Simulation-Secure Threshold PKE from LWE with Polynomial Modulus*

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November 7, 2023

Abstract

In LWE based cryptosystems, using small (polynomially large) ciphertext modulus improves both efficiency and security. In threshold encryption, one often needs *simulation security*: the ability to simulate decryption shares without the secret key. Existing lattice-based threshold encryption schemes provide one or the other but not both. Simulation security has seemed to require superpolynomial flooding noise, and the schemes with polynomial modulus use Rényi divergence based analyses that are sufficient for game-based but not simulation security.

In this work, we give the first construction of simulation-secure lattice-based threshold PKE with polynomially large modulus. The construction itself is relatively standard, but we use an improved analysis, proving that when the ciphertext noise and flooding noise are both Gaussian, simulation is possible even with very small flooding noise. Our modulus is small not just asymptotically but also concretely: this technique gives parameters roughly comparable to those of highly optimized non-threshold schemes like FrodoKEM. As part of our proof, we show that LWE remains hard in the presence of some types of leakage; these results and techniques may also be useful in other contexts where noise flooding is used.

1 Introduction

A *threshold cryptosystem* allows to share a secret (decryption) key among a set of servers, in such a way that the servers can collaboratively decrypt messages, and still no set of servers (below a given threshold) can gain any information about the encrypted messages. Threshold encryption schemes are a fundamental tool in cryptography, both in theory (e.g., as a building block used in the construction of secure multiparty computation protocols [AJL+12]) and in practice (as an effective solution to avoid the single point of failure associated to the secret key.) In order to promote further progress in the area, NIST (the National Institute of Standards and Technology) has recently issued a call for Multi-Party Threshold Schemes [BP23], aimed both at the development of a standard for threshold versions of mature "pre-quantum" schemes already standardized by NIST, and also the exploratory investigation of other primitives that are not yet a NIST standard. Of special interest among the more advanced primitives (and the main focus of this paper) is lattice-based encryption, which has already been selected as a candidate for post-quantum cryptography, and is expected to become an official standard within a year or so.

A threshold version of lattice based encryption (or, more specifically, Regev's cryptosystem [Reg09]) was first given by Bendlin and Damgard [BD10] in 2010. Based on the linear key-homomorphic properties of lattice based encryption, the BD10 scheme [BD10] gives a very elegant and relatively efficient (non-interactive) solution to the threshold decryption problem, where each server locally computes a "partial decryption", and then these partial decryptions are simply added up and rounded to the final output message.

*Research supported in part by Global Research Cluster program of Samsung Advanced Institute of Technology, Intel Cryptography Frontiers program, and NSF Award 1936703. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

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The main drawback of the BD10 scheme is that in order to protect the secret key during the partial decryption process, it uses *noise flooding*. This is a masking technique that requires the use of an exponentially large noise, and correspondingly large "ciphertext modulus" q.[1] This has a negative impact both on efficiency (requiring computations modulo a large q, and the use of "big-int" large precision arithmetic libraries) and security (requiring the assumption that lattice problems are hard to solve within superpolynomial factors $\approx q$.) So, since the publication of [BD10], it has been an important open problem in the area to develop an efficient threshold cryptosystem using a polynomial modulus q and based on the hardness of approximating lattice problems within (small) polynomial factors.

Our contribution In this paper we give an efficient lattice-based threshold encryption scheme

- satisfying a strong (simulation based) notion of security (which is relevant to the use of threshold cryptography in MPC applications)
- supporting an arbitrary (polynomial) number of decryption queries, and
- using a polynomial modulus q (which results in a standard hardness assumption of approximating lattice problems within a polynomial approximation factor.)

Some progress towards these goals had recently been obtained in [CSS$^+$22, BS23], which used Renyi divergence techniques to analyze threshold encryption with polynomial modulus, but only achieving a weaker (game-based) definition of security and assuming an a-priori polynomial upper bound on the number of decryption queries.[2] So, to the best of our knowledge, our is the first work achieving these strong security properties with a polynomial modulus and inapproximability assumption.

On the technical side, we give a general construction and analysis technique that is applicable to a broad range of lattice-based encryption schemes, including Regev's encryption [Reg09] (as already done in [BD10] using exponential noise flooding), and also more efficient variants like [LP11] that use the LWE problem both during key generation and encryption. In fact, for the case of Regev's cryptosystem, our scheme is essentially the same as BD10, and the main contribution is in the analysis technique, which may be of independent interest and find additional applications. This results in a very simple design, with a concrete efficiency (both in terms of running time, and key and ciphertext size) essentially the same as the basic (non-threshold) version of the schemes. We exemplify the practicality of the scheme considering a threshold version of Frodo [NAB$^+$20], a highly optimized scheme that was among the leading NIST candidates for post-quantum cryptography based on the general (non-ring) Learning With Errors (LWE) problem.

While in this paper we focus on LWE, most of our results are easily adapted to the Ring LWE ([SSTX09, LPR10]) setting, which provides even more efficient constructions based on stronger (but still standard) hardness assumptions on algebraic lattices. In fact, the only lemma in this paper that does not easily adapt to the ring setting is the proof that a certain variant of LWE with known error norm is equivalent to the standard LWE problem. (See Lemma 9.) In Section 5.3 we introduce the ring version of this problem as an assumption, which we call "Known-Covariance RLWE", under which we sketch an RLWE-based PKE scheme from which we can construct threshold cryptosystems. We leave it as an open problem to either provide an attack showing that this generalization of Known-Norm LWE is false in the ring setting, or demonstrate that it is equivalent to the standard Ring LWE problem and worst-case assumptions on the approximability of algebraic lattices.

[1] In lattice-based cryptography, ciphertexts consists of vectors or matrices with integer entries modulo q, for some positive q called the ciphertext modulus.

[2] Specifically, the modulus q (and lattice inapproximability factor in the complexity assumption) in these works scales with $\sqrt{\ell}$, where ℓ is the number of decryption queries performed by the attacker. So, supporting an arbitrary polynomial number of queries ℓ still requires a superpolynomial modulus q. Moreover, this limination seems intrinsic in the use of Renyi divergence techniques. See Section 1.2 for a more detailed comparison.

1.1 Technical Overview

A common framework for Threshold LWE encryption is as follows: Ciphertexts are of the form $(\mathbf{a}, \langle \mathbf{a}, \mathbf{s} \rangle + e_{ct} + msg)$, so for decryption the parties need to compute $\langle \mathbf{a}, \mathbf{s} \rangle$ collectively. Each party gets a share $\mathbf{s_i}$ of $\mathbf{s}$ under some linear secret sharing scheme. (For example, for T-out-of-T threshold encryption one could have $\sum_i \mathbf{s_i} = \mathbf{s}$.) Party i can then compute $\langle \mathbf{a}, \mathbf{s_i} \rangle$ on its own, and by linearity the shares $\langle \mathbf{a}, \mathbf{s_i} \rangle$ can be combined to reconstruct $\langle \mathbf{a}, \mathbf{s} \rangle$. However, because revealing $\langle \mathbf{a}, \mathbf{s_i} \rangle$ would leak information about $\mathbf{s_i}$, each party instead reveals a noisy version $\langle \mathbf{a}, \mathbf{s_i} \rangle + \tilde{e}$ (where $\tilde{e}$ is called the "flooding" or "smudging" noise). This means when the decryption shares are combined to reconstruct $\langle \mathbf{a}, \mathbf{s} \rangle$, the result is noisy, but since the ciphertext is noisy anyway, the parameters of the scheme just need to be adjusted so that decryption succeeds even with the extra noise.

An adversary is able to learn $(\mathbf{a}, \langle \mathbf{a}, \mathbf{s} \rangle + e_{ct})$ from the ciphertext and $\langle \mathbf{a}, \mathbf{s} \rangle + \tilde{e}$ from the decryption share. The difficulty when proving security is simulating the latter given the former, but without knowing $\mathbf{s}$ or e_{ct}.

One approach is to have $\tilde{e}$ come from a very wide distribution, say a Gaussian $\mathcal{N}_\sigma$ for large σ, such that $\mathcal{N}_{0,\sigma}$ is statistically close to $\mathcal{N}_{e_{ct},\sigma}$. Then we can simulate partial decryption by sampling $e' \leftarrow \mathcal{N}_{0,\sigma}$ and returning $\langle \mathbf{a}, \mathbf{s} \rangle + e_{ct} + e'$. This is the approach taken by [BD10]. Unfortunately, it requires σ to be superpolynomially larger than e_{ct}.

One could use smaller σ anyway, and while $\mathcal{N}_{0,\sigma}$ and $\mathcal{N}_{e_{ct},\sigma}$ have non-negligible statistical distance, they have small Rényi divergence. Under the right conditions, this means an adversary's advantage in a security game must remain small if real decryption shares are replaced with simulated ones. This is the approach taken by [CSS$^+$22, BS23]. Unfortunately, this is insufficient for simulation-based security, because the output of the simulator can still be distinguished from real decryption shares.

The main insight of this paper is that the real and simulated distributions just need to be *computationally* indistinguishable, not statistically indistinguishable, and for computationally bounded adversaries, the ciphertext looks uniform and e_{ct} is unknown. $\mathcal{N}_{0,\sigma}$ and $\mathcal{N}_{e_{ct},\sigma}$ being distinguishable is not necessarily a problem. To simulate $\langle \mathbf{a}, \mathbf{s} \rangle + \tilde{e}$, we need to know how $e_{ct} - \tilde{e}$ should be distributed. But rather than viewing e_{ct} as fixed (so that the distribution is $\mathcal{N}_{e_{ct},\sigma}$) we take the distribution over the uncertainty of e_{ct} as well as $\tilde{e}$.

For example, if $e_{ct} \leftarrow \mathcal{N}_t$ and $\tilde{e} \leftarrow \mathcal{N}_\sigma$, then the difference between $\langle \mathbf{a}, \mathbf{s} \rangle + e_{ct}$ and $\langle \mathbf{a}, \mathbf{s} \rangle + \tilde{e}$ is distributed as $\mathcal{N}_{\sqrt{\sigma^2+t^2}}$. As long as the ciphertext $(\mathbf{a}, b)$ is computationally uniform, we can sample $e' \leftarrow \mathcal{N}_{\sqrt{\sigma^2+t^2}}$ and simulate partial decryption as $b + e'$.

It's not immediately obvious that $\langle \mathbf{a}, \mathbf{s} \rangle + e_{ct}$ will still look uniform if $\langle \mathbf{a}, \mathbf{s} \rangle + \tilde{e}$ has been revealed — $\mathbf{a}$ and $\mathbf{s}$ have both been reused, and LWE might no longer be hard in the presence of this sort of leakage. We show in Theorem 1 that when e_{ct} and $\tilde{e}$ are both Gaussians, this "Reused-A LWE" problem is as hard as standard LWE with slightly smaller noise, which may be of independent interest. This allows us to use smudging noise that is polynomially large, and in fact potentially smaller than the ciphertext noise!

One technical detail is that to properly simulate, we need to know the variance of the ciphertext noise — we can't sample from $\mathcal{N}_{\sigma^2+t^2}$ without knowing t. In Section 5 we slightly modify the schemes of [Reg09] and [LP11] to produce ciphertexts with known noise variance.

1.2 Related Work

A number of works build Threshold PKE from lattices. In [BD10], Bendlin and Damgård build Threshold PKE that is UC-secure, but the modulus is exponentially large. Singh, Rangan, and Banerjee build Threshold PKE with polynomial modulus in [SRB13], but they achieve only a weak form of semantic security.

