The Systemic Errors of Banded Quantum Fourier Transformation

Zhengjun Cao and Zhenfu Cao

Abstract. Quantum Fourier Transformation (QFT) needs to construct the rotation gates with extremely tiny angles. Since it is impossible to physically manipulate such tiny angles (corresponding to extremely weak energies), those gates should be replaced by some scaled and controllable gates. The version of QFT is called banded QFT (BQFT), and can be mathematically specified by Kronecker product and binary fraction. But the systemic errors of BQFT has never been heuristically estimated. In this paper, we generate the programming code for BQFT and argue that its systemic errors are not negligible, which means the physical implementation of QFT with a huge transform size is still a challenge. To the best of our knowledge, it is the first time to obtain the result.

Keywords: Quantum Fourier Transformation, Banded Quantum Fourier Transformation, Kronecker product, transform size, systemic error.

1 Introduction

Quantum Fourier Transformation (QFT) plays a pivotal role in Shor algorithm [1], which needs to apply the rotation gates

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & \exp(-2\pi\, i/2^k) \end{bmatrix}, k = 2, 3, \cdots, n$$

to some qubits, where n is the number of qubits used for QFT. The physical manipulation of gates $R_k, k > \ell$, for some global parameter ℓ, is infeasible due to the failure of controlling extremely weak energy [2]. In 1994, Coppersmith [3] introduced a version of QFT (approximate QFT, AQFT), and claimed that the matrix entries of AQFT differ from those of QFT by a multiplicative factor of $\exp(i\epsilon)$, which leads to an overall error of a fraction of a degree in each phase angle.

In 2004, Fowler and Hollenberg [4, 5] suggested a new scaling technique for QFT, called banded QFT (BQFT). Its basic idea is to replace any gate $R_k, k > \ell$, by

$$R_{\ell,\xi} = \begin{bmatrix} 1 & 0 \\ 0 & \exp(-2\pi\, i\, \xi/2^\ell) \end{bmatrix}, \quad \text{where} \quad \xi \geq 1.$$

In 2013, Nam and Blümel [6–8] discussed the scaling laws for BQFT. But so far, the programming code for BQFT has never been exhibited, which results in the failure to estimate its systemic errors.

In this paper, we generate the programming code for BQFT and investigate its error propagation mode, which shows that some quantum states produced by BQFT are easily distinguishable from the

Z.J. Cao is with Department of Mathematics, Shanghai University, Shanghai, China. caozhj@shu.edu.cn
Z.F. Cao is with Software Engineering Institute, East China Normal University, China. zfcao@sei.ecnu.edu.cn

counterparts by QFT. We prove that the systemic errors of BQFT are not negligible. To the best of our knowledge, it is the first time to invent the estimation method for BQFT.

2 Preliminaries

The state of a qubit is described by a 2-dimensional vector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, where α and β are complex numbers such that $|\alpha|^2+|\beta|^2=1$. The basic two quantum states corresponding to the two states of a classical bit are defined by bit $0 \leftrightarrow |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, bit $1 \leftrightarrow |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The basic single-qubit operations include $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}, S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$, where H is called Hadamard gate. Clearly, $H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle), H|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$.

Given two separate qubits, the corresponding two-qubit state is given by the Kronecker product of vectors. For example,

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \otimes \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \alpha \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \\ \beta \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{bmatrix}.$$

