On the Implausibility of Constant-Round Public-Coin Zero-Knowledge Proofs[⋆]

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Abstract. We consider the problem of whether there exist non-trivial constant-round public-coin zero-knowledge (ZK) proofs. To date, in spite of high interest in the problem, there is no definite answer to the question. We focus on the type of ZK proofs that admit a universal simulator (which handles all malicious verifiers), and show a connection between the existence of such proof systems and a seemingly unrelated "program functionality distinguishing" problem: for a natural class of constant-round public-coin ZK proofs (which we call "canonical," since all known ZK protocols fall into this category), a session prefix output by the universal simulator can actually be used to distinguish a non-trivial property of the next-step functionality of the verifier's code.

Our result can be viewed as new evidence against the existence of constant-round public-coin ZK proofs, since the existence of such a proof system will bring about either one of the following: (1) a positive result for the above functionality-distinguishing problem, a typical goal in reverse-engineering attempts, commonly believed to be notoriously hard, or (2) a major paradigm shift in simulation strategies, beyond the only known (straight-line simulation) technique applicable to their argument counterpart, as we also argue. Note that the earlier negative evidence on constant-round public-coin ZK proofs is Barack, Lindell and Vadhan [FOCS '03]'s result, which was based on the incomparable assumption of the existence of certain entropy-preserving hash functions, now known not to be achievable from standard assumptions via black-box reduction.

The core of our technical contribution is showing that there exists a single verifier step for constant-round public-coin ZK proofs whose functionality (rather than its code) is crucial for a successful simulation. This is proved by combining a careful analysis of the behavior of a set of verifiers in the above protocols and during simulation, with an improved structure-preserving version of the well-known Babai-Moran Speedup (de-randomization) Theorem, a key tool of independent interest.

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1 Introduction

Goldwasser, Micali and Rackoff [24] introduced the fascinating notion of a *zero-knowledge* (ZK) interactive proof, in which a party (called the prover) wishes to convince another party (called the verifier) of some statement, in such a way that the following two properties are satisfied: (1) zero knowledge— the prover does not leak any knowledge beyond the truth of the statement being proven, and (2) soundness—no cheating prover can convince the verifier of a false statement except with small probability. A vast amount of work ensued this pioneering result. Shortly after the introduction of a ZK proof, Brassard, Chaum and Crépeau [3] defined a ZK proof system with relaxed soundness requirement, called a ZK *argument*, for which soundness is only required to hold against polynomial-time cheating provers.

The original ZK proof system for the quadratic residuosity problem presented in [24] is of a special form, in which the verifier simply sends independently random coins at each of his steps. Such a proof system is called a *public-coin* proof system, and has been found to be broadly applicable and versatile. Another notable feature of this type of proof systems is its round efficiency, as it consists of only 3 rounds, i.e., just 3 messages are exchanged in a session. This round efficiency, however, brings about a side effect of soundness error, which is too large to be used in cryptographic settings where typically a negligibly small such error is required. Indeed, there seems to be a tradeoff between round efficiency and soundness error for public-coin proof systems: we can achieve negligible soundness error by sequential repetition, but then the resulting system is no longer constant-round. This is in contrast with private-coin ZK proof systems, for which constant rounds and negligible soundness error can be achieved simultaneously.

In fact, whether constant-round public-coin ZK protocols (or even argument systems) with negligible soundness error exist for some non-trivial language was a long-standing open problem. In [22], Goldreich and Krawczyk showed that, for non-trivial languages, the zero knowledge property of such a proof system cannot be proven via black-box simulation. Black-box simulation was in fact the only known technique to demonstrate "zero-knowledgeness" for a long while, and hence the Goldreich-Krawczyk result was viewed as strong negative evidence against the existence of constant-round public-coin ZK proof systems.

A breakthrough result in 2001 changed the state of things. Indeed, in [2] Barak presented a non-black-box ZK argument in which the simulator makes use of the code of the malicious verifier in computing the prover messages (albeit without understanding it). Barak's construction follows the so-called "FLS paradigm" [19], which consists of two stages. In the first stage the prover sends a commitment c to a hash value of an arbitrary string, to which the verifier responds with a random string r; in the second stage, the prover proves using a witness indistinguishable (WI) universal argument that either the statement in question is true or c is a commitment to a hash value of some code Π, and, given input c, Π outputs r in some super-polynomial time. Note that this is a constant-round public-coin argument, and that its simulator does not "rewind" the malicious verifier (and it is hence called a *straight-line* simulator) and, furthermore, runs in strict polynomial time. These features have been proved impossible to achieve when using black-box simulation [22,7].

