

# Random Number Generators Can Be Fooled to Behave Badly

George Teşeleanu<sup>1,2</sup>

<sup>1</sup> Advanced Technologies Institute  
10 Dinu Vintilă, Bucharest, Romania  
ati@dcti.ro

<sup>2</sup> Department of Computer Science  
“Al.I.Cuza” University of Iaşi 700506 Iaşi, Romania,  
george.teseleanu@info.uaic.ro

**Abstract.** In this paper, we extend the work on purely mathematical Trojan horses initially presented by Young and Yung. This kind of mechanism affects the statistical properties of an infected random number generator (RNG) by making it very sensitive to input entropy. Thereby, when inputs have the correct distribution the Trojan has no effect, but when the distribution becomes biased the Trojan worsens it. Besides its obvious malicious usage, this mechanism can also be applied to devise lightweight health tests for RNGs. Currently, RNG designs are required to implement an early detection mechanism for entropy failure, and this class of Trojan horses is perfect for this job.

## 1 Introduction

In [15] the authors propose an interesting mechanism that blurs the line between what constitutes a Trojan horse and what does not. To detect their mechanism, a program needs to somehow differentiate between a naturally unstable random number generator (RNG) and artificially unstable one (obtained by means of certain mathematical transformations). To our knowledge, [15] is the only previous work that discusses this topic.

More precisely, in [15] a digital filter is described. Usually, digital filters are applied to RNGs to correct biases<sup>3</sup>, but this filter has an opposite purpose. When applied to a stream of unbiased bits the filter is benign. On the other hand, if applied to a stream of biased bits the filter amplifies their bias. Thereby, making the RNG worse.

In this paper we extend the filter from [15]<sup>4</sup>, provide a new class of filters and discuss some new possible applications. The main application we propose for these filters is RNG testing (*e.g.*, boosting health tests implemented in a RNG). Recent standards [11, 13] require a RNG to detect failures and one such method for early detection can be to apply an amplifier and then do some lightweight testing<sup>5</sup>. Based on the results obtained in our paper, we provide concrete examples of how to detect such failures in Section 5.1.

Due to recent events [4, 6, 7, 12] RNGs have been under a lot of scrutiny. Thus, wondering what kind of mechanisms can be implemented by a malicious third party in order to weaken or destabilize a system becomes natural. Amplifying filters provide a novel example of how one can achieve this. Based on the failure detection mechanisms proposed in Section 5.1, we show, for example, how a manufacturer can manipulate the architecture to become malicious.

*Structure of the paper.* Notations and definitions are presented in Section 2. The core of the paper consists of Sections 3 and 4 and contains two classes of filters. Experimental result and applications are given in Section 5. We conclude in Section 6.

---

<sup>3</sup> They are called randomness extractors [8].

<sup>4</sup> The filter presented in [15] corresponds to the greedy amplifier with parameter  $n = 3$  described in Section 3.

<sup>5</sup> for example the tests described in [10]

## 2 Preliminaries

Throughout the paper, we consider binary strings of length  $n$  composed of *independent and identically distributed* (i.i.d.) bits generated by a RNG. By  $0^n$  and  $1^n$  we understand the all zero and the all one strings. Also, for figures we use the standard representation of the x-axis and y-axis.

Let  $0 \leq \varepsilon \leq \frac{1}{2}$  be a real number and  $b$  a random bit. Then, without loss of generality, we denote the probability of  $b = 0$  by  $P_0 = \frac{1}{2} - \varepsilon$  and of  $b = 1$  by  $P_1 = \frac{1}{2} + \varepsilon$ . We will refer to  $\varepsilon$  as bias. The complement rule states that  $P_1 = 1 - P_0$ . Let  $P(a)$  be the probability of a random string being  $a$ . Then for any  $A \subseteq \mathbb{Z}_2^n$  we denote by  $P(A) = \sum_{a \in A} P(a)$ .

Let  $u$  be a binary string and  $A \subseteq \mathbb{Z}_2^n$ . Then  $w(u)$  denotes the hamming weight of  $u$  and  $w(A)$  the set  $\{w(a) \mid a \in A\}$ . Note that since we are working with i.i.d. bits, for any  $u, v \in \mathbb{Z}_2^n$  such that  $w(u) = w(v)$ , the equality  $P(u) = P(v)$  holds. Thus, from a probabilistic point of view, it does not matter which element of the set  $\{u \in A \mid w(u) = k\}$  we choose to work with.

The element  $\min(A)$  ( $\max(A)$ ) is the smallest (biggest) integer of the set  $A$ , while  $\min_w(A)$  ( $\max_w(A)$ ) is an element from  $A$  that has the smallest (biggest) hamming weight. We say that a pair of sets  $(S_0, S_1)$  is an equal partition of the set  $S$  if the following hold:  $S = S_1 \cup S_2$ ,  $S_1 \cap S_2 = \emptyset$  and  $|S_1| = |S_2|$ .

To ease description, we use the notation  $C_k^n$  to denote binomial coefficients. Pascal's identity states that  $C_k^n = C_k^{n-1} + C_{k-1}^{n-1}$ , where  $1 \leq k \leq n$ . Note that  $|\{u \in \mathbb{Z}_2^n \mid w(u) = k\}| = C_k^n$ .

In this paper, we consider a digital filter to be a mapping from  $\mathbb{Z}_2^n$  to  $\mathbb{Z}_2$ . If we continuously apply a filter to data generated by a RNG<sup>6</sup>, then three types of filters arise:

- **bias amplifier** - the output data has a bigger bias than the input data;
- **neutral filter** - the output data has the the same bias as the input data;
- **bias corrector**<sup>7</sup> - the output data has a smaller bias than the input data.

Let  $(S_0, S_1)$  be an equal partition of a set  $S$ . Let  $D$  be a digital filter such that it maps  $S_0$  and  $S_1$  to 0 and 1, respectively (see Table 1). Also, let  $\varepsilon_D$  be the output bias of  $D$ . We say that a **bias amplifier** is maximal if  $\varepsilon_D$  is maximal over all the equal partitions of  $\mathbb{Z}_2^n$ . To compare **bias amplifiers** we measure the distance between  $P(S_1)$  and  $P(S_0)$ .

**Table 1.** Conversion table.

Bit 0	Bit 1
$S_0$	$S_1$

Before stating our results, some restrictions are needed. If the input bits are unbiased (*i.e.*  $P_0 = \frac{1}{2}$ ) or have a maximum bias (*i.e.*  $P_0 = 0$  or  $P_1 = 0$ ) we require the filter to maintain the original bias. If one replaces a **bias corrector** with a **bias amplifier**, the amplifier must behave as the corrector when the RNG has bias 0 or  $\frac{1}{2}$ . The last requirement is that the filter amplifies the bias in the direction that it already is. Without loss of generality, we assume that the bias is towards 1.

## 3 Greedy Bias Amplifiers

In this section we generalize and improve the **bias amplifier** described in [15]. We first present a **neutral filter** and based on it we develop a maximal **bias amplifier**. We can easily transform one into the other by changing the conversion table.