Other works use lattices to build threshold versions of stronger primitives like IBE. Bendlin, Krehbiel, and Peikert in [BKP13] build IBE with threshold key generation / extraction / delegation; their construction uses polynomial modulus and they prove the security of threshold key generation / extraction / delegation in the UC framework. However, that work does not consider threshold decryption. In [KM16], Kuchta and Markowitch build IBE with threshold decryption, but under a weaker security model that does not let the adversary see partial decryption shares. In [DDE$^+$23] Dahl et al. build a simulation-secure Threshold FHE (and thus also PKE) scheme that uses a relatively small modulus during evaluation, but during partial decryption they switch to an exponentially large modulus and then bootstrap. Chowdhury et al [CSS+22] and Boudgoust and Scholl [BS23] both build Threshold FHE with polynomial modulus, proving security under slightly different game-based security definitions; both proofs use Rényi divergence arguments that are sufficient for game-based security, but insufficient for simulation security. Furthermore, because of the Rényi divergence technique, both [CSS+22] and [BS23] need a bound ℓ on the number of decryption queries to be known in advance, and the modulus scales with $\sqrt{\ell}$; in our scheme the modulus need not grow with the number of decryption queries.

The security notion we consider is CPA-like and assumes static corruptions. In [DLN+21], Devevey et al. build lattice-based threshold PKE that achieves CCA2 security against adaptive corruptions. However, their construction uses noise flooding with superpolynomial modulus-to-noise ratio. Combining our techniques with those of [DLN+21] to achieve CCA2 security with polynomially large modulus would be an interesting direction for future work.

One last technique for noise-flooding with polynomial modulus is the "gentle noise flooding", first introduced for the analysis of entropic LWE [BD20b, BD20a], and used in [dCHI+22] to achieve (non-threshold) homomorphic encryption with circuit privacy. Here the goal is to avoid leakage from the plaintext, rather than the key, and the technique does not seem applicable to achieve threshold decryption.

Threshold decryption and key generation can also be performed using general MPC techniques [KLO+19], without any noise flooding, and keeping the same LWE (polynomial) encryption modulus. It may be possible to adapt the multi-key FHE construction from MrNISC (reusable, non-interactive MPC) with polynomial modulus of [BJKL21, Shi22]. However, these methods are based on general MPC techniques and unlikely to be practical.

To the best of our knowledge, ignoring schemes based on generic MPC, ours is the first LWE-based Threshold PKE scheme with polynomially large modulus that achieves simulation security.

1.3 Outline

Section 2 recalls some results from previous works and proves some technical lemmas that will be used in our proofs.

In Section 3 we prove that LWE in the presence of certain kinds of leakage is as hard as standard LWE, which may be of independent interest.

In Section 4 we build a simulation-secure Threshold PKE scheme with polynomial modulus from any LWE-based PKE scheme satisfying certain properties. Informally, we require ciphertexts in the underlying scheme must look like fresh LWE samples (plus the encoded message), even when the secret key is known. Furthermore, the error distribution must be a (continuous) Gaussian whose standard deviation is publicly known.

In Section 5 we give two PKE schemes satisfying these conditions: a minor modification to Regev's PKE scheme from [Reg09], and a minor modification to Lindner and Peikert's scheme from [LP11].

2 Preliminaries

2.1 Gaussians

Definition 1 (Continuous Gaussians)**.** *The (one-dimensional) Gaussian measure $\rho_{c,s}$ with center c and width*[3] *s is defined as*

$$\rho_{c,s}(x) = e^{-\pi(x-c)^2/s^2}.$$

More generally in n dimensions,

$$\rho_{\mathbf{c},s}(\mathbf{x}) = e^{-\pi\|\mathbf{x}-\mathbf{c}\|^2/s^2} = \prod_{i=1}^{n} \rho_{c_i,s}(x_i).$$

[3] This is not the standard deviation; a Gaussian of width s has standard deviation $s/\sqrt{2\pi}$.

The n-dimensional spherical Gaussian distribution $\mathcal{N}^n_{\mathbf{c},s}$ is the distribution over $\mathbb{R}^n$ whose probability density is proportional to $\rho_{\mathbf{c},s}$. Equivalently,

$$\mathcal{N}^n_{\mathbf{c},s}(\mathbf{x}) = \frac{1}{s^n} \prod_{i=1}^n \rho_{c_i,s}(x_i).$$

For brevity, we will usually write $\mathcal{N}^n_s$ instead of $\mathcal{N}^n_{0,s}$ (and likewise ρ_s instead of $\rho_{0,s}$) for Gaussians centered at zero.

Definition 2 (Discrete Gaussian)**.** *The discrete Gaussian distribution over an n-dimensional lattice Λ, written $\mathcal{D}_{\Lambda,\mathbf{c},s}$, is the distribution over Λ whose probability density is proportional to $\rho_{\mathbf{c},s}$. Equivalently,*

$$\mathcal{D}_{\Lambda,\mathbf{c},s}(\mathbf{x}) = \frac{\rho_{\mathbf{c},s}(\mathbf{x})}{\rho_{\mathbf{c},s}(\Lambda)}.$$

Wide-enough discrete Gaussians behave in many ways like continuous Gaussians. To quantify how wide is wide enough, we use a lattice parameter known as the *smoothing parameter*. For a full definition of the smoothing parameter we refer the reader to [MR07]; for our purposes, the following fact will be sufficient:

Lemma 1 ([MR07, Lemma 3.3])**.** *The smoothing parameter of $\mathbb{Z}^n$, written $\eta_\epsilon(\mathbb{Z}^n)$, is $\tilde{O}(1)$ for some negligible ϵ. More precisely,*

$$\eta_\epsilon(\mathbb{Z}^n) \le \sqrt{\frac{\ln(2n(1+1/\epsilon))}{\pi}}$$

Shifting the center $\mathbf{c}$ of a discrete Gaussian $\mathcal{D}_{\mathbb{Z}^n,\mathbf{c},s}$ leaves the normalization factor $\rho_{\mathbf{c},s}(\mathbb{Z}^n)$ almost unchanged, assuming the width s is above the smoothing parameter:

Lemma 2 ([Reg09, Claim 3.8], special case where lattice is $\mathbb{Z}^n$)**.** *For any $\mathbf{c} \in \mathbb{R}^n$, $\epsilon > 0$, and $r \ge \eta_\epsilon(\mathbb{Z}^n)$,*

$$r^n(1-\epsilon) \le \rho_{\mathbf{c},r}(\mathbb{Z}^n) \le r^n(1+\epsilon).$$

The convolution of a wide-enough discrete Gaussian and a wide-enough continuous Gaussian is close to a continuous Gaussan:

Lemma 3 ([Reg09, Claim 3.9], special case where lattice is $\mathbb{Z}^n$)**.** *Let $r, s > 0$ be two reals, and let t denote $\sqrt{r^2+s^2}$. Assume that $rs/t = 1/\sqrt{1/r^2+1/s^2} \ge \eta_\epsilon(\mathbb{Z}^n) = \tilde{O}(1)$ for some $\epsilon < \frac{1}{2}$. Consider the continuous distribution Y on $\mathbb{R}^n$ obtained by sampling from $\mathcal{D}_{\mathbb{Z}^n,0,r}$ and then adding a noise vector taken from $\mathcal{N}^n_{0,s}$. Then, the statistical distance between Y and $\mathcal{N}^n_{0,t}$ is at most 4ϵ.*

We will in fact need a stronger result: suppose we have a discrete Gaussian vector $\mathbf{x}$ and add some continuous Gaussian noise $\mathbf{e}$. Assuming both Gaussians are wide enough, not only is $\mathbf{x}+\mathbf{e}$ close to a continuous Gaussian, but also if we condition on the value of $\mathbf{x}+\mathbf{e}$, then the conditional distribution of $\mathbf{x}$ will still be close to a discrete Gaussian:

Lemma 4. *Let $s, t \ge \sqrt{2}\eta_\epsilon(\mathbb{Z}^n)$ for some negligible ϵ. Then the following distributions are statistically close:*

$$D_1 = \{\mathbf{x} \leftarrow \mathcal{D}_{\mathbb{Z}^n,0,s};\ \mathbf{e} \leftarrow \mathcal{N}^n_t\ :\ (\mathbf{x}, \mathbf{x}+\mathbf{e})\}$$

and

$$D_2 = \{\mathbf{y} \leftarrow \mathcal{N}^n_\alpha;\ \mathbf{x} \leftarrow \mathcal{D}_{\mathbb{Z}^n,\beta\mathbf{y},\gamma}\ :\ (\mathbf{x}, \mathbf{y})\}$$

where $\alpha = \sqrt{s^2+t^2}$, $\beta = \frac{s^2}{s^2+t^2}$ and $\gamma = \frac{st}{\sqrt{s^2+t^2}}$.

Proof. We compute the probability density functions for D_1 and D_2 and show that they are negligibly far apart.

First we compute the density function f for D_1:

$$\begin{aligned}
f_{X,Y}(\mathbf{x},\mathbf{y}) &= f_X(\mathbf{x}) f_{Y|X=\mathbf{x}}(\mathbf{y}) \\
&= \frac{\rho_s(\mathbf{x})}{\rho_s(\mathbb{Z}^n)} \cdot \frac{\rho_t(\mathbf{y}-\mathbf{x})}{t^n} \\
&= \frac{1}{t^n \rho_s(\mathbb{Z}^n)} \exp\left(-\pi\left(\frac{\|\mathbf{x}\|^2}{s^2} + \frac{\|\mathbf{y}-\mathbf{x}\|}{t^2}\right)\right) \\
&= \frac{1}{t^n \rho_s(\mathbb{Z}^n)} \exp\left(-\pi\left(\left(\frac{1}{t^2}+\frac{1}{s^2}\right)\|\mathbf{x}\|^2 - \frac{2\langle \mathbf{x},\mathbf{y}\rangle}{t^2} + \frac{\|\mathbf{y}\|^2}{t^2}\right)\right)
\end{aligned}$$