Barak's argument system still left open the question whether non-trivial constant-round public-coin (non-black-box) ZK *proof* systems exist. At first sight, being able to extend his technique to a proof system seems challenging, mainly due to the fact that since a Turing machine or algorithm may have an arbitrarily long representation, a computationally unbounded prover may, after receiving the second verifier message r, be able to find a program Π (whose description may be different from the verifier's with which the prover is interacting) such that, $c = \mathrm{Com}(h(\Pi))$, and on input c, Π outputs r in the right amount of time.

In [9], Barak, Lindell and Vadhan showed further negative evidence for the above problem, by proving that if a certain class of entropy-preserving hash functions exist, then such a proof system cannot exist. Their formulation of entropy-preserving hash functions is mathematically simple, inspiring further research to base such hash functions on standard assumptions. Unfortunately, to our knowledge, we do not have a candidate for such functions thus far, and furthermore, as shown by Bitansky *et al.* [4], such functions *cannot* be based on any standard assumption via black-box reduction.

Our results and techniques. In this paper, we provide evidence of a different nature against the existence of constant-round public-coin ZK proof systems. We focus on the type of ZK proofs that admit a universal simulator, i.e., ZK proof systems for which there is a single simulator that can handle all malicious verifiers. (To our knowledge, all constructions of ZK proofs in the literature are of this type.)

We uncover an unexpected connection between the existence of such proof systems and a seemingly unrelated "program functionality distinguishing" problem: for a natural class of constant-round public-coin ZK proofs (which we call "canonical," as all known ZK protocols fall in this category), a universal simulator for such ZK proof system can actually be used to figure out some non-trivial property of a verifier's program functionality. (Since we will always be talking about distinguishing verifiers' programs, sometimes we will just refer to the problem as the "verifier distinguishing" problem.) More specifically, we show that, given a constant-round public-coin ZK proof system $\langle P, V\rangle$, there exist a step index k and a set of polynomial number of verifiers that share the verifier next-message functions up to the $(k-1)$-th step but have distinct k-th next-message functions—say, t, for t a polynomial, and denoted by $(V_k^1, V_k^2, ..., V_k^t)$—such that for any polynomial-time constructible code V_k^* that is promised to have the same functionality as one of V_k^i's in the above set, the universal simulator, taking V_k^* as input, can generate a session prefix before the k-th verifier step that enables us to single out a V_k^j in the set which is functionally *different* from V_k^*.

In more detail, we construct an distinguishing algorithm U which, taking only $(V_k^1, V_k^2, ..., V_k^t)$ and the session prefix output by the simulator as input, is able to pin-point an element V_k^j in the set which behaves differently from V_k^*, with probability negligibly close to 1. This means that the universal simulator must have encoded some non-trivial property of V_k^*'s functionality in the session prefix prior to the verifier's k-th step, since otherwise if the session prefix is independent of V_k^*, the success probability of U will never exceed $1 - \frac{1}{t}$ (note that U does note take V_k^* as an input). In the case of private-coin ZK protocols, encoding the functionality of the next verifier step in a session prefix is typically done by having the simulator execute V_k^* first and then redo the prefix prior to the k-th verifier step such that it can now handle the challenge from V_k^*. It should be noted that, for constant-round protocols, such a rewinding strategy seems to work only for the cases where the functionality of V_k^* is bound to some of the verifier's previous steps, and this is not the case for public-coin protocols[1].