The basis for two-qubit states consists of

$$\text{string } 00 \leftrightarrow |00\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{string } 01 \leftrightarrow |01\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\text{string } 10 \leftrightarrow |10\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{string } 11 \leftrightarrow |11\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Given a state $|x_1x_2\cdots x_n\rangle$, the below Mathematica code can convert it into a corresponding binary vector.

```
Qv[x_]:= Module[{mt, QVrule},(* convert a binary list into a vector *)
         QVrule={0->{1,0}, 1->{0,1}}; (* column vector by default *)
         mt=ToExpression[x]; (* convert charaters 0, 1 into numbers 0, 1 *)
         mt=mt/.QVrule; mt=FoldList[KroneckerProduct, mt]; Flatten[mt[[-1]]]]
x={1, 0, 1, 0, 0}; Qv[x]
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

A unitary transformation on n qubits is a matrix U of size $2^n \times 2^n$. The CNOT (controlled-NOT) gate is a commonly used two-qubit gate

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Sometimes, two-qubit gates can be described by the tensor product of some single-qubit gates. Not all two-qubit gates can be written as the tensor product of single-qubit gates. Such a gate is called an *entangling gate*, for example, the CNOT gate. The gates H, T and CNOT form a universal gate set because any general unitary transformation can be broken into a series of two qubit rotations.

The only way to change qubits without measuring is to apply a unitary operation. Quantum computations can be created by designing unitary operations in sequence, each of which is composed of smaller operations.

3 Quantum Fourier Transformation

Let n be the number of qubits used for QFT, $N = 2^n$, and $\omega = \exp(-2\pi\, i/N)$. The QFT for n-qubits is described by the matrix

$$\text{QFT}_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{bmatrix}$$

The below Mathematica code can generate such a QFT matrix.

```
QFTMatrix1[n_]:= Module[{K, w, A, B, T, i, j},
          (* without the multiplier 1/Sqrt[2^n] *)
          K=2^n; w=Exp[-2*Pi*I/K]; B={}; T=Table[1, {i, K}];
          For[j=1, j<K, j++, A=Table[w^(i*j), {i, 0, K-1}];
          B=Join[B, A]]; Partition[Join[T, B], K]]
```

The QFT can be decomposed and represented by Kronecker product and binary fraction. Let $x_1x_2\cdots x_n$ be a binary string. Its binary fraction is expressed as

$$0.x_1x_2\cdots x_n = \frac{x_1}{2} + \frac{x_2}{2^2} + \cdots + \frac{x_n}{2^n}$$

The QFT performed on state $|x_1x_2\cdots x_n\rangle$ can be represented by

$$\begin{aligned} |x_1x_2\cdots x_n\rangle \xrightarrow{\text{QFT}} & \frac{1}{\sqrt{2^n}}[|0\rangle + e^{-2\pi\, i0.x_n}|1\rangle] \otimes [|0\rangle + e^{-2\pi\, i0.x_{n-1}x_n}|1\rangle] \\ & \otimes \cdots \otimes [|0\rangle + e^{-2\pi\, i0.x_1x_2\cdots x_n}|1\rangle] \end{aligned} \quad (1)$$

and further translated into the below quantum circuit (Fig.1, Ref.[9]).

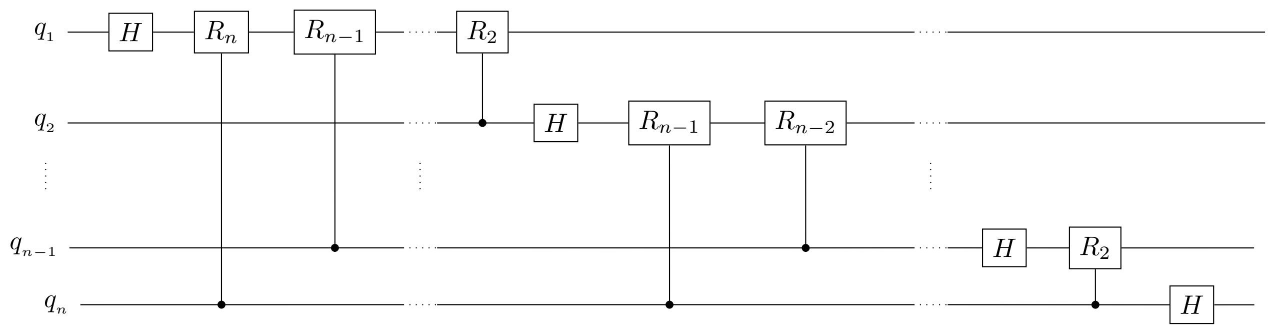

Figure 1: The QFT circuit for n qubits

The process involving only Kronecker product and binary fraction, can be converted into the below Mathematica code.