**Lemma 1.** *Let  $S_0 = \{u \in \mathbb{Z}_2^n \mid u = 0 \parallel v, v \in \mathbb{Z}_2^{n-1}\}$  and  $S_1 = \{u \in \mathbb{Z}_2^n \mid u = 1 \parallel v, v \in \mathbb{Z}_2^{n-1}\}$ . Then  $P(S_0) = P_0$  and  $P(S_1) = P_1$ .*

<sup>6</sup> Note that except for  $n = 1$  the bit rate of the RNG will drop.

<sup>7</sup> We prefer to use this notion instead of randomness extractor, because it simplifies our framework.

*Proof.* Since we are working with i.i.d. random bits the following holds

$$P(S_0) = \sum_{v \in \mathbb{Z}_2^{n-1}} P(0||v) = \sum_{v \in \mathbb{Z}_2^{n-1}} P_0 P(v) = P_0 \sum_{v \in \mathbb{Z}_2^{n-1}} P(v) = P_0.$$

Similarly, we obtain  $P(S_1) = P_1$ . □

Using Lemma 1 we can devise a **neutral filter**  $N$  by mapping all the elements of  $S_0$  and  $S_1$  to 0 and 1, respectively. Starting from the equal partition  $(S_0, S_1)$  (Lemma 1), using a greedy algorithm (Algorithm 1), we devise a new equal partition that serves as the core of a maximal **bias amplifier**.

---

**Algorithm 1.**

---

**Input:** An integer  $n$   
**Output:** An equal partition of  $\mathbb{Z}_2^n$   
1 Set  $S_0 = \{u \in \mathbb{Z}_2^n \mid u = 0||v, v \in \mathbb{Z}_2^{n-1}\}$  and  $S_1 = \{u \in \mathbb{Z}_2^n \mid u = 1||v, v \in \mathbb{Z}_2^{n-1}\}$   
2 Set  $\alpha = \max_w(S_0)$  and  $\beta = \min_w(S_1)$   
3 **while**  $w(\alpha) < w(\beta)$  **do**  
4     Set  $S_0 = (S_0 \setminus \{\alpha\}) \cup \{\beta\}$  and  $S_1 = (S_1 \setminus \{\beta\}) \cup \{\alpha\}$   
5     Update  $\alpha = \max_w(S_0)$  and  $\beta = \min_w(S_1)$   
6 **end**  
7 **return**  $(S_0, S_1)$

---

**Lemma 2.** *Let  $k$  be a positive integer and let  $(S_0, S_1)$  be the output of Algorithm 1. Then the following properties hold*

1. *If  $n = 2k + 1$  then  $S_0 = \{u \mid 0 \leq w(u) \leq k\}$  and  $S_1 = \{u \mid k + 1 \leq w(u) \leq n\}$ . Also,  $P(S_0) = \sum_{i=0}^k C_i^n (P_0)^{n-i} (P_1)^i$  and  $P(S_1) = \sum_{i=0}^k C_i^n (P_0)^i (P_1)^{n-i}$ .*
2. *If  $n = 2k$  then  $S_0 = \{u \mid 0 \leq w(u) \leq k - 1\} \cup T_0$  and  $S_1 = \{u \mid k + 1 \leq w(u) \leq n\} \cup T_1$ , where  $(T_0, T_1)$  is an equal partition of  $\{u \in \mathbb{Z}_2^n \mid w(u) = k\}$ . Also,  $P(S_0) = \sum_{i=0}^{k-1} C_i^n (P_0)^{n-i} (P_1)^i + \frac{C_k^n}{2} (P_0 P_1)^k$  and  $P(S_1) = \sum_{i=0}^{k-1} C_i^n (P_0)^i (P_1)^{n-i} + \frac{C_k^n}{2} (P_0 P_1)^k$ .*
3. *If  $\varepsilon = 0$  then  $P(S_0) = P(S_1) = \frac{1}{2}$  and if  $\varepsilon = \frac{1}{2}$  then  $P(S_0) = 0$  and  $P(S_1) = 1$ .*

*Proof.* During the while loop Algorithm 1 swaps the elements whose weight is written in Column 2, Table 2 with the elements that have their weight written in Column 3, Table 2.

**Table 2.** Operations performed during the while loop.

Number of switches	Weight of $S_0$ elements	Weight of $S_1$ elements
$C_0^{n-1}$	$n - 1$	1
$C_1^{n-1}$	$n - 2$	2
$\dots$	$\dots$	$\dots$
$C_{i-1}^{n-1}$	$n - i$	$i$
$\dots$	$\dots$	$\dots$

The while loop ends when  $w(\alpha) \geq w(\beta)$ . According to Table 2, this is equivalent with  $n - i \geq i$ . When  $n = 2k + 1$  we obtain that the while loop stops when  $i \leq k + 1$ . When  $n = 2k$  the loop stops when  $i \leq k$ . Thus, we obtain the sets  $S_0$  and  $S_1$ . The probabilities  $P(S_0)$  and  $P(S_1)$  are a direct consequence of the structure of the sets and the fact that  $C_k^n = C_{n-k}^n$ . The last item is simply a matter of computation. □

In Table 3 we present all the possible partitions of  $\mathbb{Z}_2^3$ . We mapped these partitions to 0 and 1 in such a way that  $P_1 > \frac{1}{2}$ . Note that bias amplification happens only for the partition presented in the first entry of

the table<sup>8</sup>. This is not true in general. For example, when  $n = 5$ , we obtain another bias amplifier if we map the set  $S'_1 = (S_1 \setminus \{11100\}) \cup \{11000\}$ <sup>9</sup> and  $S'_0 = \mathbb{Z}_2^5 \setminus S'_1$  to 1 and 0. We will now prove that  $(S'_0, S'_1)$  is the basis of an amplifier. Note that  $P(S'_1) = \frac{1}{2} + \frac{7}{4}\varepsilon - 4\varepsilon^3 + 4\varepsilon^5$ . Thus, we have

$$P(S'_1) - P_1 = \frac{\varepsilon}{4} (16\varepsilon^4 - 16\varepsilon^2 + 3) > 0,$$

which is equivalent with

$$16t^2 - 16t + 3 > 0, \tag{1}$$

where  $t = \varepsilon^2$ . Equation (1) has two solution  $t_1 = \frac{1}{4}$  and  $t_2 = \frac{3}{4}$ . Thus, the sign of the quadratic function is negative only when  $\frac{1}{4} < t < \frac{3}{4}$ . By taking into account the complement rule and that  $0 < \varepsilon < \frac{1}{2}$ , we obtain an amplifier by converting  $(S'_0, S'_1)$  to  $(0, 1)$ .