This is in a sharp contrast with Barak's public-coin argument system, in which the simulator does not need to "predict" the verifier's next-message functionality when computing a session prefix. Think of the first two steps in the simulation of Barak's argument, where the verifier sends a random hash function (h) and the prover replies with a commitment to a hash value of the *code* (instead of its functionality) of the next message function of the verifier's second step ($\mathrm{Com}(h(V_2^*))$). Note that when the simulator computes this session prefix it does not need to figure out the functionality of V_2^*, and in fact the functionality of V_2^* is not bound to the history prefix $(h, \mathrm{Com}(h(V_2^*)))$. Indeed, when the commitment scheme $\mathrm{Com}(h(\cdot))$ is a perfectly hiding scheme (which is allowed in Barak's argument), the message $c = \mathrm{Com}(h(V_2^*))$ can be interpreted as a commitment to *any* code of any functionality, and thus it contains zero information about V_2^*'s functionality.

Thus, our result can be viewed as further evidence against the existence of constant-round public-coin ZK proof systems. On one hand, devising a rewinding technique (to figure out the next-step functionality of the verifier) that could be used in the simulation of such a public-coin proof appears to be fairly inconceivable, as in these proofs the message (challenge) from each step of the verifier is long and hard for a cheating prover to pass, and, intuitively, in this setting the rewinding behavior of a simulator (given the code of a malicious verifier) is akin to learning an arbitrarily complicated and obfuscated verifier's next-step function (which is, as a code, independent of any previous step functions) by just sampling a few input-output pairs of this function.

On the other hand, if such a proof does admit a straight-line simulator, then our "functionality distinguishing" result described above shows that one would be able to figure out some non-trivial functionality/property of V_k^* *without executing it* (since "straight-line" typically means that in producing the session prefix before the k-th verifier step, the simulator does not run V_k^*), a problem commonly considered notoriously hard. We

[1] We note that the rewinding technique used for simulating the known public-coin protocols simply exploits the "guessing the next verifier's coins" strategy, and requires that the probability of a correct guess is very high. To meet such a requirement, the verifier's message has to be short, and as a consequence, the corresponding protocol either has large (non-negligible) soundness error, such as the original Blum's 3-round proof fro Graph Hamiltonicity [8], or is of super-constant number of rounds, such as the $\log^2 n$-fold sequential repetition of Blum's proof system.

note that exactly how hard the problem is in our concrete setting we leave as an interesting research question. (Indeed, although we do not give a definite answer to the question, we view our work as providing new negative evidence from a different angle and suggesting directions for further studies towards that goal.)

One key tool in our reduction is an improved structure-preserving version of the well-known Babai-Moran Speedup (derandomization) Theorem [1,10,11], which is of independent interest. Essentially, our result says that for a constant-round public-coin interactive proof system in which the verifier sends m messages and each of the prover messages is of length p, if the cheating probability for an unbounded prover is ϵ, then there exist $(p/O(\log \frac{1}{\epsilon}))^m$ verifier random tapes such that the cheating probability for the unbounded prover over these tapes is bounded away from 1—and this holds even when the prover knows this small set of random tapes in advance. In contrast, in our setting the original Babai-Moran theorem would yield a much larger size (namely, $(O(p))^m$) of such set of verifier random tapes. In addition, we show that this result is tight with respect to round complexity, in the sense that there are public-coin proof systems with a super-constant number of rounds for which the prover's cheating probability is 1, over any polynomial number of verifier random tapes.

The way our derandomization lemma helps in the reduction to the verifier-distinguishing problem is as follows. Intuitively, for a proof system, it seems that there should be a verifier step k for which computing a session prefix prior to this step in the simulation requires the simulator to classify the codes of the "residual" verifiers according to their functionality, since by unconditional soundness a fixed session prefix can (even for an all powerful prover) make only a few (as opposed to all efficiently computable functions) of the residual verifiers accept. Derandomization allows us to focus on those few verifiers on which the cheating probability of an all powerful prover is still bounded away from 1, and then prove the existence of the above critical verifier step.

Related work. As mentioned above, Barak, Lindell and Vadhan [9] conjectured the existence of certain entropy-preserving hash functions and proved that the conjecture's veracity would rule out the possibility of existence of constant-round public-coin ZK proof systems. Recent work by Bitansky *et al.* [4], however, showed that this conjecture cannot have a black-box reduction from any standard assumption.