```
BinaryFraction[x_]:= Module[{mt,s,t,f},
   (* the list x corresponding to a binary string *)
   f[s_,t_]:=s/2+t; mt=Reverse[x]; mt=ToExpression[mt];
   mt=FoldList[f, mt]; mt=Expand[1/2*mt]];
QFT[x_]:=Module[{mt,g,s},(* without the multiplier *)
   g[s_]={1,s}; mt=BinaryFraction[x]; mt=Map[Exp,-2*Pi*I*mt];
   mt=Map[g,mt]; mt=FoldList[KroneckerProduct, mt]; Flatten[mt[[-1]]]];
QFTMatrix2[n_]:=Module[{v,w}, v=IntegerString[Range[0,2^n-1],2];
  v=StringPadLeft[v,n,"0"]; v=MapThread[StringPartition, {v, Table[1, 2^n]}];
  MapThread[QFT,{v}]//MatrixForm]
```

4 Banded Quantum Fourier Transformation

4.1 Description of BQFT

The controlled-rotation gate R_n involves an extremely tiny angle $\pi/2^{n-1}$. Notice that the Planck constant is $h = 6.626 \times 10^{-34}$ joule second. The energy of a free particle with wave function Ae^{ikx} is $E = \frac{h^2k^2}{8\pi^2 m}$ where m is the mass of this particle, x is the distance between an observer and this particle, and k is the wave number. It's easy to find that the physical manipulation corresponding to the tiny number $\pi/2^{n-1}$ cannot be practically performed if $n > 120$.

The BQFT aims to replace those controlled-rotation gates $R_k, k > \ell$. Naturally, it can be represented as

$$\begin{aligned}|x_1x_2\cdots x_n\rangle \xrightarrow{\text{BQFT}} & \frac{1}{\sqrt{2^n}}[|0\rangle + e^{-2\pi\, i 0.x_n}|1\rangle] \otimes [|0\rangle + e^{-2\pi\, i 0.x_{n-1}x_n}|1\rangle] \otimes \cdots \\ & \otimes [|0\rangle + e^{-2\pi\, i 0.x_{n-\ell+1}x_{n-\ell+2}\cdots x_n}|1\rangle] \\ & \otimes [|0\rangle + e^{-2\pi\, i (0.x_{n-\ell}x_{n-\ell+1}\cdots x_{n-1}+\frac{r_1}{2^\ell})}|1\rangle] \\ & \otimes [|0\rangle + e^{-2\pi\, i (0.x_{n-\ell-1}x_{n-\ell}\cdots x_{n-2}+\frac{r_2}{2^\ell})}|1\rangle] \otimes \cdots \\ & \otimes [|0\rangle + e^{-2\pi\, i (0.x_1x_2\cdots x_\ell+\frac{r_{n-\ell}}{2^\ell})}|1\rangle] \end{aligned} \quad (2)$$

where $r_1, \cdots, r_{n-\ell}$ are nonnegative integers.

4.2 Truncated QFT

The noisy factor $e^{-2\pi i \frac{r}{2^\ell}}$ is regularly distributed over some components of BQFT$|x_1 x_2 \cdots x_n\rangle$. To eliminate its effect and facilitate the construction of quantum circuit, it is usual to take $r = 0$, i.e., each gate $R_k, k > \ell$ is replaced by $R_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. For convenience, we call this version truncated QFT (TQFT). Its circuit is depicted as follows (Fig.2).

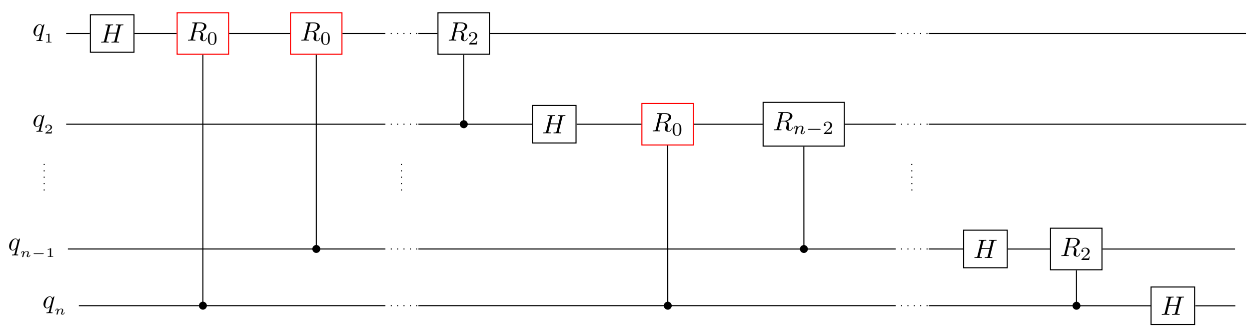

Figure 2: The TQFT circuit in which R_n, R_{n-1} are replaced by R_0

The below Mathematica code for TQFT can be easily verified.