Now we compute the density function g for D_2:

$$\begin{aligned}
g_{X,Y}(\mathbf{x},\mathbf{y}) &= g_Y(\mathbf{y}) g_{X|Y=\mathbf{y}}(\mathbf{x}) \\
&= \frac{\rho_\alpha(\mathbf{y})}{\alpha^n} \frac{\rho_\gamma(\mathbf{x}-\beta\mathbf{y})}{\rho_\gamma(\mathbb{Z}^n-\beta\mathbf{y})} \\
&= \frac{1}{\alpha^n \rho_{\beta\mathbf{y},\gamma}(\mathbb{Z}^n)} \exp\left(-\pi\left(\frac{\|\mathbf{x}-\beta\mathbf{y}\|^2}{\gamma^2} + \frac{\|\mathbf{y}\|^2}{\alpha^2}\right)\right) \\
&= \frac{1}{\alpha^n \rho_{\beta\mathbf{y},\gamma}(\mathbb{Z}^n)} \exp\left(-\pi\left(\frac{\|\mathbf{x}\|^2}{\gamma^2} - \frac{2\beta}{\gamma^2}\langle \mathbf{x},\mathbf{y}\rangle + \left(\frac{1}{\alpha^2}+\frac{\beta^2}{\gamma^2}\right)\|\mathbf{y}\|^2\right)\right) \\
&= \frac{1}{\alpha^n \rho_{\beta\mathbf{y},\gamma}(\mathbb{Z}^n)} \exp\left(-\pi\left(\frac{\|\mathbf{x}\|^2}{\frac{s^2t^2}{s^2+t^2}} - \frac{2\frac{s^2}{s^2+t^2}}{\frac{s^2t^2}{s^2+t^2}}\langle \mathbf{x},\mathbf{y}\rangle + \left(\frac{1}{s^2+t^2}+\frac{\frac{s^4}{(s^2+t^2)^2}}{\frac{s^2t^2}{s^2+t^2}}\right)\|\mathbf{y}\|^2\right)\right) \\
&= \frac{1}{\alpha^n \rho_{\beta\mathbf{y},\gamma}(\mathbb{Z}^n)} \exp\left(-\pi\left(\frac{\|\mathbf{x}\|^2}{\frac{s^2t^2}{s^2+t^2}} - \frac{2}{t^2}\langle \mathbf{x},\mathbf{y}\rangle + \left(\frac{1}{s^2+t^2}+\frac{s^2}{t^2(s^2+t^2)}\right)\|\mathbf{y}\|^2\right)\right) \\
&= \frac{1}{\alpha^n \rho_{\beta\mathbf{y},\gamma}(\mathbb{Z}^n)} \exp\left(-\pi\left(\left(\frac{1}{t^2}+\frac{1}{s^2}\right)\|\mathbf{x}\|^2 - \frac{2\langle \mathbf{x},\mathbf{y}\rangle}{t^2} + \frac{\|\mathbf{y}\|^2}{t^2}\right)\right)
\end{aligned}$$

We consider the ratio of f and g, which we want to show is close to 1:

$$\frac{f_{X,Y}(x,y)}{g_{X,Y}(x,y)} = \frac{\alpha^n \rho_{\beta\mathbf{y},\gamma}(\mathbb{Z}^n)}{t^n \rho_s(\mathbb{Z}^n)}$$

$s > \eta_\epsilon(\mathbb{Z}^n)$, and $\gamma = \frac{st}{\sqrt{s^2+t^2}} \geq \eta_\epsilon(\mathbb{Z}^n)$, so both Gaussians are wider than the smoothing parameter. Applying Lemma 2,

$$\frac{\alpha^n\gamma^n(1-\epsilon)}{t^n s^n(1+\epsilon)} \leq \frac{\alpha^n \rho_{\beta\mathbf{y},\gamma}(\mathbb{Z}^n)}{t^n \rho_s(\mathbb{Z}^n)} \leq \frac{\alpha^n\gamma^n(1+\epsilon)}{t^n s^n(1-\epsilon)}$$

and since $\alpha\gamma = st$,

$$\frac{1-\epsilon}{1+\epsilon} \leq \frac{f_{X,Y}(x,y)}{g_{X,Y}(x,y)} \leq \frac{1+\epsilon}{1-\epsilon}.$$

Since ϵ is negligible, this implies D_1 and D_2 are statistically close. □

Letting $s = t$ immediately gives the following corollary:

Corollary 1. *Let $s \geq \sqrt{2}\eta_\epsilon(\mathbb{Z}^n)$. Then the distribution*

$$\{\mathbf{r} \leftarrow \mathcal{D}_{\mathbb{Z}^n,0,s};\ \mathbf{e} \leftarrow \mathcal{N}_s^n\ :\ (\mathbf{r}, \mathbf{r}+\mathbf{e})\}$$

is statistically close to

$$\{\mathbf{v} \leftarrow \mathcal{N}_{0,\sqrt{2}s}^n;\ \mathbf{u} \leftarrow \mathcal{D}_{\mathbb{Z}^n,\mathbf{v}/2,s/\sqrt{2}}\ :\ (\mathbf{u}, \mathbf{v})\}$$

We have the following tail bound on the norm of a discrete Gaussian vector:

Lemma 5 ([Ban93, Lemma 1.5])**.** *Let $L \subset \mathbb{R}^n$ be any lattice, let $s > 0$, and let $c \geq 1/\sqrt{2\pi}$. Then* $\Pr_{\mathbf{x} \leftarrow \mathcal{D}_{L,0,s}}[\|\mathbf{x}\| > cs\sqrt{n}] < (c\sqrt{2\pi e}e^{-\pi c^2})^n$

Plugging in $c = 0.8$ gives the following corollary:

Corollary 2. *If $\mathbf{x} \leftarrow \mathcal{D}_{\mathbb{Z}^n,0,s}$ then $\|\mathbf{x}\| < 0.8s\sqrt{n}$ with probability at least $1 - 2^{-n}$.*

2.2 Min-Entropy and the Leftover Hash Lemma

Definition 3 (Min-entropy)**.** *The min-entropy of a discrete distribution P is defined as*

$$H_\infty(P) = \log \min_{x \in \mathrm{Supp}(P)} \frac{1}{P(x)}$$

The following lemma shows that the min-entropy of a discrete Gaussian distribution over a lattice is minimized when the Gaussian is centered around the origin. Notice that this statement is different from the well known fact (usually proved using Poisson summation formula) that the Gaussian sum $\rho_{\mathbf{c},s}(L)$ is maximized when $\mathbf{c} = 0$ because both the numerator and denominator of $\mathcal{D}_{L,\mathbf{c},s}(\mathbf{x}) = \rho_{\mathbf{c},s}(\mathbf{x})/\rho_{\mathbf{c},s}(L)$ depend on $\mathbf{c}$.

Lemma 6. *The min-entropy of a discrete Gaussian over a lattice L is lowest when centered at the origin. That is, for all $s > 0$ and $\mathbf{c} \in \mathbb{R}^n$, we have*

$$H_\infty(\mathcal{D}_{L,\mathbf{c},s}) \geq H_\infty(\mathcal{D}_{L,0,s}).$$

For simplicity, we prove the statement only for $L = \mathbb{Z}^n$, as this is all that we need in this paper.

Proof. Since $\mathbb{Z}^n$ has an orthogonal basis, we have $\mathcal{D}_{\mathbb{Z}^n,\mathbf{c},s}(\mathbf{x}) = \prod_{i=1}^n \mathcal{D}_{\mathbb{Z},c_i,s}(x_i)$, and it suffices to prove the lemma for $n = 1$. Without loss of generality assume $c \in [-\frac{1}{2}, \frac{1}{2}]$, so that $\mathcal{D}_{\mathbb{Z},c,s}(x)$ is maximized at $x = 0$, and the min-entropy $H_\infty(\mathcal{D}_{\mathbb{Z},c,s})$ is $\log(1/\mathcal{D}_{\mathbb{Z},c,s}(0))$. We wish to show that (for fixed s) $1/\mathcal{D}_{\mathbb{Z},c,s}(0)$ is minimized at $c = 0$.

$$\begin{aligned}\frac{1}{\mathcal{D}_{\mathbb{Z},c,s}(0)} = \frac{\rho_{c,s}(\mathbb{Z})}{\rho_{c,s}(0)} &= \frac{\sum_{x \in \mathbb{Z}} e^{-\pi(x-c)^2/s^2}}{e^{-\pi c^2/s^2}} \\ &= \sum_{x \in \mathbb{Z}} e^{(-\pi/s^2)((x-c)^2 - c^2)} \\ &= \sum_{x \in \mathbb{Z}} e^{(-\pi/s^2)(x^2 - 2xc)}\end{aligned}$$

Merging the summands for $+x$ and $-x$,

$$\begin{aligned}&= 1 + \sum_{x=1}^{\infty} (e^{(-\pi/s^2)(x^2-2xc)} + e^{(-\pi/s^2)(x^2+2xc)}) \\ &= 1 + \sum_{x=1}^{\infty} e^{(-\pi/s^2)x^2}\left(e^{2\pi xc/s^2} + e^{-2\pi xc/s^2}\right) \\ &= 1 + \sum_{x=1}^{\infty} e^{(-\pi/s^2)x^2}\left(y + \frac{1}{y}\right)\end{aligned}$$

where $y = e^{2\pi xc/s^2}$. By symmetry/convexity, this quantity is minimized when $y = 1$, i.e., $c = 0$. Thus $1/\mathcal{D}_{\mathbb{Z},c,s}(0)$ is minimized at $c = 0$, and so is the min-entropy of $\mathcal{D}_{\mathbb{Z},c,s}$. □

The proof can be generalized to arbitrary lattices as follows: without loss of generality, rotate the lattice so that the optimal $\mathbf{c}$ is on the line through the origin and $(1, 0, \ldots, 0)$, so that we can treat c as a scalar. Then take the derivative of $1/\mathcal{D}_{L,c,s}(0)$ with respect to c to show that it is minimized at $c = 0$. We omit the details since $L = \mathbb{Z}^n$ suffices for this paper.

Lemma 7. *Let $s \geq 3$, and let $\mathbf{c} \in \mathbb{R}^n$ be any vector. Then $\mathcal{D}_{\mathbb{Z}^n,\mathbf{c},s}$ has at least n bits of min-entropy.*

Proof. By Lemma 6 we can assume $\mathbf{c} = 0$. In one dimension we can compute numerically that $\rho_{0,3}(0)/\rho_{0,3}(\mathbb{Z}) < 1/2$. So in n dimensions we have $\mathcal{D}_{\mathbb{Z}^n,\mathbf{0},3}(0) < 1/2^n$, and so $\mathcal{D}_{\mathbb{Z}^n,\mathbf{0},3}$ has more than n bits of min-entropy. □

We will use the following formulation of the Leftover Hash Lemma:

Lemma 8 ([BD20a, Lemma 2.1])**.** *Let q be prime and let m, n be integers. Let $\mathbf{r}$ be a random variable defined on $\mathbb{Z}_q^m$ and let $\mathbf{A} \xleftarrow{\$} \mathbb{Z}_q^{m \times n}$ be chosen uniformly at random. Then*

$$\Delta((\mathbf{A}, \mathbf{r}^T\mathbf{A}), (\mathbf{A}, \mathrm{Unif}(\mathbb{Z}_q^n))) \leq \sqrt{q^n \cdot 2^{-H_\infty(\mathbf{r})}}$$

2.3 Simulation Security

We will use the following security notion for a T-out-of-T threshold PKE scheme. We assume an honest-but-curious adversary that knows all but one keyshare, and can ask the honest party to partially decrypt honestly-generated encryptions of adversarially chosen messages. Informally, simulation security means the adversary can by themself simulate the honest party's partial decryptions, without the help of the honest party or any access to the honest party's secret keyshare, as long as the adversary knows the underlying plaintext. This implies that the partial decryptions reveal nothing more than the underlying plaintexts and cannot help the adversary break the scheme.

More formally,

Definition 4. *A threshold PKE scheme is* simulation secure *if*

- *the scheme without decryption is IND-CPA secure, and*
- *there is an efficient algorithm* Sim *such that, with overwhelming probability over* $(\mathsf{pk}, \mathsf{sk}^{hon}, \{\mathsf{sk}^{mal}\}) \leftarrow \mathsf{KeyGen}$*, and for all (possibly adaptively chosen) sequences of messages* $\{m_i\}$*,*

  $$\big\{\mathsf{ct}_i \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_i)\,\forall i \quad : \quad (\mathsf{pk}, \mathsf{sk}^{mal}, \{m_i, \mathsf{ct}_i, \mathsf{Dec}_{\mathsf{sk}^{hon}}(\mathsf{ct}_i)\})\big\}$$

  is computationally indistinguishable from

  $$\big\{\mathsf{ct}_i \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_i)\,\forall i \quad : \quad (\mathsf{pk}, \mathsf{sk}^{mal}, \{m_i, \mathsf{ct}_i, \mathsf{Sim}_{\mathsf{pk},\mathsf{sk}^{mal}}(\mathsf{ct}_i, m_i)\})\big\}.$$

We remark that in general one would also let the adversary call Dec on the same ciphertext more than once, but in our constructions Dec will be deterministic, so this definition is equivalent.