**Table 3.** Trigraph conversion table.

Bit 0	Bit 1	$P_1$	Corrector	Amplifier
000 & 001 & 010 & 100	111 & 110 & 101 & 011	$\frac{1}{2} + \frac{3}{2}\varepsilon - 2\varepsilon^3$	-	$0 < \varepsilon < \frac{1}{2}$
000 & 001 & 010 & 011	111 & 110 & 101 & 100	$\frac{1}{2} + \varepsilon$	-	-
000 & 001 & 101 & 011	111 & 110 & 010 & 100	$\frac{1}{2} + \frac{1}{2}\varepsilon + 2\varepsilon^3$	$0 < \varepsilon < \frac{1}{2}$	-
000 & 110 & 101 & 011	111 & 001 & 010 & 100	$\frac{1}{2} + 4\varepsilon^3$	$0 < \varepsilon < \frac{1}{2}$	-
000 & 001 & 010 & 111	100 & 110 & 101 & 011	$\frac{1}{2} + \frac{1}{2}\varepsilon + 2\varepsilon^2 - 2\varepsilon^3$	$0 < \varepsilon < \frac{1}{2}$	-
011 & 110 & 010 & 100	000 & 001 & 101 & 111	$\frac{1}{2} + 2\varepsilon^2$	$0 < \varepsilon < \frac{1}{2}$	-
011 & 001 & 010 & 100	000 & 110 & 101 & 111	$\frac{1}{2} + \frac{1}{2}\varepsilon + 2\varepsilon^2 - 2\varepsilon^3$	$0 < \varepsilon < \frac{1}{2}$	-

**Lemma 3.** *Let  $(S_0, S_1)$  be the output of Algorithm 1. If we map all the elements of  $S_0$  and  $S_1$  to 0 and 1, respectively, then we obtain a maximal bias amplifier  $G$ .*

*Proof.* According to Lemma 2 all the lowest and highest probability elements are in  $S_0$  and  $S_1$ , respectively. Thus, the statement is true.  $\square$

**Lemma 4.** *Let  $(S_0^n, S_1^n)$  be the output of Algorithm 1 for  $n = 2k + 1$ . Then the following hold*

1.  $P(S_0^n) = P(S_0^{n+1})$  and  $P(S_1^n) = P(S_1^{n+1})$ .
2.  $P(S_0^n) - P(S_0^{n+2}) = P(S_1^{n+2}) - P(S_1^n) = 2\varepsilon C_k^n (P_0 P_1)^{k+1}$ .
3.  $P(S_0^n) > P(S_0^{n+2})$  and  $P(S_1^n) < P(S_1^{n+2})$ .
4.  $P(S_1^n) - P(S_0^n) < P(S_1^{n+2}) - P(S_0^{n+2})$ .

*Proof.* We prove the first statement using induction. When  $k = 1$  we have  $S_0^1 = \{0\}$ ,  $S_1^1 = \{1\}$ ,  $S_0^2 = \{00, 01\}$  and  $S_1^2 = \{10, 11\}$ . Using Lemma 1, we obtain  $P(S_0^1) = P_0 = P(S_0^2)$  and  $P(S_1^1) = P_1 = P(S_1^2)$ . Thus, proving the statement for the case  $k = 1$ .

<sup>8</sup> The authors of [15] call it a RNG biasing Trojan horse.

<sup>9</sup>  $S_1$  is the set from Lemma 2.

We now assume that the statement is true for  $k$  (*i.e.*  $P(S_0^n) = P(S_0^{n+1})$  and  $P(S_1^n) = P(S_1^{n+1})$ ) and we it for  $k + 1$ . Applying Pascal's identity twice to  $P(S_0^{n+2})$  we obtain

$$\begin{aligned} P(S_0^{n+2}) &= \sum_{i=0}^{k+1} C_i^{n+2} (P_0)^{n+2-i} (P_1)^i = (P_0)^{n+2} + (n+2)(P_0)^{n+1} P_1 \\ &\quad + \sum_{i=2}^{k+1} (C_i^n + 2C_{i-1}^n + C_{i-2}^n) (P_0)^{n+2-i} (P_1)^i. \end{aligned} \quad (2)$$

We rewrite Equation (2) as a sum of  $S^1, S^2, S^3$  (described next):

$$S^1 = (P_0)^{n+2} + n(P_0)^{n+1} P_1 + \sum_{i=2}^{k+1} C_i^n (P_0)^{n+2-i} (P_1)^i = (P_0)^2 P(S_0^n) + C_{k+1}^n (P_0)^{n+1-k} (P_1)^{k+1}, \quad (3)$$

$$S^2 = 2(P_0)^{n+1} P_1 + 2 \sum_{i=2}^{k+1} C_{i-1}^n (P_0)^{n+2-i} (P_1)^i = 2 \sum_{i=0}^k C_i^n (P_0)^{n+1-i} (P_1)^{i+1} = 2P_0 P_1 P(S_0^n), \quad (4)$$

$$S^3 = \sum_{i=2}^{k+1} C_{i-2}^n (P_0)^{n+2-i} (P_1)^i = \sum_{i=0}^{k-1} C_i^n (P_0)^{n-i} (P_1)^{i+2} = (P_1)^2 P(S_0^n) - C_k^n (P_0)^{n-k} (P_1)^{k+2}. \quad (5)$$

Reassembling Equations (3) to (5) we obtain

$$P(S_0^{n+2}) = P(S_0^n) + C_{k+1}^n (P_0)^{n+1-k} (P_1)^{k+1} - C_k^n (P_0)^{n-k} (P_1)^{k+2} = P(S_0^n) - 2\varepsilon C_k^n (P_0 P_1)^{k+1}. \quad (6)$$

Applying Pascal's identity twice to  $P(S_0^{n+3})$  we obtain

$$\begin{aligned} P(S_0^{n+3}) &= \sum_{i=0}^{k+1} C_i^{n+3} (P_0)^{n+3-i} (P_1)^i + \frac{C_{k+2}^{n+3}}{2} (P_0 P_1)^{k+2} = (P_0)^{n+3} + (n+3)(P_0)^{n+2} P_1 \\ &\quad + \sum_{i=2}^{k+1} (C_i^{n+1} + 2C_{i-1}^{n+1} + C_{i-2}^{n+1}) (P_0)^{n+3-i} (P_1)^i + \frac{C_{k+2}^{n+3}}{2} (P_0 P_1)^{k+2}. \end{aligned} \quad (7)$$

Let  $\alpha = \sum_{i=0}^k C_i^{n+1} (P_0)^{n+1-i} (P_1)^i$ . We rewrite Equation (7) as a sum of  $S^4, S^5, S^6$  (described next):

$$S^4 = (P_0)^{n+3} + (n+1)(P_0)^{n+2} P_1 + \sum_{i=2}^{k+1} C_i^{n+1} (P_0)^{n+3-i} (P_1)^i = (P_0)^2 \alpha + C_{k+1}^{n+1} (P_0)^{n+2-k} (P_1)^{k+1}, \quad (8)$$

$$S^5 = 2(P_0)^{n+2} P_1 + 2 \sum_{i=2}^{k+1} C_{i-1}^{n+1} (P_0)^{n+3-i} (P_1)^i = 2 \sum_{i=0}^k C_i^{n+1} (P_0)^{n+2-i} (P_1)^{i+1} = 2P_0 P_1 \alpha, \quad (9)$$

$$S^6 = \sum_{i=2}^{k+1} C_{i-2}^{n+1} (P_0)^{n+3-i} (P_1)^i = \sum_{i=0}^{k-1} C_i^{n+1} (P_0)^{n+1-i} (P_1)^{i+2} = (P_1)^2 \alpha - C_k^{n+1} (P_0)^{n+1-k} (P_1)^{k+2}. \quad (10)$$