```
TruncatedBinaryFraction[x_, k_]:= Module[{mt, ht, u, n, f, s, t},
(* k is the number of truncated fraction types *)
   f[s_, t_]:=s/2+t; n=Length[x]; u=1/2^(n-k+1); mt=Reverse[x]; ht=Take[mt,k];
   mt=ToExpression[mt]; mt=FoldList[f,mt]; mt=Expand[1/2*mt];
   ht=ToExpression[ht]; ht=FoldList[f,ht]; ht=Expand[u*ht];
   ht=PadLeft[ht,n]; mt-ht];
TQFT[x_, k_]:= Module[{mt, g}, g[s_]:={1,s}; mt=TruncatedBinaryFraction[x,k];
   mt=Map[Exp, -2*Pi*I*mt]; mt=Map[g, mt];
   mt=FoldList[KroneckerProduct, mt]; Flatten[mt[[-1]]]]

x={a, b, c, d, h}; k=0; TQFT[x, k]
k=1; TQFT[x, k]
```

For example, given the 5-qubit state $|abcdh\rangle$, QFT will generate the following state (without the multiplier $\frac{1}{\sqrt{2^5}}$)

$$\{\, 1, e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\frac{h}{32}\right)\pi}, e^{-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\frac{h}{32}\right)\pi-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi}, e^{-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\frac{h}{32}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi}, e^{-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\frac{h}{32}\right)\pi-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi}, e^{-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\frac{h}{32}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi}, e^{-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi},$$

$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\frac{h}{32}\right)\pi-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi}, e^{-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\frac{h}{32}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi}, e^{-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\frac{h}{32}\right)\pi-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi}, e^{-ih\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\frac{h}{32}\right)\pi-ih\pi}, e^{-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-ih\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\frac{h}{32}\right)\pi-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-ih\pi}, e^{-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-ih\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\frac{h}{32}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-ih\pi}, e^{-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-ih\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\frac{h}{32}\right)\pi-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-ih\pi}, e^{-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi-ih\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\frac{h}{32}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi-ih\pi}, e^{-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi-ih\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\frac{h}{32}\right)\pi-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi-ih\pi}, e^{-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi-ih\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\frac{h}{32}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi-ih\pi},$$
$$e^{-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi-ih\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\frac{h}{32}\right)\pi-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi-ih\pi}\ \}$$

If the gate R_5 is replaced by the unit gate R_0, the TQFT will generate the below state

$$\{\ 1, e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}\right)\pi}, e^{-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}\right)\pi-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi}, e^{-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi}, e^{-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}\right)\pi-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi}, e^{-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi}, e^{-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}\right)\pi-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi}, e^{-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi}, e^{-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}\right)\pi-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi}, e^{-ih\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}\right)\pi-ih\pi}, e^{-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-ih\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}\right)\pi-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-ih\pi}, e^{-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-ih\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-ih\pi}, e^{-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-ih\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}\right)\pi-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-ih\pi}, e^{-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi-ih\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi-ih\pi}, e^{-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi-ih\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}\right)\pi-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi-ih\pi}, e^{-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi-ih\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi-ih\pi},$$
$$e^{-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi-ih\pi},$$
$$e^{-2i\left(\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}\right)\pi-2i\left(\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\frac{h}{16}\right)\pi-2i\left(\frac{c}{2}+\frac{d}{4}+\frac{h}{8}\right)\pi-2i\left(\frac{d}{2}+\frac{h}{4}\right)\pi-ih\pi}\ \}$$

The factor $e^{-2\pi\, i \frac{h}{32}}$ in 16 components are eliminated.

The below Mathematica code can be used to construct TQFT matrix.

