, \cdot)$ runs in time quadratic to its input, and using this fact, $p''(\cdot)$ can be made explicit depending on the length of the extended domain that we require from h'', once that length is explicitly specified. This is enough to get an upper bound for the running time of $h''(k, \cdot)$.

The universal hash function is described by a collection of $2 \cdot m + 1$ strings:

$$h_{\mathsf{uni}} = (i_1, \ldots, i_m, k_1, \ldots, k_m, x).$$

The numbers i_1, \ldots, i_m (represented as strings) are the indices to Turing Machines (assume a canonical ordering of TMs) that describe *m* different hash function families $h_{i_j}'': \{0,1\}^{n'} \rightarrow \{0,1\}^{n-1}, j = 1, \ldots, m$. The keys k_j define the hash functions $h_{i_j}''(k_j, \cdot)$. Finally, for $j = 1, \ldots, m$, we run each $h_{i_j}''(k_j, x)$ for at most p''(|x|) steps and output:

$$h_{\mathsf{uni}}(x) = h_{i_1}^{\prime\prime}(k_1, x) \circ \cdots \circ h_{i_m}^{\prime\prime}(k_m, x).$$

If $h_{i_j}''(k_j, x)$ does not terminate after p''(|x|), we output \perp for that *j*. We can see that if at least one hash function family h_{i_j}'' is collision-resistant, then so is h_{uni} . At a high level, if at least one hash function family h_{i_j} is collision-resistant then so is h_{i_j}' , and moreover so is h_{i_j}'' by the security of domain extension (e.g. Merkle–Damgård). Without loss of generality we assume that all families h_{i_j}'' are defined on the same domain and range. Finally, the simple hash function combiner in which we concatenate the output of *m* different hash functions $h_{i_j}''(k_j, \cdot)$, is collision-resistant as long as at least one hash function family h_{i_j}'' is collision-resistant.