Reassembling Equations (8) to (10) we obtain

$$\begin{aligned} P(S_0^{n+3}) &= P(S_0^{n+1}) + C_{k+1}^{n+1} (P_0)^{n+2-k} (P_1)^{k+1} - C_k^{n+1} (P_0)^{n+1-k} (P_1)^{k+2} - \frac{C_{k+1}^{n+1}}{2} (P_0 P_1)^{k+1} \\ &\quad + \frac{C_{k+2}^{n+3}}{2} (P_0 P_1)^{k+2} \\ &= P(S_0^{n+1}) - C_k^n (P_0 P_1)^{k+1} \left\{ \frac{n+1}{k+1} \left[ (P_0)^2 - \frac{1}{2} \right] + P_0 P_1 \left[ -\frac{n+1}{k+2} + \frac{(n+1)(n+2)(n+3)}{2(k+1)(k+2)(k+2)} \right] \right\} \\ &= P(S_0^{n+1}) - C_k^n (P_0 P_1)^{k+1} \left\{ 2 \left[ (P_0)^2 - \frac{1}{2} \right] + 2P_0 P_1 \right\} = P(S_0^{n+1}) - 2\varepsilon C_k^n (P_0 P_1)^{k+1}. \end{aligned} \quad (11)$$

Applying the induction step to Equations (6) and (11) we obtain that  $P(S_0^{n+2}) = P(S_0^{n+3})$ . The following equality is a consequence of the complement rule

$$P(S_1^{n+2}) = 1 - P(S_0^{n+2}) = 1 - P(S_0^{n+3}) = P(S_1^{n+3}).$$

This completes the proof the first statement. The remaining statements are a direct consequence of Equation (6) and the complement rule.  $\square$

**Corollary 1.** *Let  $(S_0^n, S_1^n)$  be the output of Algorithm 1 for  $n = 2k + 1$ . Then  $P(S_0^n) - P(S_0^{n+2}) > P(S_0^{n+2}) - P(S_0^{n+4})$  and  $P(S_1^{n+2}) - P(S_1^n) > P(S_1^{n+4}) - P(S_1^{n+2})$ .*

*Proof.* Using Lemma 4 we obtain that  $P(S_0^n) - P(S_0^{n+2}) > P(S_0^{n+2}) - P(S_0^{n+4})$  is equivalent with  $2\varepsilon C_k^n (P_0 P_1)^{k+1} > 2\varepsilon C_{k+1}^{n+2} (P_0 P_1)^{k+2}$ . Rewriting the inequality we obtain

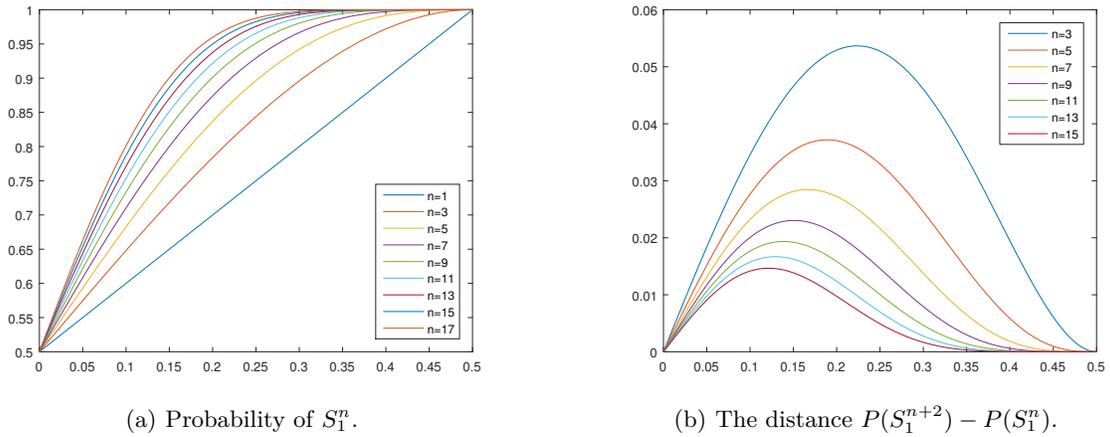
$$1 > \frac{(2k+2)(2k+3)}{(k+1)(k+2)} P_0 P_1.$$

The proof is concluded by observing that

$$\frac{(2k+2)(2k+3)}{(k+1)(k+2)} P_0 P_1 < 4 \left( \frac{1}{4} - \varepsilon^2 \right) = 1 - 4\varepsilon^2 \leq 1.$$

$\square$

Figure 1(a) and Figure 1(b) are a graphical representation of Lemma 4 ( $n \leq 17$ ) and Corollary 1 ( $n \leq 15$ ), respectively. The x-axis represents the original bias  $\varepsilon$ , while the y-axis represents  $P(S_1^n)$  (Figure 1(a)) and  $P(S_1^{n+2}) - P(S_1^n)$  (Figure 1(b)).



**Fig. 1.** Greedy amplifier.

Using the properties stated in Lemmas 2 and 4, we will next describe an equivalent and simplified version of Algorithm 1. Note that devising a greedy **bias amplifier** only makes sense when  $n$  is odd.

---

**Algorithm 2.**

---

**Input:** An odd integer  $n$   
**Output:** An equal partition of  $\mathbb{Z}_2^n$

```

1 Set  $S_0 = S_1 = \emptyset$ 
2 for  $i = 0, \dots, 2^n - 1$  do
3   if  $w(i) \leq k$  then
4      $S_0 = S_0 \cup \{i\}$ 
5   end
6   else
7      $S_1 = S_1 \cup \{i\}$ 
8   end
9 end
10 return  $(S_0, S_1)$ 

```

---

## 4 Von Neumann Bias Amplifier

Von Neumann introduced in [14] a simple, yet effective method for correcting the bias of a RNG. Each time the RNG generates two random bits  $b_1$  and  $b_2$ , the filter outputs  $b_1$  if and only if  $b_1 \neq b_2$ . It is easy to see that  $P(b_1 b_2 = 01) = P(b_1 b_2 = 10) = P_0 P_1$ . Thus, the bias of the output data is 0. We further generalize Von Neumann's method and explain how to replace it's conversion table in order to obtain a maximal **bias amplifier**. Through this section we will restrict  $n$  to be of the form  $2k$ , where  $k$  is a positive integer.

**Lemma 5.** *Let  $V = \{u \in \mathbb{Z}_2^n \mid w(u) = k\}$  and let  $(V_0, V_1)$  be an equal partition of  $V$ . Then  $P(V_0) = P(V_1) = \frac{C_k^n}{2} (P_0 P_1)^k$ .*

*Proof.* Since  $(V_0, V_1)$  is an equal partition of  $V$ , we obtain that  $|V_0| = |V_1| = \frac{|V|}{2} = \frac{C_k^n}{2}$ . Note that  $P(u) = (P_0 P_1)^k$ , for any  $u \in V$ . Combining these two facts we obtain the statement of the lemma.  $\square$

Using Lemma 5 we can devise a **corrector filter**<sup>10</sup>  $V_c$  by mapping all the elements of  $V_0$  and  $V_1$  to 0 and 1, respectively. In Algorithm 3 we provide an example of how to generate a pair  $(V_0, V_1)$ .