```
GetMatrix[n_, k_]:= Module[{v, w}, (* n is the number of qubits,
      k is the number of truncated fraction types *)
  v=IntegerString[Range[0,2^n-1],2]; v=StringPadLeft[v,n,"0"];
  v=MapThread[StringPartition,{v,Table[1,2^n]}];
  w=ToExpression[Table[k,2^n]]; MapThread[TQFT,{v,w}]//MatrixForm]
```

For example, GetMatrix$[4,1]$ outputs the below matrix (Fig.3). By induction, we have the following result.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & e^{-\frac{i\pi}{4}} & e^{-\frac{i\pi}{4}} & -i & -i & -i\,e^{-\frac{i\pi}{4}} & -i\,e^{-\frac{i\pi}{4}} & -1 & -1 & -e^{-\frac{i\pi}{4}} & -e^{-\frac{i\pi}{4}} & i & i & i\,e^{-\frac{i\pi}{4}} & i\,e^{-\frac{i\pi}{4}} \\
1 & e^{-\frac{i\pi}{4}} & -i & -i\,e^{-\frac{i\pi}{4}} & -1 & -e^{-\frac{i\pi}{4}} & i & i\,e^{-\frac{i\pi}{4}} & 1 & e^{-\frac{i\pi}{4}} & -i & -i\,e^{-\frac{i\pi}{4}} & -1 & -e^{-\frac{i\pi}{4}} & i & i\,e^{-\frac{i\pi}{4}} \\
1 & e^{-\frac{i\pi}{4}} & e^{-\frac{3i\pi}{4}} & -1 & i & i\,e^{-\frac{i\pi}{4}} & i\,e^{-\frac{3i\pi}{4}} & -i & -1 & -e^{-\frac{i\pi}{4}} & -e^{-\frac{3i\pi}{4}} & 1 & -i & -i\,e^{-\frac{i\pi}{4}} & -i\,e^{-\frac{3i\pi}{4}} & i \\
1 & -i & -1 & i & 1 & -i & -1 & i & 1 & -i & -1 & i & 1 & -i & -1 & i \\
1 & -i & e^{\frac{3i\pi}{4}} & -i\,e^{\frac{3i\pi}{4}} & -i & -1 & -i\,e^{\frac{3i\pi}{4}} & -e^{\frac{3i\pi}{4}} & -1 & i & -e^{\frac{3i\pi}{4}} & i\,e^{\frac{3i\pi}{4}} & i & 1 & i\,e^{\frac{3i\pi}{4}} & e^{\frac{3i\pi}{4}} \\
1 & e^{-\frac{3i\pi}{4}} & i & i\,e^{-\frac{3i\pi}{4}} & -1 & -e^{-\frac{3i\pi}{4}} & -i & -i\,e^{-\frac{3i\pi}{4}} & 1 & e^{-\frac{3i\pi}{4}} & i & i\,e^{-\frac{3i\pi}{4}} & -1 & -e^{-\frac{3i\pi}{4}} & -i & -i\,e^{-\frac{3i\pi}{4}} \\
1 & e^{-\frac{3i\pi}{4}} & e^{\frac{i\pi}{4}} & -i & i & i\,e^{-\frac{3i\pi}{4}} & i\,e^{\frac{i\pi}{4}} & 1 & -1 & -e^{-\frac{3i\pi}{4}} & -e^{\frac{i\pi}{4}} & i & -i & -i\,e^{-\frac{3i\pi}{4}} & -i\,e^{\frac{i\pi}{4}} & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & e^{-\frac{i\pi}{4}} & -e^{-\frac{i\pi}{4}} & -i & i & -i\,e^{-\frac{i\pi}{4}} & i\,e^{-\frac{i\pi}{4}} & -1 & 1 & -e^{-\frac{i\pi}{4}} & e^{-\frac{i\pi}{4}} & i & -i & i\,e^{-\frac{i\pi}{4}} & -i\,e^{-\frac{i\pi}{4}} \\
1 & e^{\frac{3i\pi}{4}} & -i & -i\,e^{\frac{3i\pi}{4}} & -1 & -e^{\frac{3i\pi}{4}} & i & i\,e^{\frac{3i\pi}{4}} & 1 & e^{\frac{3i\pi}{4}} & -i & -i\,e^{\frac{3i\pi}{4}} & -1 & -e^{\frac{3i\pi}{4}} & i & i\,e^{\frac{3i\pi}{4}} \\