3 LWE Variants

For security we will rely on a few variants of the usual LWE assumption. All are provably as hard as the standard LWE assumption.

3.1 Reused-A LWE

Definition 5. *The Reused-A LWE distribution with parameters n, m, σ_1, σ_2 is defined as*

$$\left\{\mathbf{A} \xleftarrow{\$} \mathbb{Z}_q^{m\times n};\ \mathbf{s} \xleftarrow{\$} \mathbb{Z}_q^n;\ \mathbf{e_1} \leftarrow \mathcal{N}_{\sigma_1}^m;\ \mathbf{e_2} \leftarrow \mathcal{N}_{\sigma_2}^m\ :\ (\mathbf{A}, \mathbf{As}+\mathbf{e_1}, \mathbf{As}+\mathbf{e_2})\right\}.$$

The Search Reused-A LWE problem is to recover $\mathbf{s}$.

The Decision Reused-A LWE problem is to distinguish the distribution from

$$\left\{\mathbf{A} \xleftarrow{\$} \mathbb{Z}_q^{m\times n};\ \mathbf{b}' \xleftarrow{\$} \mathbb{R}_q^m;\ \mathbf{c} \leftarrow \mathcal{N}_{\sqrt{\sigma_1^2+\sigma_2^2}}^m\ :\ (\mathbf{A}, \mathbf{b}', \mathbf{b}'+\mathbf{c})\right\}.$$

For notational convenience we present this in "matrix form" where $\mathbf{A}$ is a matrix of m vectors $\mathbf{a_i}$. We can also consider the case where m is polynomially large but not known in advance – i.e., the adversary can make polynomially-many queries to an oracle that outputs samples $(\mathbf{a}, \langle\mathbf{a},\mathbf{s}\rangle + e_1, \langle\mathbf{a},\mathbf{s}\rangle + e_2)$ for fixed unknown $\mathbf{s}$. All of the below reductions for Reused-**A** LWE are sample-preserving and so extend to the unknown-m case as well.

Theorem 1. *For all $\mathbf{A} \in \mathbb{Z}_q^{m\times n}$ and $\mathbf{s} \in \mathbb{Z}_q^n$, the following two distributions are identical:*

$$\left\{\mathbf{e_1} \leftarrow \mathcal{N}_{\sigma_1}^m;\ \mathbf{e_2} \leftarrow \mathcal{N}_{\sigma_2}^m\ :\ (\mathbf{A}, \mathbf{As}+\mathbf{e_1}, \mathbf{As}+\mathbf{e_2})\right\}$$

and

$$\left\{\mathbf{e} \leftarrow \mathcal{N}_{\sigma_b}^m;\ \mathbf{e}' \leftarrow \mathcal{N}_1^m;\ \mathbf{b} \leftarrow \mathbf{As}+\mathbf{e}\ :\ (\mathbf{A}, \mathbf{b}+\sigma_3\mathbf{e}', \mathbf{b}-\sigma_4\mathbf{e}')\right\}$$

where

$$\sigma_b = \frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}} = \frac{1}{\sqrt{\sigma_1^{-2}+\sigma_2^{-2}}}$$
$$\sigma_3 = \frac{\sigma_1^2}{\sqrt{\sigma_1^2+\sigma_2^2}}$$
$$\sigma_4 = \frac{\sigma_2^2}{\sqrt{\sigma_1^2+\sigma_2^2}}$$

Proof. It suffices to show that the joint distribution D_1 of (e_1, e_2) is identical to the joint distribution D_2 of $(e+\sigma_3 e', e-\sigma_4 e')$. Since these are all linear transformations of continuous Gaussians, the distributions are completely specified by their covariance matrices.

The covariance matrix of D_1 is

$$\begin{pmatrix}\sigma_1^2 & \\ & \sigma_2^2\end{pmatrix}.$$

For D_2, we start with (e, e') which has covariance matrix

$$\begin{pmatrix}\sigma_b^2 & \\ & 1\end{pmatrix}$$

and then apply the transformation

$$T = \begin{pmatrix}1 & \sigma_3 \\ 1 & -\sigma_4\end{pmatrix}.$$

The resulting covariance matrix of D_2 is

$$T\begin{pmatrix}\sigma_b^2 & \\ & 1\end{pmatrix}T^\top = \begin{pmatrix}\sigma_b^2+\sigma_3^2 & \sigma_b^2-\sigma_3\sigma_4 \\ \sigma_b^2-\sigma_3\sigma_4 & \sigma_b^2+\sigma_4^2\end{pmatrix} = \begin{pmatrix}\sigma_1^2 & 0 \\ 0 & \sigma_2^2\end{pmatrix}$$

which is the same as the covariance matrix of D_1. So the two distributions are identical.

□

Corollary 3. *Search Reused-**A** LWE is as hard as search LWE with Gaussian noise of width* $\frac{1}{\sqrt{\sigma_1^{-2}+\sigma_2^{-2}}}$*. Furthermore, if decision LWE is hard with Gaussian noise of width* $\frac{1}{\sqrt{\sigma_1^{-2}+\sigma_2^{-2}}}$*, then decision Reused-**A** LWE is also hard.*

Proof. To reduce LWE to Reused-A LWE, we take an LWE instance $(\mathbf{A}, \mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e})$ where $\mathbf{e}$ is continuous Gaussian with width $\sigma_b = \frac{1}{\sqrt{\sigma_1^{-2}+\sigma_2^{-2}}}$. We sample $\mathbf{e}' \leftarrow \mathcal{N}_1^m$ ourselves and compute $(\mathbf{A}, \mathbf{b} + \sigma_3\mathbf{e}', \mathbf{b} - \sigma_4\mathbf{e}')$, with σ_3 and σ_4 defined as in Theorem 1. By the theorem, this is distributed exactly as a Reused-**A** LWE sample with noise parameters σ_1 and σ_2, with the same secret as our input LWE instance. This gives a reduction from search LWE to search Reused-**A** LWE.

For the decision version, by Theorem 1 the Reused-A LWE distribution is identical to $(\mathbf{A}, \mathbf{b} + \sigma_3\mathbf{e}', \mathbf{b} - \sigma_4\mathbf{e}')$, where $\mathbf{e}' \leftarrow \mathcal{N}_1^m$ and $(\mathbf{A}, \mathbf{b})$ are an LWE instance with noise width $\sigma_b = \frac{1}{\sqrt{\sigma_1^{-2}+\sigma_2^{-2}}}$. But $(\mathbf{A}, \mathbf{b})$ look uniform by decision LWE. Letting $\mathbf{b}' = \mathbf{b} + \sigma_3\mathbf{e}'$ (which also looks uniform), our distribution is $(\mathbf{A}, \mathbf{b}', \mathbf{b}' - (\sigma_3 + \sigma_4)\mathbf{e}')$, and $\sigma_3 + \sigma_4 = \sqrt{\sigma_1^2 + \sigma_2^2}$, completing the proof. □

As an aside, we remark that this proof does not go through if the noise is non-Gaussian. In fact, for polynomially large bounded uniform noise, there is an explicit attack!

Claim 1. *There is a poly-time adversary that solves Search Reused-**A** LWE if errors, instead of Gaussian, are uniform in the interval* $[-B, B]$ *for polynomially large* B*, given* $m = \Omega(B^2 n)$ *samples.*

Proof. For each sample $(\mathbf{a}, b_1 = \langle \mathbf{a}, \mathbf{s} \rangle + e_1, b_2 = \langle \mathbf{a}, \mathbf{s} \rangle + e_2)$, check whether $b_2 - b_1 = 2B + 1$. If so, it must be the case that $e_2 = B$ and $e_1 = -B$, and we learn the value of $\langle \mathbf{a}, \mathbf{s} \rangle$. This will happen with probability $\frac{1}{(2B+1)^2} = O(1/B^2)$ — non-negligible because B is only polynomially large. With $\Omega(B^2 n)$ samples we expect to recover the exact value of $\langle \mathbf{a}, \mathbf{s} \rangle$ for n different values of $\mathbf{a}$, from which we can recover $\mathbf{s}$. □

3.2 Known-Norm LWE

Definition 6 (LWE with known norm)**.** *Given integers* n, m, q*, with* m *and* q *polynomially large in* n*, and an error distribution* χ *whose support is in* $\mathbb{Z}_q$*, the Known-Norm LWE Distribution is defined as*

$$\left\{\mathbf{s} \leftarrow \chi^n;\ \mathbf{e} \leftarrow \chi^m;\ \mathbf{A} \leftarrow \mathbb{Z}_q^{m \times n}\ :\ (\mathbf{A}, \mathbf{A}\mathbf{s} + \mathbf{e}, \|\mathbf{s}\|^2 + \|\mathbf{e}\|^2)\right\}.$$

In other words, Known-Norm LWE is small-secret LWE except the adversary is also given the ℓ_2 *norm of the vector* $[\mathbf{s}|\mathbf{e}] \in \mathbb{Z}^{n+m}$.

The decisional Known-Norm LWE assumption is that this distribution is computationally indistinguishable from

$$\left\{\mathbf{s} \leftarrow \chi^n;\ \mathbf{e} \leftarrow \chi^m;\ \mathbf{A} \leftarrow \mathbb{Z}_q^{m \times n}\ :\ (\mathbf{A}, \mathrm{Unif}, \|\mathbf{s}\|^2 + \|\mathbf{e}\|^2)\right\}.$$

Lemma 9. *The decisional Known-Norm LWE problem is as hard as decisional small-secret LWE with the same parameters, up to a polynomial factor in the advantage.*

Proof. $\|\mathbf{s}\|^2 + \|\mathbf{e}\|^2$ is no more than $q^2(m + n)$, which is polynomially large. Given a small-secret LWE instance, an adversary that solves the Known-Norm LWE problem can just guess the value of $\|\mathbf{s}\|^2 + \|\mathbf{e}\|^2$, and will guess correctly with at least $1/poly(n)$ probability. □

The above argument is extremely loose. Concretely, when χ is a discrete Gaussian of small width, $\|\mathbf{s}\|^2 + \|\mathbf{e}\|^2$ will with overwheming probability be much smaller than $q^2(m + n)$, allowing better concrete parameters for a given security level.

We can also define a version of Known-Norm LWE where the secret $\mathbf{s}$ is uniform instead of from the error distribution, and the norm given is $\|\mathbf{e}\|^2$ instead of $\|\mathbf{s}\|^2 + \|\mathbf{e}\|^2$. The same proof technique shows that this version is as hard as standard LWE (again up to a polynomial factor in the advantage).