---

**Algorithm 3.**

---

**Input:** An integer  $n$   
**Output:** An equal partition of  $V$

```

1 Set  $V_0 = V_1 = \emptyset$  and  $V = \{u \in \mathbb{Z}_2^n \mid w(u) = k\}$ 
2 Set  $\alpha = \max(V)$  and  $\beta = \min(V)$ 
3 for  $i = 1, \dots, C_k^n/2$  do
4   Set  $V_0 = V_0 \cup \{\beta\}$  and  $V_1 = V_1 \cup \{\alpha\}$ 
5   Update  $V = V \setminus \{\alpha, \beta\}$ 
6   Set  $\alpha = \max(V)$  and  $\beta = \min(V)$ 
7 end
8 return  $(V_0, V_1)$ 

```

---

We further show that the probabilities  $V_0$  and  $V_1$  get smaller if we increase  $n$ . This translates in a lower bit rate if we apply  $V_c$ . Note that increasing  $n$  does not change the bias of the output data, thus making  $V_c$ <sup>11</sup> useless in practice if used only for correcting biases.

<sup>10</sup> with the bias of the output data 0

<sup>11</sup> for  $n \geq 4$

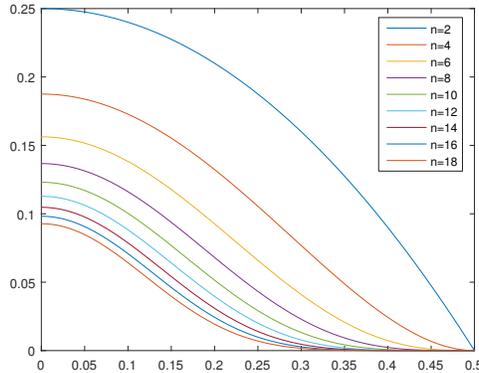
**Lemma 6.** Let  $(V_0^n, V_1^n)$  be the output of Algorithm 3 for  $n = 2k$ . Then  $P(V_0^n) > P(V_0^{n+2})$ .

*Proof.* We remark that  $P(V_0^n) > P(V_0^{n+2})$  is equivalent with

$$1 > \frac{(2k+1)(2k+2)}{(k+1)(k+1)} P_0 P_1.$$

The proof is now similar to Corollary 1 and thus is omitted.  $\square$

Figure 2 is a graphical representation of Lemma 6 ( $n \leq 18$ ). The x-axis represents the original bias  $\epsilon$ , while the y-axis represents  $P(V_0^n)$ .



**Fig. 2.** Probability of  $V_0^n$  and  $V_1^n$ .

Note that when  $P_0 = 0$  or  $P_1 = 0$  we obtain  $P(V_0) = P(V_1) = 0$ . When constructing a **bias amplifier**  $V_a$  we must have the same behavior. Thus, the strings we use to construct  $V_a$  need to contain at least a 0 and an 1. When  $n = 2$  the only strings that contain 0 and 1 are 01 and 10, but these are the basis for the Von Neumann **bias corrector**. Hence, when  $n = 2$  there are no **bias amplifiers**. This leads to the restriction  $n \geq 4$ . We again use a greedy approach (Algorithm 4) and devise a core for  $V_a$ .

---

**Algorithm 4.**

---

**Input:** An integer  $n$

**Output:** Two sets  $V_0$  and  $V_1$

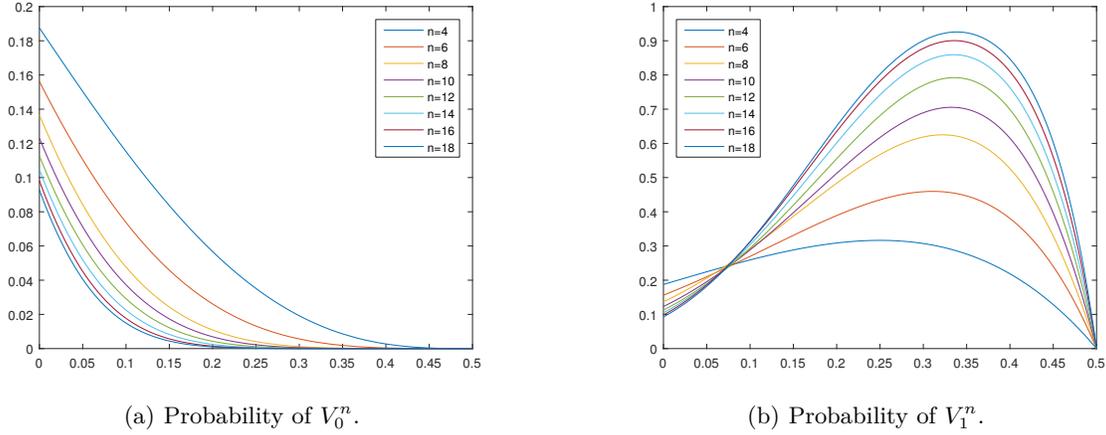
- 1 Set  $V_0 = V_1 = \emptyset$  and  $W = \mathbb{Z}_2^n \setminus \{0^n, 1^n\}$
  - 2 Set  $\alpha = \min_w(W)$  and  $\beta = \max_w(W)$
  - 3 **for**  $i = 1, \dots, C_k^n/2$  **do**
  - 4     Set  $V_0 = V_0 \cup \{\alpha\}$  and  $V_1 = V_1 \cup \{\beta\}$
  - 5     Update  $W = W \setminus \{\alpha, \beta\}$
  - 6     Set  $\alpha = \min_w(W)$  and  $\beta = \max_w(W)$
  - 7 **end**
  - 8 **return**  $(V_0, V_1)$
- 

**Lemma 7.** Let  $x$  be an integer such that  $\sum_{i=1}^x C_i^n < C_k^n/2 < \sum_{i=1}^{x+1} C_i^n$ . Define  $y = C_k^n/2 - \sum_{i=1}^x C_i^n$ ,  $W_0 \subset \{u \in \mathbb{Z}_2^n \mid w(u) = x+1\}$ ,  $W_1 \subset \{u \in \mathbb{Z}_2^n \mid w(u) = n-x-1\}$ , such that  $|W_0| = |W_1| = y$ . Also, let  $(V_0, V_1)$  be the output of Algorithm 4. Then the following properties hold

1.  $V_0 = \{u \mid 1 \leq w(u) \leq x\} \cup W_0$  and  $V_1 = \{u \mid n - x \leq w(u) \leq n - 1\} \cup W_1$ .
2.  $P(V_0) = \sum_{i=1}^x C_i^n (P_0)^{n-i} (P_1)^i + y (P_0)^{n-x-1} (P_1)^{x+1}$  and  $P(V_1) = \sum_{i=1}^x C_i^n (P_0)^i (P_1)^{n-i} + y (P_0)^{x+1} (P_1)^{n-x-1}$ .
3. If  $\varepsilon = 0$  then  $P(S_0) = P(S_1) = \frac{1}{2}$  and if  $\varepsilon = \frac{1}{2}$  then  $P_0 = 0$  and  $P_1 = 1$ .

*Proof.* The proof is a direct consequence of Algorithm 4 and thus is omitted.  $\square$

Figure 3 is a graphical representation of Lemma 7 ( $n \leq 18$ ). The x-axis represents the original bias  $\varepsilon$ , while the y-axis in Figure 3(a) and Figure 3(b) represents  $P(V_0^n)$  and  $P(V_1^n)$ , respectively.