A somewhat related problem to our functionality-distinguishing problem is program obfuscation, the theoretical study of which was initiated by Barak *et al.* [6]. At a high level, an obfuscator is an efficient compiler that takes a program as input and outputs an "unreadable" program with the same functionality as the input program. Hada [26], in particular, showed that the existence of a certain type of ZK protocol is tightly related to the existence of an obfuscator for some specific functionality. Unfortunately, for a large class of functionalities, it has been shown that obfuscators do not exist, and it is not clear whether the recent and exciting formulation and constructions of indistinguishability obfuscators (cf. [21] and numerous follow-ups) imply a negative answer to our problem[2].

Organization of the paper. Preliminaries, notation and definitions that are used throughout the paper are presented in Section 2. Definitions of *canonical* ZK proofs and of the *verifier-distinguishing* problem are formulated in Section 3. The improved derandomization lemma is presented in Section 4, and the reduction of constant-round public-coin ZK proofs to the verifier-distinguishing problem, which makes use of it, in Section 5. For the sake of readability, some of the proofs presented in the main body are only sketches; the full proofs can be found in the appendix, as well as a counterexample for superconstant-round proof systems.

2 Preliminaries

In this section we recall some definitions and introduce notation that will be used throughout the paper.

[2] Very recently, Canetti *et al.* [12] constructed a family of "correlation intractable" hash functions based on sub-exponentially secure indistinguishability obfuscation and input-hiding obfuscators, but as the authors point out, their result does not directly imply the non-existence of 3-round public-coin ZK proofs.

We say that function $\mathsf{neg}(n)$ is *negligible* if for every polynomial $q(n)$ there exists an N such that for all $n \geq N$, $\mathsf{neg}(n) \leq 1/q(n)$. Throughout this paper, polynomials always refer to polynomials in the security parameter n of a proof system.

When referring to a Turing machine M, we will slightly abuse notation and use M to represent both its code and its functionality. Specifically, if we write $M \in \mathcal{G}$ for some set $\mathcal{G}$, we will mean that there is a Turing machine in $\mathcal{G}$ whose code is identical to the code of M; on the other hand, if we say that M^* is "functionally equivalent" to M (as defined below), both M^* and M will clearly refer to their functionality.

We think of an interactive Turing machine as a machine that computes a collection of next-message functions. (We refer the reader to [20] for a rigorous definition.)

Definition 1. *For two deterministic (interactive) Turing machines M^1 and M^2, we say M^1 and M^2* have the same functionality, *or* are functionally equivalent *if they compute the same collection of next-message functions. That is, for any input* hist, *the next message produced by M^1 is identical to the one produced by M^2—i.e., $M^1(\mathsf{hist}) = M^2(\mathsf{hist})$.*

We will use $M^1 \overset{\mathrm{f}}{=} M^2$ as a shorthand for the above, and $M^1 \overset{\mathrm{f}}{\neq} M^2$ as its negation.

An *interactive proof system* $\langle P, V\rangle$ for a language L is a pair of interactive Turing machines in which the prover P wishes to convince the verifier V of some statement $x \in L$. In an interaction between P and V, the *view* of V, denoted by View_V^P, consists of the common input x,V's random tape, and all the prover messages it received. The *round complexity* of an interactive proof system $\langle P, V\rangle$ is the number of messages exchanged in an execution of $\langle P, V\rangle$. Without loss of generality, in this paper we assume that the verifier V sends the first message; thus, if the verifier sends m messages in total, the round complexity of this proof system is $2m$.

Definition 2 (Interactive Proofs). *A pair of interactive Turing machines $\langle P, V\rangle$ is called an* interactive proof system for language L *if V is a probabilistic polynomial-time (PPT) machine and the following conditions hold:*

- COMPLETENESS: *For every $x \in L$,* $\Pr[\langle P, V\rangle(x) = 1] = 1$.
- SOUNDNESS: *For every $x \notin L$, and every (unbounded) prover P^*,* $\Pr[\langle P^*, V\rangle(x) = 1] < \mathsf{neg}(|x|)$.

Public-coin proof systems and verifier decomposition. An interactive proof system is called *public-coin* if at every verifier step, the verifier sends only truly random messages.