3.3 Fixed-Matrix Shifted LWE

Definition 7 (Fixed-Matrix Shifted LWE)**.** *Let $n \in \mathbb{Z}$; $q \in poly(n)$; $\gamma \in \mathbb{R}$; and let Ψ be an arbitrary distribution over $\mathbb{R}^n$. The Fixed-Matrix Shifted LWE problem is as follows: Fix a public random matrix $\mathbf{A} \xleftarrow{\$} \mathbb{Z}_q^{n\times n}$. Given $\mathbf{A}$, and given polynomially many samples all from either*

$$D_{real} = \{\mathbf{c} \leftarrow \Psi;\ \mathbf{d} \leftarrow \Psi;\ \mathbf{r} \leftarrow \mathcal{D}_{\mathbb{Z}^n,\mathbf{c},\gamma};\ \mathbf{f} \leftarrow \mathcal{D}_{\mathbb{Z}^n,\mathbf{d},\gamma};\ :\ (\mathbf{rA}+\mathbf{f},\mathbf{c},\mathbf{d})\}$$

or

$$D_{random} = \left\{\mathbf{c} \leftarrow \Psi;\ \mathbf{d} \leftarrow \Psi;\ \mathbf{a} \xleftarrow{\$} \mathbb{Z}_q^n;\ :\ (\mathbf{a},\mathbf{c},\mathbf{d})\right\},$$

determine whether the samples were from D_{real} or D_{random}.

We remark that if the $\mathbf{r}$ and $\mathbf{f}$ vectors came from zero-centered discrete Gaussians, this would be Matrix LWE: the problem of distinguishing $(\mathbf{A}, \mathbf{SA}+\mathbf{E})$ from uniform. Matrix LWE is known to be as hard as standard LWE by a hybrid argument. [LP11] If the shifts $\mathbf{c}$ and $\mathbf{d}$ were integer vectors, we could add or subtract $\mathbf{cA}+\mathbf{d}$ ourselves to shift the centers. The only subtlety is dealing with the fractional parts of the $\mathbf{c}$ and $\mathbf{d}$, which we can do by adding small noise with the appropriate mean.

Lemma 10. *If Matrix LWE is hard in dimension n with the secret and noise taken from discrete Gaussians of parameter σ, then Fixed-Matrix Shifted LWE is hard with noise parameter $\sqrt{\sigma^2 + \eta_\epsilon(\mathbb{Z}^n)^2}$.*

Proof. We are given a Matrix LWE instance $(\mathbf{A}, \mathbf{B} = \mathbf{AS}+\mathbf{E}) \in \mathbb{Z}_q^{n\times n} \times \mathbb{Z}_q^{n\times m}$, where columns of $\mathbf{S}$ and $\mathbf{E}$ come from $\mathcal{D}_{\mathbb{Z}^n,0,\sigma}$. Sample m pairs of vectors $(\mathbf{c_i}, \mathbf{d_i})$ from Ψ. Then, for $i \in [m]$, sample $\bar{\mathbf{s}}_\mathbf{i} \leftarrow \mathcal{D}_{\mathbb{Z}^n,\mathbf{c_i},\eta_\epsilon(\mathbb{Z}^n)}$ and $\bar{\mathbf{e}}_\mathbf{i} \leftarrow \mathcal{D}_{\mathbb{Z}^n,\mathbf{d_i},\eta_\epsilon(\mathbb{Z}^n)}$.

Concatenate the $\bar{\mathbf{s}}_\mathbf{i}$ into an $n \times m$ matrix $\bar{\mathbf{S}}$ and the $\bar{\mathbf{e}}_\mathbf{i}$ into $\bar{\mathbf{E}}$. Now

$$(\mathbf{A}, \mathbf{B}+\mathbf{A}\bar{\mathbf{S}}+\bar{\mathbf{E}}, \{(\mathbf{c_i},\mathbf{d_i})\}) = (\mathbf{A}, \mathbf{A}(\mathbf{S}+\bar{\mathbf{S}})+(\mathbf{E}+\bar{\mathbf{E}}), \{(\mathbf{c_i},\mathbf{d_i})\})$$

which is a Fixed-Matrix Shifted LWE instance with $\gamma = \sqrt{\sigma^2 + \eta_\epsilon(\mathbb{Z}^n)^2}$. If $\mathbf{B}$ is instead uniform, then the result of this transformation is still uniform. Thus Fixed-Matrix Shifted LWE with noise parameter $\gamma = \sqrt{\sigma^2 + \eta_\epsilon(\mathbb{Z}^n)^2}$ is as hard as Matrix LWE with noise parameter σ. □

4 Threshold PKE from PKE

We now show how to transform any PKE scheme satisfying certain properties into a Threshold PKE scheme where the amount of "smudging noise" can be extremely small and the modulus can be polynomially large.

Let PKE be an IND-CPA secure public key encryption scheme with the following properties:

- The secret key is a vector $s \in \mathbb{Z}_q^n$.
- Even when the secret key (and public key) are known, the output of $\mathsf{Enc}(msg)$[4] is indistinguishable from
  $$\{\mathbf{a} \xleftarrow{\$} \mathbb{Z}_q^n;\ e_{ct} \leftarrow \mathcal{N}_{\sigma_{ct}}\ :\ (\mathbf{a}, \langle \mathbf{a}, \mathbf{s}\rangle + e_{ct} + msg)\}$$
  where σ_{ct} is publicly known. In other words, each ciphertext looks like the message plus a fresh LWE sample with continuous Gaussian noise of width σ_{ct}.
- $\sigma_{ct} \geq \sqrt{2}\sigma_{dLWE}$. That is, ciphertext noise has at least twice the variance needed for the Decisional LWE assumption to hold.
- Decryption should be possible for noise slightly wider than σ_{ct}; in particular, to build T-out-of-T threshold PKE, we will have noise width $\sqrt{\sigma_{ct}^2 + 2T\sigma_{dLWE}^2}$. (We remark that not only is this polynomially large, but it is also concretely quite small.)

[4]Here msg is already scaled or otherwise encoded; we ignore the details of the encoding.

We will show in Section 5 that, with slight modifications, both standard Regev PKE ([Reg09]) and the compact variant of Lindner and Peikert ([LP11]) satisfy these conditions, with polynomially large modulus.

Then we can implement (T, T)-threshold PKE as follows:

Key generation: s and pk are generated as in the underlying PKE scheme. Each party's secret key is an additive share of $\mathbf{s}$: $\mathbf{s} = \sum_i \mathbf{s_i}$.

Encryption: Same as the underlying PKE scheme.

Partial Decryption: To decrypt $(\mathbf{a}, b)$ using keyshare $\mathbf{s_i}$: Sample $\tilde{e} \leftarrow \mathcal{N}_{\sigma_{sm}}$, where $\sigma_{sm} = \sqrt{2}\sigma_{dLWE}$. Output $\langle \mathbf{a}, \mathbf{s_i} \rangle + \tilde{e}$. If given the same input $\mathbf{a}$ multiple times, give the same output every time (e.g., by keeping a table of previous inputs and outputs, or by keeping some secret PRF key k and using $\mathrm{PRF}_k(\mathbf{a})$ as a PRG seed for sampling $\tilde{e}$.)

Reconstruction: Add the partial decryptions $\{\langle \mathbf{a}, \mathbf{s_i} \rangle + \tilde{e}_i\}_{i \in [T]}$ to recover $\langle \mathbf{a}, \mathbf{s} \rangle + \sum_i \tilde{e}_i$. Subtract this from the b component of the ciphertext to recover

$$b - \langle \mathbf{a}, \mathbf{s} \rangle - \sum_i \tilde{e}_i = msg + e_{ct} - \sum_i \tilde{e}_i$$

and then error-correct to recover the underlying message.

Theorem 2. *Assuming the underlying PKE scheme satisfies the above conditions, the above Threshold PKE construction satisfies simulation security.*

Proof. The PKE scheme is already IND-CPA secure; we must show that partial decryption queries can be simulated without knowing the full secret key, and that the adversary's view with a real Dec oracle is computationally indistinguishable from the view with the simulated Dec oracle. Let $\mathbf{s}^{hon}$ be the honest party's secret keyshare, and let $\mathbf{s}^{mal}$ be the sum of the adversary's keyshares, such that $\mathbf{s}^{hon} + \mathbf{s}^{mal} = \mathbf{s}$.

The adversary's view in the real world is

$$\begin{aligned} &\left\{\mathbf{s}^{mal}, \mathsf{pk}, \{\mathsf{msg}_i, \mathsf{ctxt}_i, \mathsf{Dec}(\mathsf{ctxt}_i)\}_i\right\} \\ =&\left\{\mathbf{s}^{mal}, \mathsf{pk}, \{\mathsf{msg}_i, \mathbf{a_i}, \langle \mathbf{a_i}, \mathbf{s} \rangle + e_i + \mathsf{msg}_i, \langle \mathbf{a_i}, \mathbf{s}^{hon} \rangle + \tilde{e}_i\}_i\right\} \end{aligned}$$

Subtracting the known values $\{\mathsf{msg}_i + \langle \mathbf{a_i}, \mathbf{s}^{mal} \rangle\}$ from each ciphertext (but not from the partial decryptions), we can equivalently write the view as

$$= \left\{\mathbf{s}^{mal}, \mathsf{pk}, \{\mathsf{msg}_i, \mathbf{a_i}, \langle \mathbf{a_i}, \mathbf{s}^{hon} \rangle + e_i, \langle \mathbf{a_i}, \mathbf{s}^{hon} \rangle + \tilde{e}_i\}_i\right\}$$

We remark that without loss of generality we can assume each ciphertext is unique and is decrypted exactly once, since any future calls to Dec on the same ciphertext give the same output.

The $\{\mathbf{a_i}\}$ are indistinguishable from i.i.d. uniform, and the $\{e_i\}$ are indistinguishable from i.i.d. samples from $\mathcal{N}_{\sigma_{ct}}$. (This is true even in the presence of pk and the partial decryptions: by assumption it would be true even if $\mathbf{s}$ were known, which would be enough for the adversary to compute Dec on their own.) The $\{\tilde{e}_i\}$ are distributed as $\mathcal{N}_{\sigma_{sm}}$.

So $\{\mathbf{a_i}, \langle \mathbf{a_i}, \mathbf{s}^{hon} \rangle + e_i, \langle \mathbf{a_i}, \mathbf{s}^{hon} \rangle + \tilde{e}_i\}_i$ follows the Reused-A LWE distribution, with $\sigma_1 = \sigma_{ct}$ and $\sigma_2 = \sigma_{sm}$. We apply Corollary 3: both σ_{ct} and σ_{sm} are at least $\sqrt{2}\sigma_{dLWE}$, so $\frac{1}{\sigma_{sm}^{-2} + \sigma_{ct}^{-2}} \geq \sigma_{dLWE}^2$, large enough that decision LWE is hard. The conditions for Corollary 3 are thus satisfied, and distribution of this part of the view is computationally indistinguishable from

$$\{\mathbf{a} \xleftarrow{\$} \mathbb{Z}_q^n;\ b \xleftarrow{\$} \mathbb{R}_q^m;\ c \leftarrow \mathcal{N}_{\sqrt{\sigma_{ct}^2 + \sigma_{sm}^2}}\ :\ (\mathbf{a}, b, b + c)\}.$$

In particular, conditioned on b (and the rest of the view), the output of Dec is b plus a Gaussian of known width.