**Fig. 3.** Von Neumann amplifier.

**Lemma 8.** Let  $(V_0, V_1)$  be the output of Algorithm 4. If we map all the elements of  $V_0$  and  $V_1$  to 0 and 1, respectively, then we obtain a maximal bias amplifier  $V_a$ .

*Proof.* According to Lemma 7 all the lowest and highest probability elements are in  $V_0$  and  $V_1$ , respectively. Thus, the statement is true.  $\square$

Unfortunately, due to the nature of  $x$  and  $y$ , the best we could do is to heuristically provide a graphical representation of Conjecture 1 (Figure 4). We could not theoretically prove it in general.

**Conjecture 1.** Let  $n$  be even,  $(S_0^{n-1}, S_1^{n-1})$  be the output of Algorithm 1 for  $n - 1$  and  $(V_0^n, V_1^n)$  be the output of Algorithm 4 for  $n$ . Denote by  $M^n = [P(V_1^n) - P(V_0^n)]/[P(V_1^n) + P(V_0^n)]$ . Then  $M^n < M^{n+2}$  and  $P(S_1^{n-1}) - P(S_0^{n-1}) < M^n$ .

Note that in the case of greedy amplifiers the metric  $[P(S_1^{n-1}) - P(S_0^{n-1})]/[P(S_1^{n-1}) + P(S_0^{n-1})]$  is equal to  $P(S_1^{n-1}) - P(S_0^{n-1})$ . Thus, Conjecture 1 states that the Von Neumann amplifier for a given  $n$  is better at amplifying  $\varepsilon$  than its greedy counterpart. We chose to state the conjecture such that it is true for all  $n \geq 4$ , but, from Figure 4, we can observe that as  $n$  grows the Von Neumann amplifier becomes better at amplifying  $\varepsilon$ <sup>12</sup>. Note that in Figure 4 the x-axis represents the original bias  $\varepsilon$ , while the y-axis represents the values  $P(S_1^{n-1}) - P(S_0^{n-1})$  (interrupted line) and  $M^n$  (continuous line).

<sup>12</sup> e.g. the Von Neumann amplifier for  $n = 8$  is better than the greedy amplifiers for  $n = 3, \dots, 17$

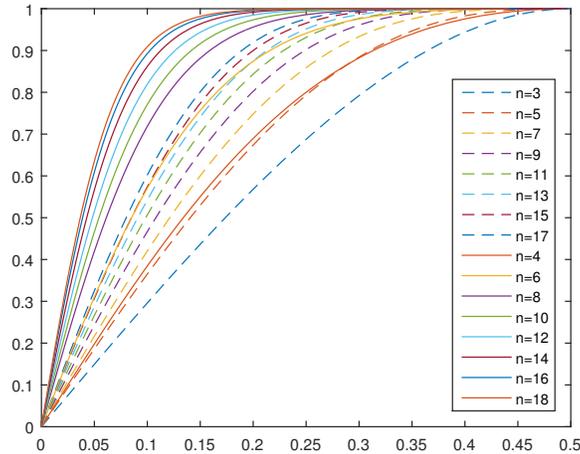


Fig. 4. Comparing greedy amplifiers (interrupted line) with Von Neumann amplifiers (continuous line).

## 5 Applications

### 5.1 The Good

RNG standards [11, 13] require manufactures to implement some early detection mechanism for entropy failure. Health tests represent one such method for detecting major failures. There are two categories of health tests: startup tests and continuous tests. The former are one time tests conducted before the RNG starts producing outputs, while the latter are tests performed in the background during normal operation.

We propose a generic architecture for implementing health tests (Figure 5). We first store data  $D$  (obtained from the noise source) in a buffer, then we apply a **bias amplifier** to it and obtain data  $D_a$ . Next, we apply some lightweight tests on  $D_a$ . If the tests are passed, the RNG outputs  $D$ , otherwise  $D$  is discarded. Note that the **bias amplifier** can be implemented as a lookup table, thus obtaining no processing overhead at the expense of  $\mathcal{O}(2^n)$  memory.

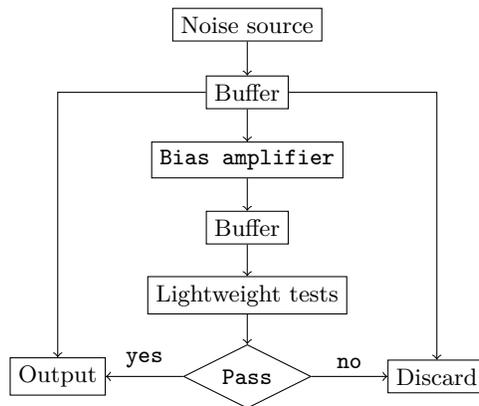


Fig. 5. Generic architecture for implementing health tests.

In our instantiations we used the health tests implemented in Intel’s processors [10]. Intel’s health tests  $H_i$  use a sliding window and count how many times each of the six different bit patterns (Column 1, Table 4) appear in a 256 bit sample. An example of allowable margins for the six patterns can be found in Column 2, Table 4. The thresholds mentioned in Tables 4 and 6 were computed using  $10^6$  256 bit samples generated using the default RNG from the GMP library [3].

We first propose a continuous test using the greedy amplifiers described in Section 3. Depending on the available memory we can use one of the greedy amplifiers and then apply  $H_i$ . Note that  $n$  should be odd due to Lemma 4. If the health test are implemented in a processor it is much easier to use  $n = 4, 8, 16$ . From the health bounds presented in Table 4, we can observe that the differences between data without amplification and data with amplification are not significant. Thus, one can easily update an existing good RNG<sup>13</sup> by adding an extra buffer and an amplification module, while leaving the health bounds intact. Note that due to the unpredictable number of output bits produced by a Von Neumann amplifier, greedy amplifiers are better suited for continuous testing.

**Table 4.** Health bounds for greedy amplifiers (amp.).

Bit pattern	Allowable number of occurrences per sample			
	without amp.	$n = 3$ amp.	$n = 5$ amp.	$n = 7$ amp.
1	$90 < m < 165$	$88 < m < 165$	$89 < m < 167$	$90 < m < 165$
01	$45 < m < 83$	$45 < m < 82$	$46 < m < 83$	$45 < m < 83$
010	$8 < m < 59$	$9 < m < 62$	$10 < m < 58$	$7 < m < 60$
0110	$1 < m < 38$	$2 < m < 34$	$2 < m < 35$	$2 < m < 34$
101	$10 < m < 59$	$10 < m < 61$	$10 < m < 60$	$9 < m < 63$
1001	$1 < m < 35$	$2 < m < 36$	$0 < m < 35$	$1 < m < 35$

To test our proposed configuration and obtain some metrics (Table 5) we conducted a series of experiments. More precisely, we generated 105000 256 bit samples using the Bernoulli distribution instantiated with the Mersenne Twister engine (mt19937) found in the C++ random library [1]. Then, we applied the bias amplifying filters from Table 4 and counted how many samples are marked **pass**. In the case of raw data, a sample is marked **pass**<sup>14</sup> if it passes the  $H_i$  test from Column 1, Table 4. In the case of bias amplification, if a 256 bit buffer  $b_a$  from  $D_a$  passes  $H_i$ , all the input buffers that were used to produce  $b_a$  are marked **pass**. Note that to implement our filters we used lookup tables and thus we had no performance overhead.