1 & e^{\frac{3i\pi}{4}} & e^{-\frac{3i\pi}{4}} & 1 & i & i\,e^{\frac{3i\pi}{4}} & i\,e^{-\frac{3i\pi}{4}} & i & -1 & -e^{\frac{3i\pi}{4}} & -e^{-\frac{3i\pi}{4}} & -1 & -i & -i\,e^{\frac{3i\pi}{4}} & -i\,e^{-\frac{3i\pi}{4}} & -i \\
1 & i & -1 & -i & 1 & i & -1 & -i & 1 & i & -1 & -i & 1 & i & -1 & -i \\
1 & i & e^{\frac{3i\pi}{4}} & i\,e^{\frac{3i\pi}{4}} & -i & 1 & -i\,e^{\frac{3i\pi}{4}} & e^{\frac{3i\pi}{4}} & -1 & -i & -e^{\frac{3i\pi}{4}} & -i\,e^{\frac{3i\pi}{4}} & i & -1 & i\,e^{\frac{3i\pi}{4}} & -e^{\frac{3i\pi}{4}} \\
1 & e^{\frac{i\pi}{4}} & i & i\,e^{\frac{i\pi}{4}} & -1 & -e^{\frac{i\pi}{4}} & -i & -i\,e^{\frac{i\pi}{4}} & 1 & e^{\frac{i\pi}{4}} & i & i\,e^{\frac{i\pi}{4}} & -1 & -e^{\frac{i\pi}{4}} & -i & -i\,e^{\frac{i\pi}{4}} \\
1 & e^{\frac{i\pi}{4}} & e^{\frac{i\pi}{4}} & i & i & i\,e^{\frac{i\pi}{4}} & i\,e^{\frac{i\pi}{4}} & -1 & -1 & -e^{\frac{i\pi}{4}} & -e^{\frac{i\pi}{4}} & -i & -i & -i\,e^{\frac{i\pi}{4}} & -i\,e^{\frac{i\pi}{4}} & 1
\end{pmatrix}$$

Figure 3: The matrix for 4-qubit TQFT with 1 truncated fraction type

Theorem 1. *The TQFT matrix is unitary, symmetric, and invertible. Its inverse transformation has the same decomposition as TQFT, using Kronecker product and binary fraction, but in which -2π is replaced with 2π. The quantum circuit of TQFT likes that of QFT, except replacing any gate $R_k, k > \ell$ with the unit gate R_0.*

4.3 The systemic errors of TQFT

It is easy to find the difference between QFT and TQFT for n-qubit state $|x_1 x_2 \cdots x_n\rangle$, namely, the factors

$$\begin{aligned}
&e^{-2\pi\, i \frac{x_n}{2^{\ell+1}}}, \\
&e^{-2\pi\, i \frac{x_{n-1}}{2^{\ell+1}}}, e^{-2\pi\, i \frac{x_n}{2^{\ell+2}}}, \\
&e^{-2\pi\, i \frac{x_{n-2}}{2^{\ell+1}}}, e^{-2\pi\, i \frac{x_{n-1}}{2^{\ell+2}}}, e^{-2\pi\, i \frac{x_n}{2^{\ell+3}}}, \\
&\qquad\qquad \vdots \\
&e^{-2\pi\, i \frac{x_{\ell+1}}{2^{\ell+1}}}, \cdots, e^{-2\pi\, i \frac{x_{n-1}}{2^{n-1}}}, e^{-2\pi\, i \frac{x_n}{2^n}}
\end{aligned}$$

are eliminated. Notice that the second component of $\text{QFT}|x_1x_2\cdots x_n\rangle$ has the angle

$$\alpha_2 := -2\pi\left(\frac{x_1}{2}+\frac{x_2}{2^2}+\cdots+\frac{x_\ell}{2^\ell}+\cdots+\frac{x_n}{2^n}\right)$$

whlie the counterpart of $\text{TQFT}|x_1x_2\cdots x_n\rangle$ is

$$\beta_2 := -2\pi\left(\frac{x_1}{2}+\frac{x_2}{2^2}+\cdots+\frac{x_\ell}{2^\ell}\right)$$

The third component of $\text{QFT}|x_1x_2\cdots x_n\rangle$ has the angle

$$\alpha_3 := -2\pi\left(\frac{x_2}{2}+\frac{x_3}{2^2}+\cdots+\frac{x_\ell}{2^{\ell-1}}+\cdots+\frac{x_n}{2^{n-1}}\right)$$

while its counterpart is