Recalling that we earlier subtracted $\langle \mathbf{a}, \mathbf{s}^{mal} \rangle + \mathsf{msg}$ from the b component of each ciphertext, we can simulate decryption to a given message as follows:

$$\mathsf{Sim}_{\mathsf{pk},\mathsf{sk}^{mal}}((\mathbf{a}, b), \mathsf{msg}) = b - \langle \mathbf{a}, \mathbf{s}^{mal} \rangle - \mathsf{msg} + e_{sim}$$

where $e_{sim} \leftarrow \mathcal{N}_{\sqrt{\sigma_{ct}^2 + \sigma_{sm}^2}}$. As shown, this is computationally indistinguishable from the real output of Dec when the underlying message is msg. Thus the scheme is simulation secure. □

We make a few remarks:

- The smudging noise has width $\sigma_{sm} = \sqrt{2}\sigma_{dLWE}$ — only $\sqrt{2}$ times wider than the smallest possible noise under which LWE can be hard. In contrast to previous schemes where smuding noise was superpolynomially larger than ciphertext noise, here the smudging noise is *smaller* than the ciphertext noise. σ_{sm} being so small is especially helpful for supporting a large number of parties, because when partial decryptions from all T parties are added together during reconstruction, the width of the noise will be $\sqrt{\sigma_{ct}^2 + T\sigma_{sm}^2}$.

- The number of decryption queries need not be known in advance, as Reused-A LWE remains hard with arbitrary polynomially many samples.

- The proof requires that σ_{ct} (or more precisely, $\sigma_{ct}^2 + \sigma_{sm}^2$) be a publicly known value, as otherwise the simulator won't know how much noise to add. A natural question is whether a bound on σ_{ct} is sufficient, rather than the actual value, as long as the noise is still Gaussian. This would be a useful property for building threshold homomorphic encryption: the precise noise variance after some operations (like bootstrapping) depends on secret information, but it can be publicly bounded.

  Unfortunately, this is not a proof artifact: simulation security does require the actual value. In the real world, given a ciphertext $(a, b = \langle a, s \rangle + e)$ and partial decryption $c = \langle a, s^{hon} \rangle + \tilde{e}$, the adversary can compute $b - \langle a, s^{mal} \rangle - c = e - \tilde{e}$, which is distributed as a Gaussian with width $\sqrt{\sigma_{ct}^2 + \sigma_{sm}^2}$. The adversary could repeat this for many ciphertexts and measure the variance of the resulting distribution to learn $\sigma_{ct}^2 + \sigma_{sm}^2$.

- The proof of Theorem 2 easily adapts to the Ring LWE setting, where the ciphertext noise should now be a spherical Gaussian of known variance.

- If Dec is called on the same ciphertext twice, it must always give the same output. Otherwise, an adversary could ask for many partial decryptions of the same ciphertext to gather a large number of samples $\{\langle \mathbf{a}, \mathbf{s} \rangle + \tilde{e}_i\}$ and average them, effectively reducing the width of the smudging noise below the $\sqrt{2}\sigma_{dLWE}$ needed for security.

- This Threshold PKE construction is linearly homomorphic (assuming the underlying PKE scheme is) but is not secure as-is as a Threshold Linearly Homomorphic Encryption scheme. In particular, if we allow homomorphic evaluation, the $\mathbf{a}$ vectors in each ciphertext may no longer be independent. For example, $\mathsf{Dec}(ct_1) + \mathsf{Dec}(ct_2) \approx \mathsf{Dec}(ct_1 + ct_2)$ but with different smudging noise. This effectively makes Dec no longer deterministic: the adversary can see many samples $\{\langle \mathbf{a}, \mathbf{s} \rangle + \tilde{e}_i\}$ with different $\tilde{e}_i$ and average them.

5 Schemes Satisfying the Conditions of Section 4

5.1 Regev-like PKE

The following PKE scheme, similar to the scheme of [Reg09], satisfies the properties needed for Theorem 2, and so can be transformed into a simulation secure threshold PKE scheme. Like in the original scheme, here the $\mathbf{a}$ vector of each ciphertext is statistically close to uniform by the Leftover Hash Lemma. We add some additional noise to the b component of the ciphertext to make the ciphertext noise distribution a continuous Gaussian. We also reveal the ℓ_2 norm of the error vector of the public key, so that the width of the ciphertext noise can be publicly known.

Parameters: Let λ be the security parameter. Let q be a polynomially large prime. Let n and σ_{dLWE} be such that LWE in dimension n with Gaussian noise of width σ_{dLWE} is secure. (We remark that for provable security σ_{dLWE} can be $O(\sqrt{n})$ [Reg09] but in practice $\sigma_{dLWE} \sim O(1)$ is often used.[ACC+18]) Let $\sigma_{pk} = \sigma_{dLWE}$, and $\sigma_r = \sigma_e = \sqrt{2} \cdot \max(3, \eta_\epsilon(\mathbb{Z}^n)) = \tilde{O}(1)$. Let $m = n \log q + \lambda$.

KeyGen: Sample $\mathbf{s} \xleftarrow{\$} \mathbb{Z}_q^n, \mathbf{A} \xleftarrow{\$} \mathbb{Z}_q^{m \times n}, \mathbf{e_{pk}} \leftarrow \mathcal{D}^m_{\mathbb{Z}^m,0,\sigma_{pk}}$.

The public key is $(\mathbf{A}, \mathbf{b_{pk}} = \mathbf{As} + \mathbf{e_{pk}}, \|\mathbf{e_{pk}}\|)$.

$\mathsf{Enc}_{\mathsf{pk}}(msg)$: Sample $\mathbf{r} \leftarrow \mathcal{D}_{\mathbb{Z}^m,0,\sigma_r}, e' \leftarrow \mathcal{N}_{\sigma_e \cdot \|e_{pk}\|}$. Output $(\mathbf{r}^\top \mathbf{A}, \mathbf{r}^\top \mathbf{b_{pk}} + e' + msg)$

Theorem 3. *For the above scheme, conditioned on the secret key* $\mathbf{s}$ *and the public key* $(\mathbf{A}, \mathbf{b}, c)$*, the output of* $\mathsf{Enc}(msg)$ *is indistinguishable from*

$$\{\mathbf{a} \xleftarrow{\$} \mathbb{Z}_q^n;\ e_{ct} \leftarrow \mathcal{N}_{\sigma_{ct}};\ (\mathbf{a}, \langle \mathbf{a}, \mathbf{s} \rangle + e_{ct} + msg)\}$$

where $\sigma_{ct} = \sqrt{2}\sigma_e \|\mathbf{e_{pk}}\| > \sqrt{2}\sigma_{dLWE}$*. Furthermore, for any polynomially large* T*, polynomially large* q *is sufficient to allow correct decryption of ciphertexts with noise of width* $\sqrt{\sigma_{ct}^2 + 2T\sigma_{dLWE}^2}$*.*

Proof. We first remark that revealing $\|\mathbf{e_{pk}}\|$ can increase the adversary's advantage by no more than a polynomial factor (this is Known-Norm LWE with uniform secrets).

We will show that the output distribution of Enc is statistically close to (the message plus) a fresh LWE sample with continuous Gaussian noise of width $\sigma_e \sqrt{2}\|\mathbf{e_{pk}}\|$:

$$\begin{aligned}
\{\mathsf{Enc}(msg)\} &= \{\mathbf{r} \leftarrow \mathcal{D}_{\mathbb{Z}^m,0,\sigma_r};\ e' \leftarrow \mathcal{N}_{\sigma_e \cdot \|e_{pk}\|}\ :\ (\mathbf{r}^T A, \mathbf{r}^T(\mathbf{As} + \mathbf{e_{pk}}) + e' + msg)\} \\
&= \{\mathbf{r} \leftarrow \mathcal{D}_{\mathbb{Z}^m,0,\sigma_r};\ \mathbf{e} \leftarrow \mathcal{N}^m_{0,\sigma_e}\ :\ (\mathbf{r}^T A, \mathbf{r}^T \mathbf{As} + \langle \mathbf{r}, \mathbf{e_{pk}} \rangle + \langle \mathbf{e}, \mathbf{e_{pk}} \rangle + msg)\} \\
&= \{\mathbf{r} \leftarrow \mathcal{D}_{\mathbb{Z}^m,0,\sigma_r};\ \mathbf{e} \leftarrow \mathcal{N}^m_{0,\sigma_e}\ :\ (\mathbf{r}^T \mathbf{A}, \mathbf{r}^T \mathbf{As} + \langle \mathbf{r} + \mathbf{e}, \mathbf{e_{pk}} \rangle + msg)\}
\end{aligned}$$

We apply Corollary 1:

$$\approx_s \{\mathbf{t} \leftarrow \mathcal{N}^m_{0,\sqrt{2}\sigma_e};\ \mathbf{r}' \leftarrow \mathcal{D}_{\mathbb{Z}^m,\mathbf{t}/2,\sigma_e/\sqrt{2}}\ :\ (\mathbf{r}'^T \mathbf{A}, \mathbf{r}'^T \mathbf{As} + \langle \mathbf{t}, \mathbf{e_{pk}} \rangle + msg)\}$$

By Lemma 7, $\mathbf{r}'$ has at least m bits of min-entropy; by the Leftover Hash Lemma (Lemma 8) the statistical distance between $(\mathbf{A}, \mathbf{r}'^T \mathbf{A})$ and $(\mathbf{A}, \mathrm{Unif}(\mathbb{Z}_q^n))$ is no more than $2^{-\lambda}$.

$$\begin{aligned}
\{\mathsf{Enc}(msg)\} &\approx_s \{\mathbf{a}' \xleftarrow{\$} \mathbb{Z}_q^n;\ \mathbf{t} \leftarrow \mathcal{N}^m_{0,\sqrt{2}\sigma_e}\ :\ (\mathbf{a}', \langle \mathbf{a}', \mathbf{s} \rangle + \langle \mathbf{t}, \mathbf{e_{pk}} \rangle + msg)\} \\
&= \{\mathbf{a}' \xleftarrow{\$} \mathbb{Z}_q^n;\ \tilde{e} \leftarrow \mathcal{N}_{0,\sqrt{2}\sigma_e \cdot \|\mathbf{e_{pk}}\|}\ :\ (\mathbf{a}', \langle \mathbf{a}', \mathbf{s} \rangle + \tilde{e} + msg)\}.
\end{aligned}$$

So the output of encryption is statistically close to an LWE sample (plus the message) where the noise is a continuous Gaussian with width $\sigma_{ct} = \sigma_e \cdot \sqrt{2}\|\mathbf{e_{pk}}\|$. Since $\sigma_e > \sigma_{dLWE}$, we have $\sigma_{ct} \geq \sqrt{2}\sigma_{dLWE}$ as desired.

Now all we have left to show is that polynomially large q is sufficient to allow decryption of ciphertexts with noise of width $\sqrt{\sigma_{ct}^2 + 2T\sigma_{dLWE}^2}$ (for some given polynomially large T). Since $\mathbf{e}_{\mathsf{pk}}$ comes from $\mathcal{D}_{\mathbb{Z}^m,0,\sigma_{dLWE}}$, by Corollary 2 we have that $\|\mathbf{e_{pk}}\| < 0.8\sqrt{m}\sigma_{dLWE}$ with overwhelming probability. Now $\sigma_{ct} = \sigma_e \sqrt{2}\|\mathbf{e_{pk}}\| \in \tilde{O}(\sqrt{n}\sigma_{dLWE})$. Since σ_{dLWE}, n, and T are all polynomially large, so is $\sigma_{ct}^2 + 2T\sigma_{dLWE}^2$, and a polynomially large modulus suffices to allow decryption to succeed with overwhelming probability.