We will use boldface lowercase letters to refer to the verifier's random tapes (e.g., $\mathbf{r}$), and italic for each verifier message (e.g., r). Thus, for a $2m$-round public-coin interactive proof system $\langle P, V\rangle$, we have $\mathbf{r} = [r_1, r_2, ..., r_m]$, where r_i is the i-th verifier message. We use superscripts to distinguish different verifier's random tapes; e.g., $\mathbf{r}^i$, $\mathbf{r}^j$, etc.

Given a random tape $\mathbf{r} = [r_1, r_2, ..., r_m]$, we can "decompose" the verifier $V(\mathbf{r})$ into a collection of next-message functions, $V = [V_1, V_2, ..., V_m]$, with each V_i being defined as:

$$r_i \text{ or } \perp \leftarrow\ V_i(\mathsf{hist}, r_1, r_2, ..., r_i),$$

where hist refers to the current history up to the $(i-1)$-st prover step ; that is, given hist, $V_i(\mathsf{hist}, r_1, r_2, ..., r_i)$ outputs r_i if hist is accepting, or aborts if not. Note that the next message function V_i needs the randomness $[r_1, r_2, ..., r_{i-1}]$ of previous verifier steps in order to check whether the current history is accepting or not.

We will sometimes abbreviate and use superscripts to distinguish verifiers running on different random tapes; that is, given two random tapes $\mathbf{r}^i = [r_1^i, r_2^i, ..., r_m^i]$ and $\mathbf{r}^j = [r_1^j, r_2^j, ..., r_m^j]$, we will use V^i and V^j as a shorthand for $V(\mathbf{r}^i)$ and $V(\mathbf{r}^j)$, respectively. Similarly, we will use V_k^i to denote the k-th next-message function of the verifier $V(\mathbf{r}^i)$.

Now, given a verifier $V^i = [V_1^i, ..., V_m^i]$, we will use $V_{[j,k]}^i$ to denote the partial verifier strategy starting with the j-th next message function and up to the k-th next message function. We will typically be concerned with the following partial strategies:

$$\text{prefix strategy: } V^i_{[1,k]} \triangleq [V^i_1, V^i_2, ..., V^i_k];$$
$$\text{suffix strategy: } V^i_{[k,m]} \triangleq [V^i_k, V^i_{k+1}, ..., V^i_m].$$

ZK proofs with universal simulator. We first present the standard definition of ZK proofs.

Definition 3 (Zero-Knowledge Proofs). *An interactive proof system $\langle P, V\rangle$ for a language L is said to be* zero-knowledge *if for any probabilistic polynomial-time V^*, there exists a probabilistic polynomial-time algorithm S such that the distribution $\{\mathrm{View}^P_{V^*}\}_{x\in L}$ is computationally indistinguishable from the distribution $\{S(x, V^*)\}_{x\in L}$.*

The standard simulation process for a malicious verifier V^* is typically as follows. The PPT simulator S, taking the common input x and V^*'s code as inputs, is to output a session transcript. S treats V^* as a subroutine, interacting (with possible "rewinds") with it *internally,* and outputting a view of V^* as the result of the interaction. Without loss of generality, one can think of the output of the simulator as the final (internal) interaction between $S(x, V^*)$ and V^*. In this paper, we wish to be able to obtain prover messages from S one by one, rather than obtaining the entire session transcript at once. For this purpose, we make the above (final) internal interaction "external," by casting the simulation process for a malicious verifier V^* as a real interaction between $S(x, V^*)$ (playing the role of the prover) and an external V^*, and whenever S wants to rewind V^*, it does it on its own copy of V^*. We denote this interaction by $(S(x, V^*) \Leftrightarrow V^*)$, and the view of V^* resulting from this interaction by $\{\mathrm{View}^{S(x,V^*)}_{V^*}\}_{x\in L}$. (For brevity, we will sometimes drop x from the above notation.)

The following fact is easy to verify.

Fact 1. For any x and any V, V^* such that $V \overset{\mathrm{f}}{=} V^*$, $(S(x, V^*) \Leftrightarrow V^*)$ generates the same session transcript as $(S(x, V^*) \Leftrightarrow V)$.