From Table 5 we can easily see that when the bias is increased, the number of samples that are marked **pass** is lower than  $H_i$  in the case of greedy amplifiers. Also, note that the rejection rate is higher as  $n$  increases. Thus, enabling us to have an early detection mechanism for RNG failure.

If the design of the RNG has a Von Neumann module, then Von Neumann amplifiers can be used to devise a startup test. Before entering normal operation, the Von Neumann module can be instantiated using the conversion table of the corresponding amplifier. For example, when  $n = 4$  one would use  $V_0 = \{0001, 0010, 0100\}$  and  $V_1 = \{0111, 1011, 1101\}$ <sup>15</sup> instead of  $V_0 = \{0011, 0101, 0110\}$  and  $V_1 = \{1001, 1010, 1100\}$ <sup>16</sup>. The resulting data can then be tested using  $H_i$  and if the test pass the RNG will discard the data and enter normal operation. Note that the first buffer from Figure 5 is not necessary in this case. Note that Von Neumann amplifiers require  $n > 2$ , thus the speed of the RNG will drop. This can be acceptable if the data speed needed for raw data permits it, the RNG generates data much faster than the connecting cables are able to transmit or the raw data is further used by a pseudo-random number generator (PRNG).

<sup>13</sup> that already has  $H_i$  implemented

<sup>14</sup> The terminology used by Intel is that the sample is "healthy".

<sup>15</sup> the sets used to define the maximal Von Neumann amplifier

<sup>16</sup> the sets used to define the Von Neumann corrector

**Table 5.** Greedy amplifiers (amp.) metrics.

$\epsilon$	Number of samples marked pass			
	without amp.	$n = 3$ amp.	$n = 5$ amp.	$n = 7$ amp.
0.00	104999	104997	105000	105000
0.01	104999	104991	104990	105000
0.02	104996	104979	104945	104965
0.03	104988	104925	104685	104384
0.04	104949	104631	103545	101661
0.05	104856	103620	99370	91413
0.06	104598	100668	88845	69832
0.07	104002	93840	69810	41286
0.08	102763	81660	46110	17724
0.09	100411	64332	23460	5404
0.10	96381	44262	9005	1043
0.11	89967	26142	2625	105
0.12	80849	12882	570	0
0.13	69164	5253	65	0
0.14	55856	1704	0	0
0.15	41777	420	0	0
0.16	29039	87	0	0
0.17	18410	21	0	0
0.18	10470	6	0	0
0.19	5331	0	0	0
0.20	2393	0	0	0
0.21	992	0	0	0
0.22	335	0	0	0
0.23	102	0	0	0
0.24	32	0	0	0
0.25	11	0	0	0
0.26	2	0	0	0

**Table 6.** Health bounds for Von Neumann correctors (corr.) and amplifiers (amp.).

Bit pattern	Allowable number of occurrences per sample				
	$n = 2$ corr.	$n = 4$ corr.	$n = 4$ amp.	$n = 6$ corr.	$n = 6$ amp.
1	$91 < m < 166$	$88 < m < 166$	$91 < m < 167$	$89 < m < 167$	$90 < m < 168$
01	$45 < m < 84$	$43 < m < 83$	$44 < m < 83$	$44 < m < 85$	$45 < m < 82$
010	$9 < m < 59$	$9 < m < 59$	$10 < m < 60$	$7 < m < 58$	$9 < m < 60$
0110	$2 < m < 37$	$1 < m < 33$	$1 < m < 36$	$2 < m < 35$	$2 < m < 33$
101	$8 < m < 61$	$10 < m < 58$	$11 < m < 61$	$10 < m < 57$	$8 < m < 60$
1001	$2 < m < 35$	$0 < m < 34$	$2 < m < 35$	$1 < m < 34$	$1 < m < 34$

We also conducted a series of experiments to test the performance of the proposed startup test. This time, we generated data until we obtained 1000 256-bit samples, applied the bias correcting/amplifying filters from Table 6 and counted how many of these samples pass the  $H_i$  test from Column 1, Table 4. Another metric that we computed is the number of input bits required to generate one output bit.

Note that in Table 7 we only wrote the  $n = 2$  corrector, since for  $n = 4, 6$  the results are almost identical. From Table 7 we can easily observe that when the bias is increased the number of samples that pass  $H_i$  is lower than the corrector in the case of Von Neumann amplifiers. As in the case of greedy amplifiers, we can observe that the rejection rate is higher as  $n$  increases. The experimental data also shows that Von Neumann amplifiers perform better than the greedy amplifiers when rejecting bad samples.

**Table 7.** Von Neumann correctors (corr.) and amplifiers (amp.) metrics.

$\epsilon$	Number of samples that pass $H_i$		
	$n = 2$ corr.	$n = 4$ amp.	$n = 6$ amp.
0.00	1000	1000	1000
0.01	1000	1000	1000
0.02	1000	1000	995
0.03	1000	998	940
0.04	1000	981	721
0.05	1000	919	322
0.06	1000	806	79
0.07	1000	567	7
0.08	1000	310	0
0.09	1000	134	0
0.10	1000	53	0
0.11	1000	11	0
0.12	1000	2	0

In Table 8 we can see that more data is required to generate one bit as  $n$  grows. When the bias increases, we can observe that compared to Von Neumann correctors the throughput of the corresponding amplifiers is better. Thus, besides having an early detection mechanism, it also takes less time to detect if an RNG is broken if we use a Von Neumann amplifier.

**Table 8.** Von Neumann correctors (corr.) and amplifiers (amp.) throughput.

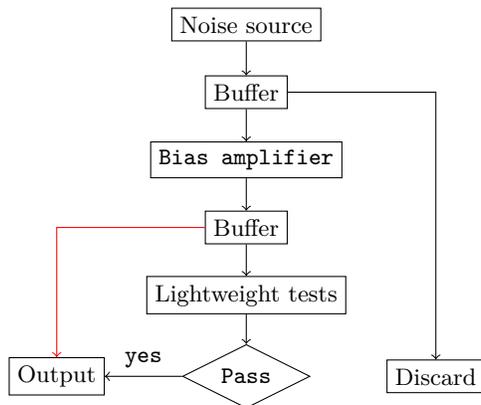
$\epsilon$	Number of input bits per number of output bits				
	$n = 2$ corr.	$n = 4$ corr.	$n = 4$ amp.	$n = 6$ corr.	$n = 6$ amp.
0.00	3.9958	10.6646	10.6751	19.1374	19.2776
0.01	3.9978	10.6690	10.6817	19.1873	19.2548
0.02	4.0044	10.6852	10.6885	19.2513	19.2017
0.03	4.0106	10.7272	10.6873	19.3623	19.0892
0.04	4.0202	10.7956	10.6900	19.5129	18.9534
0.05	4.0352	10.8755	10.6952	19.7228	18.7933
0.06	4.0531	10.9713	10.6980	20.0087	18.5889
0.07	4.0755	11.1025	10.6876	20.3259	18.3405
0.08	4.1013	11.2489	10.6709	20.7180	18.0855
0.09	4.1264	11.3916	10.6841	21.1418	17.8104
0.10	4.1594	11.5733	10.6823	21.6591	17.5187
0.11	4.1956	11.7824	10.6862	22.2298	17.2154
0.12	4.2362	12.0062	10.7001	22.8814	16.9006

## 5.2 The Bad

One can easily turn the benign architecture presented in Figure 5 into a malicious architecture (Figure 6). In the new proposed configuration, health tests always output **pass** and instead of outputting  $D$  the system outputs  $D_a$ .