□

5.2 Scheme based on LP11

We now present a PKE scheme (a slight modification of the scheme of [LP11]) that satisfies the conditions needed for the Theorem 2 construction.

Parameters Let n, σ_{dLWE}, and q be chosen such that small-secret LWE in dimension n with modulus q and discrete Gaussian noise of parameter σ_{dLWE} is secure. (In theory, σ_{dLWE} is $O(\sqrt{n})$; in practice it is $O(1)$.) Let $\sigma_{pk} = \sigma_{dLWE}$, and $\sigma_r = \sigma_e = 2\max(\eta_\epsilon(\mathbb{Z}^{2n}), \sigma_{dLWE})$.

Key generation Sample a square matrix $\mathbf{A} \xleftarrow{\$} \mathbb{Z}_q^{n\times n}$; sample vectors $\mathbf{s}$ and $\mathbf{e_{pk}}$ from $\mathcal{D}_{\mathbb{Z}^n,0,\sigma_{pk}}$. The secret key is $\mathbf{s} \in \mathbb{Z}_q^n$, and the public key is $(\mathbf{A}, \mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e_{pk}}, c = \|\mathbf{s}\|^2 + \|\mathbf{e_{pk}}\|^2) \in \mathbb{Z}_q^{n\times n} \times \mathbb{Z}_q^n \times \mathbb{Z}$.

Encryption On encoded input message msg and public key $(\mathbf{A}, \mathbf{b}, c)$, Enc first samples vectors $\mathbf{r}$ and $\mathbf{f}$ from $\mathcal{D}_{\mathbb{Z}^n,0,\sigma_r}$ and a (continuous) scalar $e' \leftarrow \mathcal{N}_{\sigma_e\sqrt{c}}$. The ciphertext is $(\mathbf{r}^\top\mathbf{A} + \mathbf{f}, \mathbf{r}^\top\mathbf{b} + e' + msg)$.

Theorem 4. *For the above scheme, conditioned on the secret key* $\mathbf{s}$ *and the public key* $(\mathbf{A}, \mathbf{b}, c)$*, the output of* $\mathsf{Enc}(msg)$ *is indistinguishable from*

$$\{\mathbf{a} \xleftarrow{\$} \mathbb{Z}_q^n;\ e_{ct} \leftarrow \mathcal{N}_{\sigma_{ct}};\ (\mathbf{a}, \langle\mathbf{a}, \mathbf{s}\rangle + e_{ct} + msg)\}$$

where $\sigma_{ct} = \sqrt{c(\sigma_r^2 + \sigma_e^2)}$*. Moreover,* σ_{ct} *is (with overwhelming probability) less than* $3.2\sqrt{n}\sigma_{dLWE}\max(\eta_\epsilon(\mathbb{Z}^{2n}), \sigma_{dLWE})$*, which is polynomially large.*

Proof. We can rewrite the output of Enc as

$$(\mathbf{r}^\top\mathbf{A} + \mathbf{f}, \mathbf{r}^\top\mathbf{b} + e' + msg) = (\mathbf{r}^\top\mathbf{A} + \mathbf{f}, (\mathbf{r}^\top\mathbf{A} + \mathbf{f})\mathbf{s} - \langle\mathbf{f}, \mathbf{s}\rangle + \langle\mathbf{r}, \mathbf{e_{pk}}\rangle + e' + msg).$$

We can't yet say that $\mathbf{r}^\top\mathbf{A} + \mathbf{f}$ is uniform by decisional LWE, as $\mathbf{r}$ and $\mathbf{f}$ appear elsewhere.

The vectors $\mathbf{s}$ and $\mathbf{e_{pk}}$ are fixed; $\mathbf{r}$ and $\mathbf{f}$ each come from $\mathcal{D}_{\mathbb{Z}^n,0,\sigma_r}$; e' comes from $\mathcal{N}_{\sigma_e\sqrt{c}}$. We can view the concatenation of $\mathbf{r}$ and $\mathbf{f}$ as a single vector from $\mathcal{D}_{\mathbb{Z}^{2n},0,\sigma_r}$, and the concatenation of $\mathbf{e_{pk}}$ and $\mathbf{s}$ as a vector of norm $\sqrt{\|\mathbf{s}\|^2 + \|\mathbf{e_{pk}}\|^2} = \sqrt{c}$. Our ciphertext distribution is:

$$\left\{[\mathbf{r}|\mathbf{f}] \leftarrow \mathcal{D}_{\mathbb{Z}^{2n},0,\sigma_r};\ e' \leftarrow \mathcal{N}_{\sigma_e\sqrt{c}}\ :\ \Big(\mathbf{r}\mathbf{A} + \mathbf{f}, (\mathbf{r}\mathbf{A} + \mathbf{f})\mathbf{s} + \Big\langle[\mathbf{r}|\mathbf{f}],\ [\mathbf{e_{pk}} \mid -\mathbf{s}]\Big\rangle + e' + msg\Big)\right\}$$

$$= \left\{[\mathbf{r}|\mathbf{f}] \leftarrow \mathcal{D}_{\mathbb{Z}^{2n},0,\sigma_r};\ \mathbf{e} \leftarrow \mathcal{N}_{\sigma_e}^{2n}\ :\ \Big(\mathbf{r}\mathbf{A} + \mathbf{f}, (\mathbf{r}\mathbf{A} + \mathbf{f})\mathbf{s} + \Big\langle[\mathbf{r}|\mathbf{f}] + \mathbf{e},\ [\mathbf{e_{pk}} \mid -\mathbf{s}]\Big\rangle + msg\Big)\right\}$$

We now apply Lemma 4 to the joint distribution of $([\mathbf{r}|\mathbf{f}], [\mathbf{r}|\mathbf{f}] + \mathbf{e})$, where the values of s and t in the lemma are σ_r and σ_e, respectively. (Both σ_r and σ_e are more than $\sqrt{2}\eta_\epsilon(\mathbb{Z}^{2n})$ so the lemma applies.)

$$\approx_s \left\{\mathbf{y} \leftarrow \mathcal{N}_\alpha^{2n};\ [\mathbf{r}|\mathbf{f}] \leftarrow \mathcal{D}_{\mathbb{Z}^{2n},\beta\mathbf{y},\gamma}\ :\ \Big(\mathbf{r}\mathbf{A} + \mathbf{f}, (\mathbf{r}\mathbf{A} + \mathbf{f})\mathbf{s} + \Big\langle\mathbf{y},\ [\mathbf{e_{pk}} \mid -\mathbf{s}]\Big\rangle + msg\Big)\right\}$$

where

$$\alpha = \sqrt{\sigma_r^2 + \sigma_e^2} \qquad \beta = \frac{\sigma_r^2}{\sigma_r^2 + \sigma_e^2} \qquad \gamma = \frac{\sigma_r\sigma_e}{\sqrt{\sigma_r^2 + \sigma_e^2}}.$$

Now that $\mathbf{r}$ and $\mathbf{f}$ are used only in the expression $\mathbf{r}\mathbf{A}+\mathbf{f}$, we can argue by decisional LWE that $\mathbf{r}\mathbf{A}+\mathbf{f}$ is computationally uniform even given $\mathbf{A}$. In particular, this is Fixed-Matrix Shifted LWE, and we apply Lemma 10. (In particular, for our chosen parameters, we have $\gamma = (\sigma_r^{-2} + \sigma_e^{-2})^{-1/2} = \sqrt{2}\max(\eta_\epsilon(\mathbb{Z}^{2n}), \sigma_{dLWE})$, which is larger than the $\sqrt{\sigma_{dLWE}^2 + \eta_\epsilon(\mathbb{Z}^n)}$ necessary for Fixed-Matrix Shifted LWE to be hard.)

$$\approx_c \left\{\mathbf{y} \leftarrow \mathcal{N}_\alpha^{2n};\ \mathbf{a} \xleftarrow{\$} \mathbb{Z}_q^n\ :\ \Big(\mathbf{a}, \langle\mathbf{a}, \mathbf{s}\rangle + \Big\langle\mathbf{y},\ [\mathbf{e_{pk}} \mid -\mathbf{s}]\Big\rangle + msg\Big)\right\}$$

Finally, now that $\mathbf{y}$ appears nowhere else, we can replace $\langle \mathbf{y}, [\mathbf{e_{pk}}|\mathbf{s}]\rangle$ with a one-dimensional continuous Gaussian:

$$= \left\{ e_{ct} \leftarrow \mathcal{N}_{\alpha\sqrt{c}};\ \mathbf{a} \xleftarrow{\$} \mathbb{Z}_q^n\ :\ (\mathbf{a}, \langle \mathbf{a}, \mathbf{s}\rangle + e_{ct} + msg) \right\}$$

In particular, even conditioned on the secret (and public) keys, the output of $\mathsf{Enc}(msg)$ is a fresh LWE sample (plus the message) where the ciphertext noise comes from a continuous Gaussian of width $\alpha\sqrt{c} = \sqrt{c(\sigma_r^2 + \sigma_e^2)} = \sigma_{ct}$.

To complete the proof, we show that for our choice of parameters, σ_{ct} is polynomially large with overwhelming probability, allowing polynomially large q. In particular, using the extremely loose bound of Lemma 2, we have $\sqrt{c} < 0.8\sqrt{2n}\sigma_{dLWE}$ with overwhelming probability, so

$$\begin{aligned} \sigma_{ct} &< 0.8\sqrt{2n}\sigma_{dLWE}(2\sqrt{2}\max(\eta_\epsilon(\mathbb{Z}^{2n}), \sigma_{dLWE})) \\ &= 3.2\sqrt{n}\sigma_{dLWE}\max(\eta_\epsilon(\mathbb{Z}^{2n}), \sigma_{dLWE}) \end{aligned}$$

which is polynomially large. □

5.3 RLWE-based PKE

We now sketch a ring-based PKE scheme satisfying the properties needed in Theorem 2. Its security depends on the (conjectured but unproven) hardness of a ring analog of the Known-Norm LWE problem that we call the Known-Covariance RLWE problem.

5.3.1 Ring Background

We use the following facts about power-of-2 cyclotomics. Let n be a power of 2, and let $\mathcal{R} = \frac{\mathbb{Z}[x]}{(x^n+1)}$.

- A ring element is a degree $< n$ polynomial a, which can be viewed as an n-dimensional coefficient vector.
- Multiplication by a ring element a is a linear operation, which can be described as multiplication by an n-by-n negacyclic matrix whose top row is the coefficients of a. In a slight abuse of notation, we identify a ring element with its corresponding matrix.
- If A is the matrix for multiplication by $a(x)$, then its transpose $A^\top$ corresponds to multiplication by the ring element $a(x^{-1})$. We write $\overline{a(x)} = a(x^{-1})$.
- Embed $\mathcal{R}$ into $\mathbb{R}^n$. For any ring element a, multiplying a spherical Gaussian by a gives a (non-spherical) Gaussian with covariance matrix $a\overline{a}$. That is, the distribution of $\{e \leftarrow \mathcal{N}_1^n\ :\ ae\}$ is a continuous Gaussian whose covariance matrix is (the matrix corresponding to) $a\overline{a}$.