We conclude this section with the following definition of *ZK proof with universal simulator,* which differs from the standard ZK definition in the order of quantifiers ("$\exists S \forall V^*$" instead of "$\forall V^* \exists S$").[3]

Definition 4 (Zero-Knowledge Proofs with Universal Simulator). *An interactive proof system $\langle P, V\rangle$ for a language L is said to be* zero-knowledge with universal simulator *if there exists a probabilistic polynomial-time algorithm S such that for any probabilistic polynomial-time V^*, the distribution $\{\mathrm{View}^P_{V^*}\}_{x\in L}$ is computationally indistinguishable from the distribution $\{\mathrm{View}^{S(V^*)}_{V^*}\}_{x\in L}$.*

Remark 1. By definition, a simulator for a ZK proof system needs to handle arbitrary verifiers. Throughout the paper we just deal with the arbitrary code of an *honest* verifier, which strengthens the result.

3 Canonical ZK Proofs and the *Verifier-Distinguishing* Problem

In this paper we will focus on ZK proof systems with a certain property, which we call "canonical," since all known constructions (see below) fall in this category. We first give some intuition behind it. (To simplify notation, from here on we will drop the common input x from $(S(x, V^*) \Leftrightarrow V^*)$, and write the simulation as $(S(V^*) \Leftrightarrow V^*)$.) We observe that for many ZK protocols, if the simulation is formulated as an interaction between $S(V^*)$ and V^*, as in the previous section, then for a successful simulation to take place it is sufficient to feed S with only the *partial* code of V^*, rather than with its entire code.

Examples are those ZK proofs following the popular "FLS paradigm" [19], and Blum's ZK proof systems for Graph Hamiltonicity [8]. We elaborate on those systems in detail in Section 3.2, but as an illustrative warm-up, recall that in the FLS paradigm, a ZK proof consists of two stages: in the first stage, the prover and

[3] To our knowledge, all known ZK proofs admit a universal simulator, satisfying this stronger requirement.

the verifier set up a trapdoor (which is useful for the simulation), and then, in the second stage, the prover proves that either the statement being proven is true or that he knows the trapdoor in a WI protocol. Hence, it is easy to see that if the code V^* of a malicious verifier is given by two separate specifications $V_{\rm I}^*$ and $V_{\rm II}^*$, representing the first and second stages of V^*, respectively, then the simulator can perform a successful simulation given only $V_{\rm I}^*$, since it can extract the trapdoor from it, which, no matter what $V_{\rm II}^*$ is, enables it to simulate the second stage in a straight-line fashion. That is, using the notation from last section, for any second-stage honest verifier $V_{\rm II}$ (which may have a different functionality from $V_{\rm II}^*$'s), both interactions $(S(V^*) \Leftrightarrow [V_{\rm I}^*, V_{\rm II}])$ and $(S(V_{\rm I}^*) \Leftrightarrow [V_{\rm I}^*, V_{\rm II}])$ are accepting. This, in a nutshell, is what the canonical property says—if the former interaction is accepting for any $V_{\rm II}$, so is the second interaction. We now state this more formally.

3.1 Canonical ZK proofs

We mentioned partial code of V^* above. The following definition about session prefixes of proof systems will become handy.

Definition 5 **(*Good/bad* session prefix).** *Let $\langle P, V\rangle$ be a $2m$-round public-coin proof system for a language L, and let $\mathcal{V}_{[1,\ell]}$ denote the set of verifiers that share the same verifier prefix strategy $V_{[1,\ell]}$, for some $1 \le \ell \le m$. We call a session prefix $(r_1, p_1, ..., p_\ell)$* good *with respect to $\mathcal{V}_{[1,\ell]}$ if there is a residual (unbounded) prover strategy with auxiliary input $\mathcal{V}_{[1,\ell]}$ which, based on this session prefix, can make a verifier randomly chosen from $\mathcal{V}_{[1,\ell]}$ accept with probability 1. Otherwise, we call the session prefix* bad *with respect to $\mathcal{V}_{[1,\ell]}$.*

A more detailed formulation and properties of good/bad prefixes, as well as an illustration (Figure 3), are presented in Appendix A.