The malicious configuration can be justified as a bug and can be obtained from the original architecture either by commenting some code lines (similarly to [6]) or by manipulating data buffers (similarly to [7]). Note that code inspection or reverse engineering will reveal these so called bugs. A partial solution to detection can be implementing the architecture in a tamper proof device and deleting the code if someone tinkers with the device. Another partial solution is embedding the architecture as a submodule in a more complex

architecture (similarly to [6]). This solution is plausible due to the sheer complexity of open-source software and the small number of experts who review them [5].



**Fig. 6.** Generic architecture for infecting RNGs.

Another problem is that the RNG will output  $D_a$ s instead of  $D$ s and this translates to lower data rates. A possible solution to this problem is to use  $D_a$  as a seed for a PRNG and then output the data produced by the PRNG. Thus, raw data is never exposed. A problem with this approach is that in most cases the PRNG will also mask the bias. The only case that is compatible with this approach is when the bias is large. Therefore one can simply use an intelligent brute force to find the seed. Hence, breaking the system.

A more suitable approach to the aforementioned problem is to use a pace regulator [9]. This method uses an intermediary buffer to store data and supplies the data consumer with a constant stream of bits. Unfortunately, if data requirements are high, then the regulator will require a lot of memory and in some cases the intermediary buffer will be depleted. Thus, failing to provide data.

A solution specific to greedy amplifiers is to implement in all devices a **neutral filter** after  $D$  and output the resulting data  $D_n$ . Thus, when a malicious version of the RNG is required, one can simply replace the conversion table of the **neutral filter** with the conversion table of the corresponding **bias amplifier**. For example, when  $n = 3$  one would change  $S_0 = \{000, 001, 010, 100\}$  and  $S_1 = \{111, 110, 101, 100\}$ <sup>17</sup> with  $S_0 = \{000, 001, 010, 100\}$  and  $S_1 = \{111, 110, 101, 011\}$ <sup>18</sup>. It is easy to see that in this case both  $D_n$  and  $D_a$  have the same frequency.

Since we are modifying the statistical properties of the raw data, a simple method for detecting changes is black box statistical testing (for example using [2]). Thus, if a user is getting suspicious he can detect the "bugs". Again, a partial solution is to implement the malicious architecture as a submodule inside a more complex architecture either in tamper proof devices, either in complex software. Thus, eliminating the user's access to raw data.

## 6 Conclusions

In our paper we studied and extended **bias amplifiers**, compared their performance and provided some possible applications for them. Even thou in its infancy, this area of research provides insight into what can go wrong with a RNG.

<sup>17</sup> the sets used to define the **neutral filter**

<sup>18</sup> the sets used to define the maximal greedy amplifier

*Future Work.* A possible future direction would be to extend our results to other randomness extractors. Of particular interest, is finding a method to turn a block cipher or a hash function<sup>19</sup> into an amplifier.

## Acknowledgments

The author would like to thank Diana Maimuț and the anonymous reviewers for their helpful comments.

## References

1. C++ Random Library. [www.cplusplus.com/reference/random/](http://www.cplusplus.com/reference/random/).
2. NIST SP 800-22: Download Documentation and Software. <https://csrc.nist.gov/Projects/Random-Bit-Generation/Documentation-and-Software>.
3. The GNU Multiple Precision Arithmetic Library. <https://gmplib.org/>.
4. James Ball, Julian Borger, and Glenn Greenwald. Revealed: How US and UK Spy Agencies Defeat Internet Privacy and Security. *The Guardian*, 6, 2013.
5. Mihir Bellare, Kenneth G. Paterson, and Phillip Rogaway. Security of Symmetric Encryption Against Mass Surveillance. In *CRYPTO 2014*, volume 8616 of *Lecture Notes in Computer Science*, pages 1–19. Springer, 2014.
6. Luciano Bello. DSA-1571-1 OpenSSL—Predictable Random Number Generator. <https://www.debian.org/security/2008/dsa-1571>, 2008.
7. Stephen Checkoway, Jacob Maskiewicz, Christina Garman, Joshua Fried, Shaanan Cohny, Matthew Green, Nadia Heninger, Ralf-Philipp Weinmann, Eric Rescorla, and Hovav Shacham. A Systematic Analysis of the Juniper Dual EC Incident. In *ACM-CCS 2016*, pages 468–479. ACM, 2016.
8. Yevgeniy Dodis, Rosario Gennaro, Johan Håstad, Hugo Krawczyk, and Tal Rabin. Randomness Extraction and Key Derivation Using the CBC, Cascade and HMAC Modes. In *CRYPTO 2004*, volume 3152 of *Lecture Notes in Computer Science*, pages 494–510. Springer, 2004.
9. Houda Ferradi, Rémi Géraud, Diana Maimuț, David Naccache, and Amaury de Wargny. Regulating the Pace of von Neumann Correctors. *Journal of Cryptographic Engineering*, pages 1–7, 2017.
10. Mike Hamburg, Paul Kocher, and Mark E Marson. Analysis of Intel’s Ivy Bridge Digital Random Number Generator. [http://www.rambus.com/wp-content/uploads/2015/08/Intel\\_TRNG\\_Report\\_20120312.pdf](http://www.rambus.com/wp-content/uploads/2015/08/Intel_TRNG_Report_20120312.pdf), 2012.
11. Wolfgang Killmann and Werner Schindler. A Proposal for: Functionality Classes for Random Number Generators, version 2.0. [https://www.bsi.bund.de/SharedDocs/Downloads/DE/BSI/Zertifizierung/Interpretationen/AIS\\_31\\_Functionality\\_classes\\_for\\_random\\_number\\_generators\\_e.pdf?\\_\\_blob=publicationFile](https://www.bsi.bund.de/SharedDocs/Downloads/DE/BSI/Zertifizierung/Interpretationen/AIS_31_Functionality_classes_for_random_number_generators_e.pdf?__blob=publicationFile), 2011.
12. Nicole Perlroth, Jeff Larson, and Scott Shane. NSA Able to Foil Basic Safeguards of Privacy on Web. *The New York Times*, 5, 2013.
13. Meltem Sönmez Turan, Elaine Barker, John Kelsey, Kerry McKay, Mary Baish, and Mike Boyle. NIST DRAFT Special Publication 800-90B: Recommendation for the Entropy Sources Used for Random Bit Generation. <https://nvlpubs.nist.gov/nistpubs/SpecialPublications/NIST.SP.800-90B.pdf>, 2012.
14. John Von Neumann. Various Techniques Used in Connection with Random Digits. *Applied Math Series*, 12:36–38, 1951.
15. Adam Young and Moti Yung. *Malicious Cryptography: Exposing Cryptovirology*. John Wiley & Sons, 2004.

---

<sup>19</sup> For a formal treatment of how one can use a block cipher or a hash function to extract randomness we refer the reader to [8].