5.3.2 Known-Covariance RLWE

Definition 8. *Let $\mathcal{R} = \frac{\mathbb{Z}[x]}{(x^n+1)}$ be a power-of-2 cyclotomic ring. Let χ be an error distribution whose support is in $\mathcal{R}$. Let q be a (polynomially large) modulus. Let m be polynomially large.*

The Known-Covariance RLWE Distribution is defined as

$$\left\{ s \leftarrow \chi;\ \mathbf{e} \leftarrow \chi^m;\ \mathbf{a} \xleftarrow{\$} \mathcal{R}^m\ :\ (\mathbf{a}, s\mathbf{a} + \mathbf{e}, s\overline{s} + \sum_{i=1}^m e_i\overline{e_i}) \right\}.$$

The decisional Known-Covariance RLWE assumption is that this distribution is computationally indistinguishable from

$$\left\{ s \leftarrow \chi;\ \mathbf{e} \leftarrow \chi^m;\ \mathbf{a} \xleftarrow{\$} \mathcal{R}^m\ :\ (\mathbf{a}, Unif, s\overline{s} + \sum_{i=1}^m e_i\overline{e_i}) \right\}.$$

In Known-Norm LWE we revealed $\|\mathbf{s}\|^2 + \|\mathbf{e}\|^2$, which gives the variance of $\langle[\mathbf{s}|\mathbf{e}], \mathbf{z}\rangle$ when $\mathbf{z} \leftarrow \mathcal{N}_1^{n+m}$.

Analogously in the ring setting, $s\overline{s} + \sum_i e_i\overline{e_i}$ gives the covariance matrix of $\langle[s|\mathbf{e}], \mathbf{z}\rangle$ when $\mathbf{z} \leftarrow (\mathcal{N}_1^n)^{1+m}$. So this is a natural generalization of Known-Norm LWE to the ring setting.

In the plain LWE case Lemma 9 shows Known-Norm LWE is as hard as standard LWE. Unfortunately, that proof does not carry over to the ring setting. It seems plausible that the problem remains hard (for appropriate values of m), but proving or disproving the Known-Covariance RLWE assumption is left open for future work.

5.3.3 Construction

We now sketch our RLWE-based PKE scheme. At a high level, it is similar to the LP11-like scheme from Section 5.2. But directly mapping that construction to the ring setting would give non-spherical ciphertext noise – its covariance matrix will depend on the secret key. So we will add extra noise whose covariance matrix is chosen to cancel this out and make the result spherical, and reveal in the public key the information necessary to compute this covariance matrix. Assuming Known-Covariance RLWE is hard, the scheme remains secure with this leakage.

Let m be such that Known-Covariance RLWE is (conjectured) hard. Let the distribution χ be a discrete Gaussian on $\mathcal{R}$ with width σ_r. Let α be a fixed public parameter.

The secret key is a short ring element s. The public key contains a vector of m RLWE samples $(\mathbf{a}, \mathbf{b} = s\mathbf{a} + \mathbf{e})$. (We remark that $m = 1$ is sufficient if we assume Known-Covariance RLWE is hard when $m = 1$.) The public key also includes a ring element F and an integer λ computed as follows: Let $E = s\overline{s} + \sum_i e_i\overline{e_i}$. Let $\lambda \geq (\sigma_r^2 + \alpha)\lambda_1$, where λ_1 is the largest eigenvalue of E (viewing E as a matrix). Let $F = \lambda\mathbf{I}_n - \sigma_r^2 E$. The public key is $(\mathbf{a}, \mathbf{b}, F, \lambda)$.

Observe that F has the same eigenvectors as E, and all its eigenvalues are non-negative (so F is positive semi-definite). Moreover, observe that the sum of a Gaussian with covariance $\sigma_r^2 E$ and an independent Gaussian with covariance F will be spherical, having covariance $\lambda\mathbf{I}_n$.

Like in the rest of the paper, for ease of presentation ciphertexts will use continuous rather than discrete Gaussian noise.[5]

Encryption is as follows: sample $\mathbf{r} \leftarrow \chi^m$ and $r_0 \leftarrow \chi$. Sample f from an n-dimensional non-spherical Gaussian with covariance matrix F.[6] Output ciphertext $(a' = \langle\mathbf{r}, \mathbf{a}\rangle - r_0, b' = \langle\mathbf{r}, \mathbf{b}\rangle + f + msg)$.

Claim 2. *In the above scheme, conditioned on the secret key, the distribution of a fresh ciphertext is indistinguishable from $(a', a's + e' + msg)$ where a' is a uniform $\mathcal{R}$ element and e' is an (independent) spherical Gaussian of variance λ.*

Proof sketch:

$$\begin{aligned}
b' &= \langle\mathbf{r}, \mathbf{b}\rangle + f + msg \\
&= s\langle\mathbf{r}, \mathbf{a}\rangle + \langle\mathbf{r}, \mathbf{e}\rangle + f + msg \\
&= sa' + sr_0 + \langle\mathbf{r}, \mathbf{e}\rangle + f + msg \\
e' &= b' - a's - msg = \langle\mathbf{r}, \mathbf{e}\rangle + sr_0 + f \\
&= \langle[\mathbf{r}|r_0],\ [\mathbf{e}|s]\rangle + f
\end{aligned}$$

Since $\langle[\mathbf{r}|r_0], [\mathbf{e}|s]\rangle$ has covariance matrix $\sigma_r^2 E$, and f is independent with covariance matrix F, the resulting ciphertext noise e' has covariance matrix $\sigma_r^2 E + F = \lambda I_n$, so is a spherical Gaussian of variance λ (all over the randomness of $\mathbf{r}$, r_0, and f).

On its own, $a' = \langle\mathbf{r}, \mathbf{a}\rangle - r_0$ would be computationally uniform even given $\mathbf{a}$ under Decisional RLWE.[7] However, this is insufficient, because we need to show that a' and e' are (computationally) independent. So we need to analyze the joint distribution (a', e').

[5] The public key still uses discrete noise, like in the rest of the paper, and it does not need to be Gaussian.

[6] For example, sample from $\mathcal{N}_\sigma^n$ and multiply by the matrix square root $\sqrt{\mathbf{F}}$.

[7] If $m > 1$ we could instead assume Module LWE, but Ring LWE suffices.

We can write $e = e_1 + e_2$ where e_1 has covariance αE and e_2 has covariance $F - \alpha E$.

$$(a', e') = (\langle \mathbf{r}, \mathbf{a} \rangle - r_0, \langle [r_0|\mathbf{r}],\ [s|\mathbf{e}] \rangle + e_1 + e_2)$$

e_1 has covariance αE, and $\langle \mathbf{r}, \mathbf{e} \rangle$ has covariance $\sigma_r^2 E$; we can combine:

$$= (\langle \mathbf{r}, \mathbf{a} \rangle - r_0, \langle [r_0|\mathbf{r}] + \mathbf{y},\ \mathbf{e} \rangle + e_2)$$

where $\mathbf{y}$ is a spherical Gaussian with variance α.

The rest is similar to the proof of Theorem 4 for the plain case. We apply Lemma 4 to argue that the joint distribution of $([\mathbf{r}|r_0], [\mathbf{r}|r_0] + \mathbf{y})$ is the same as $(\mathbf{r}', \mathbf{y}')$ where $\mathbf{y}'$ is a zero-centered Gaussian and $\mathbf{r}'$ is a discrete Gaussian whose center depends on $\mathbf{y}'$. As long as α and σ_r aren't too small, the distribution of $\mathbf{r}'$ is wide enough to argue that a' is computationally uniform under a ring version of Fixed-Matrix Shifted LWE (Lemma 10), which essentially just says that RLWE remains hard if the secrets and error come from discrete Gaussians that aren't centered at zero. This makes a' computationally independent of e', completing the proof. $\square$

6 Example concrete parameters

We now present some example concrete parameters for the threshold version of the LP11-like scheme from Section 5.2, showing that simulation-secure Threshold PKE is possible with parameters roughly similar to those of FrodoKEM[NAB+20].

Frodo-640 targets NIST Level 1, "matching or exceeding the brute-force security of AES-128." It uses LWE secret dimension $n = 640$ and ciphertext modulus $q = 32768$. The plaintext modulus is 4 — Frodo encodes a 128-bit message as an 8-by-8 matrix of 2-bit values.

For our comparison we will use plaintext modulus 4 as well. Our construction from Section 5.2 encrypts individual scalars rather than 8-by-8 matrices; adapting the construction to matrices we believe should be straightforward but we leave it for future work. Our example parameters are not highly optimized and are for rough comparison only.

First we set n, q, and σ_{dLWE}. For the 128-bit security level, we set $n = 640$ (to match Frodo-640) and $q = 65536$ (twice the modulus of Frodo-640) and $\sigma_{dLWE} = 5$ (giving standard deviation ≈ 1.99, somewhat smaller than Frodo-640's standard deviation of 2.8, although Frodo's noise is not exactly Gaussian). According to Albrecht, Player, and Scott's Lattice Estimator[APS15][8], the estimated complexity of attacking small-secret LWE with these parameters is roughly $2^{141.4}$ operations. Applying Lemma 5 with $c = 0.509$, we get that $\sqrt{\|\mathbf{s}\|^2 + \|\mathbf{e_{pk}}\|^2}$ is less than 91.053 with probability at least $1 - 2^{-128}$. In particular, if we assume we lose $2\lg(91.053) \approx 13.02$ bits of security in the reduction from Known-Norm LWE to LWE, then the complexity of attacking the public key is still at least $2^{128.3}$. So $\sigma_{dLWE} = 5$ is sufficient for the 128-bit security level.

By Lemma 1, for $\epsilon = 2^{-128}$, we have $\eta_\epsilon(\mathbb{Z}^{2n}) \leq 5.545$. We set $\sigma_r = \sigma_e = 2\max(\eta_\epsilon(\mathbb{Z}^{2n}), \sigma_{dLWE}) \approx 11.09$, and we set $\sigma_{pk} = \sigma_{dLWE} = 5$. With these parameters, and using the fact that $\sqrt{\|\mathbf{s}\|^2 + \|\mathbf{e_{pk}}\|^2} < 91.053$ with overwhelming probability, we have

$$\sigma_{ct} < \sqrt{91.053 \cdot (\sigma_r^2 + \sigma_e^2)} \approx 1428.$$

Now we verify that decryption will be correct (for 2-bit messages) with these parameters. For threshold decryption with T parties, the noise during reconstruction will have width $\sigma_d = \sqrt{\sigma_{ct}^2 + 2T\sigma_{dLWE}^2} < \sqrt{1428^2 + 50T}$. To successfully decrypt 2-bit messages, we need the size of the noise to be less than $65536/8 = 8192$ with high probability.

$$\Pr_{x \leftarrow \mathcal{N}_{\sigma_d}}[|x| > 8192] = \Pr_{x \leftarrow \mathcal{N}_1}[|x| > 8192/\sigma_d].$$

[8] `https://github.com/malb/lattice-estimator`, commit `fd4a460c`

We can compute numerically that this decryption failure probability will be less than 2^{-128} as long as $\sigma_d < 1566$. This means these parameters will support as many as $T = 8263$ parties.

We remark that while Frodo-640 ostensibly targets 2^{128} security, its parameters are chosen to withstand quantum attacks and include a large security margin — the Lattice Estimator estimates 2^{163} operations to break Frodo-640. Still, this rough comparison to a highly optimized scheme like FrodoKEM shows that our technique allows Threshold PKE for large numbers of parties with concrete parameters of practical size.

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