We are now ready to define what we call *canonical* ZK proofs; these proofs are defined conditionally, predicated on the existence of a good prefix. Roughly speaking, the property states that if a simulator $S([V_{[1,k-1]}^*, V_k^*])$, taking the partial code $[V_{[1,k-1]}^*, V_k^*]$ as input, can generate a session prefix up to the $(k-1)$-th prover step that is good for verifiers with a k-th step function different from V_k^*, then S can do the same *without* being given verifier code V_k^*. Next, we present the definition of a canonical ZK proof system with an arbitrary (constant) number of rounds; in Section 3.2 we analyze concrete examples (e.g., 3-round proof systems).

Definition 6 **(*Canonical* ZK Proofs).** *Let $\langle P, V\rangle$ be a $2m$-round universally simulatable ZK proof system for a language L (Definition 4), S be the associated simulator and t be some polynomial. We call $\langle P, V\rangle$* canonical *if for any common input x (not necessarily in L), every set $\mathcal{V}_{[1,k-1]}$ of verifiers that share prefix strategy $V_{[1,k-1]}$, $2 \le k \le m$ (as in Definition 5), but with t distinct k-th step strategies $V_k^1, V_k^2, ..., V_k^t$, the following holds.*

For any verifier code $V_{[1,k-1]}^*$ satisfying $V_{[1,k-1]}^* \overset{\rm f}{=} V_{[1,k-1]}$, if, for some $1 \le i \le t$, there exists $V_k^* \overset{\rm f}{=} V_k^i$ such that the session prefix $(r_1, p_1, ..., p_{k-1}) \leftarrow (S([V_{[1,k-1]}^*, V_k^*]) \Leftrightarrow [V_{[1,k-1]}^*, V_k^*])$ is good with respect to $\mathcal{V}_{[1,k-1]}$, then S, taking only $V_{[1,k-1]}^*$ as input, can also produce a session prefix (i.e., $(r_1, p_1', ..., p_{k-1}') \leftarrow (S(V_{[1,k-1]}^*) \Leftrightarrow V_{[1,k-1]}^*)$) which is good with respect to $\mathcal{V}_{[1,k-1]}$.

A canonical ZK proof is depicted in Figure 1.

Remark 2. We stress that, for a zero knowledge proof, the canonical property above makes a demand on the simulator only when the condition of the if clause holds. We also note that the ability of the simulator $S([V_{[1,k-1]}^*, V_k^*])$ to produce a good session prefix may depend on the common input x (see the examples in the next section).[4]

[4] Further, looking ahead, the only place where this property will be used is in the proof of Lemma 3, where we fix a false statement x first and then discuss the properties of the simulator.

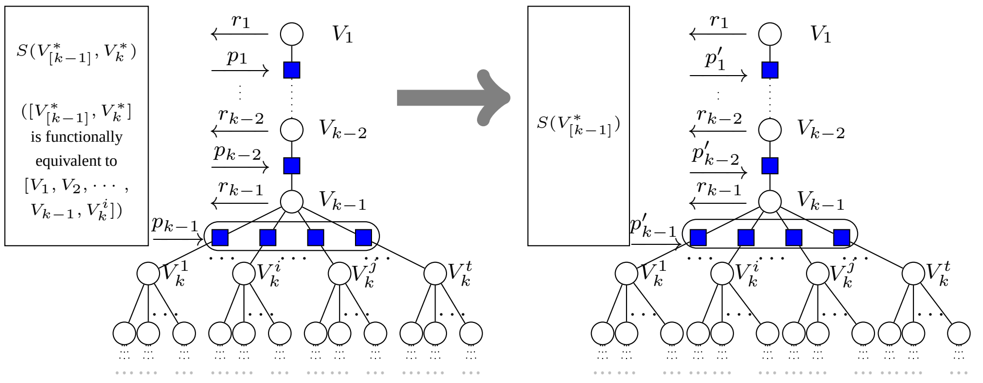


Fig. 1. A canonical ZK proof.

3.2 Canonical ZK proofs: Examples

To our knowledge, all constructions of ZK proofs enjoy this property—cf. the FLS proof system [19] example at the beginning of the section, as well as those protocols that do not follow the FLS paradigm, such as, for example, Blum's 3-round ZK proof for Graph Hamiltonicity [8] (and its sequential repetition version), which we now analyze in more detail.