$$\beta_3 := -2\pi\left(\frac{x_2}{2}+\frac{x_3}{2^2}+\cdots+\frac{x_\ell}{2^{\ell-1}}\right)$$

If $x_1 = x_2 = \cdots = x_{\ell-1} = 0, x_\ell = \cdots = x_n = 1$ and $n \gg \ell$, then

$$\alpha_2 = -2\pi\left(\frac{1}{2^\ell}+\cdots+\frac{1}{2^n}\right) \approx -\frac{\pi}{2^{\ell-2}}, \quad \beta_2 = -\frac{\pi}{2^{\ell-1}}, \ \alpha_2/\beta_2 \approx 2$$

Likewise, we have $\alpha_3/\beta_3 \approx 2$, etc. The ratio is too big to neglect.

Theorem 2. *Let ℓ be the threshold value for rotation gates $R_k, \ell < k \le n$, where n is the number of qubits for QFT. Then the systemic error distribution of TQFT is unbalanced. For some inputs, the ratios of some rotation angles in state $QFT|x_1x_2\cdots x_n\rangle$ to their counterparts in state $TQFT|x_1x_2\cdots x_n\rangle$ approximate 2.*

5 Further discussions

As we see, the QFT is not used in isolation. Its inverse QFT^{-1} should be used later. For example,

$$\begin{aligned}\text{binary string } x_1x_2\cdots x_n &\xrightarrow[\text{[phase-1]}]{\text{modulation}} |x_1x_2\cdots x_n\rangle \\ &\xrightarrow[\text{[phase-2]}]{\text{QFT}} \frac{1}{\sqrt{2^n}}\sum_{i=0}^{2^n-1} e^{-2\pi\, i\omega_i}|\chi_i\rangle \\ &\xrightarrow[\text{[phase-3]}]{\text{QFT}^{-1}} \frac{1}{\sqrt{2^n}}\sum_{i=0}^{2^n-1} e^{-2\pi\, i\vartheta_i}|\xi_i\rangle \\ &\xrightarrow[\text{[phase-4]}]{\text{measurement}} \text{binary string } y_1y_2\cdots y_n\end{aligned}$$

Only phase-1 and phase-2 can be numerically simulated. The phase-3 cannot be simulated, because its input is a superposition. So, the overall error of TQFT and TQFT^{-1} cannot be estimated mathematically. Can the current quantum computers, including IBM Heron 133-qubit processor and Google Sycamore quantum chip, be used to test QFT or TQFT to certify themselves?

6 Conclusion

We generate the programming codes to test the TQFT and show its systemic errors are not negligible. We want to stress that it is impossible to efficiently implement QFT with a big transform size using the current quantum technology.

Acknowledgements

We thank the Key Programs of National Natural Science Foundation of China, Grant 62132005.

References

[1] P. Shor. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. SIAM J. Comput., 26(5), 1484-1509, 1997.

[2] Z. J. Cao and Z. F. Cao. The Planck constant and quantum Fourier transformation. Preprint, https://eprint.iacr.org/2023/1971

[3] D. Coppersmith. An Approximate Fourier Transform Useful in Quantum Factoring, IBM Research Report RC 19642, (1994), https://arxiv.org/abs/quant-ph/0201067

[4] A. G. Fowler and L. C. L. Hollenberg. Scalability of Shor's algorithm with a limited set of rotation gates. Phys. Rev. A 70, 032329, 2004.

[5] A. G. Fowler and L. C. L. Hollenberg. Erratum: Scalability of Shor's algorithm with a limited set of rotation gates. [Phys. Rev. A 70, 032329 (2004)] Phys. Rev. A 75, 029905, 2007.

[6] Y. S. Nam and R. Blümel. Scaling laws for Shor's algorithm with a banded quantum Fourier transform. Phys. Rev. A 87, 032333, 2013.

[7] Y. S. Nam and R. Blümel. Robustness of the quantum Fourier transform with respect to static gate defects. Phys. Rev. A, Atomic, molecular, and optical physics, 89(4), 2014.

[8] Y. S. Nam and R. Blümel. Performance scaling of the quantum Fourier transform with defective rotation gates. Quantum Information & Computation, 15(9-10):721-736, 2015.

[9] D. Camps, R. Beeumen, C. Yang. Quantum Fourier transform revisited, arXiv:2003.03011v2, 2020.

Appendix: Approximate Quantum Fourier Transformation

The AQFT introduced by Coppersmith [3] is a version of QFT. The matrix entries of AQFT differ from those of QFT by a multiplicative factor of $\exp(i\epsilon)$. For example, the AQFT matrix on 3 electrons can be decomposed as below (see Ref.[3]), where $\omega = \exp(-2\pi i/8)$, and the rows are numbered in bit-reversed order (04261537).

$$\frac{e^{\pi\, i(2\epsilon_1+\epsilon_2)}}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega & \omega^6 & \omega^3 \\ 1 & \omega^3 & \omega^6 & \omega & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix}$$

$$=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \times e^{\pi\, i\, \epsilon_1}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 \end{bmatrix}$$

$$\times e^{\pi\, i\, \epsilon_2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega \end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\times e^{\pi\, i\, \epsilon_1}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 \end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

It is worth noting that the AQFT matrix is eventually specified by matrix decomposition and bit-reversed order. There is at least one sparse matrix involving the least rotation angle $\pi/2^{n-1}$, when n qubits are used. The least angle can be scaled to $\pi\varphi := \pi(\epsilon + \frac{1}{2^{n-1}}) > \pi/2^{\ell}$ by choosing proper ϵ. Some entries of value 1 or -1 in the same sparse matrix are simultaneously scaled to $\pi\,\epsilon = \pi\varphi - \frac{\pi}{2^{n-1}} > \pi/2^{\ell}$. But so far, the quantum circuit for AQFT has never been discovered.