Blum's Graph Hamiltonicity ZK proof. Consider Blum's 3-round ZK proof system for Graph Hamiltonicity (with soundness error $\frac{1}{2}$). In this case, we denote by V_1^1 and V_1^2 the verifiers that produce challenges 1 and 0, respectively[5]. Suppose that when the verifier sends challenge 1, the prover needs to reveal the isomorphism between the common input graph and the graph committed in the first prover message p_1. Note that when the simulator S takes any V_1^* that is functionally equivalent to V_1^1, then it will simply choose an isomorphism and commit to a new graph isomorphic to the common input graph in the message p_1 (i.e., it acts as an honest prover in the first prover step). For this proof system, the "if" clause of Definition 6 holds depending on whether the common input graph is Hamiltonian or not:

- If the common input graph is Hamiltonian, then the "if" clause holds: Given a verifier code V_1^* that is functionally equivalent to V_1^1 as input (i.e., $i = 1$ in Definition 6), $S(V_1^*)$ will generate a first prover message p_1, for which an unbounded prover can answer both challenges 1 and 0 (from V_1^1 and V_1^2, resp.), since the graph committed by $S(V_1^*)$ in p_1 is also Hamiltonian. In this case, the simulator S, without being given the code V_1^*, can also act as an honest prover in the first prover step and generate p_1 that will enable an unbounded prover to answer both challenges 1 and 0[6]—i.e., prefix p_1 is *good* with respect to the verifier set $\{V_1^1, V_1^2\}$.
- If the common input graph is not Hamiltonian then the "if" clause does not hold: For $t \in \{1, 2\}$, given a verifier code V_1^* that is functionally equivalent to V_1^t as input, $S(V_1^*)$ will generate the first prover message p_1 for which an unbounded prover can only answer a challenge from V_1^t, since the graph committed in p_1 is either a graph isomorphic to the common input graph or a Hamiltonian graph (which is not isomorphic to the common input graph)—i.e., p_1 is *bad* with respect to the verifier set $\{V_1^1, V_1^2\}$.

In sum, Blum's 3-round ZK proof system for Graph Hamiltonicity is canonical according to Definition 6: whenever the definition's "if" clause is satisfied, i.e., the simulator $S(V_1^*)$ can generates p_1 that is good with respect to both verifier challenges, then S, without being given the code V_1^* as input, can also generate a good prefix p_1.

[5] To match our definition, we can think of these protocols as being of even number rounds by letting the verifier send a dummy message in the first step of the protocol, and denote by V_2^i the challenge step of the verifier.
[6] Recall that an honest prover can compute p_1 without knowledge of the corresponding witness

FLS-type ZK proofs. The classical FLS-type ZK proofs [19] are also canonical. Recall how these proofs work. In the first stage, the verifier sends a perfectly hiding commitment c_1 to a random string, followed by a perfectly binding commitment c_2 to a random string from the prover, after which the verifier opens the commitment sent at its first step. In the second stage, the prover proves that the common input $x \in L$ or that the random string committed in c_1 matches the random string committed in c_2 via a Blum 3-round proof system as above (but with negligible soundness error). We view the two verifier steps in the first stage as a single step[7], and denote it by V_1, and denote by V_2 the verifier step in stage 2. We now analyze what happens at each step.

It is easy to verify that the second verifier step ($k = 2$) satisfies the definition, based on the following observation. Fix a code V_1^* of the first verifier step (recall that, by definition, we consider only on the set of verifiers sharing the same first verifier step that is functionally equivalent to V_1^*.) Observe that the simulator, given only a code V_1^* of a first stage verifier that is functionally equivalent to some V_1 as input, can generate a good first stage prefix (by rewinding the first stage verifier V_1^*) that will enable an unbounded prover answer any challenge from the second verifier step (since an unbounded prover can always recover the corresponding trapdoor from the transcript of the first stage and carry out the second stage in a straight-line fashion). I.e., the unbounded prover will, based on the first stage transcript output by $S(V_1^*)$, make a random verifier that share the same prefix V_1 accept.

For the first verifier step V_1, the "if" condition is satisfied depending on whether $x \in L$ or not:

- When $x \in L$, the "if" clause holds, since an unbounded prover can, based on any first stage transcript output by the simulator $S(V_1^*)$, make a random verifier (that may have a prefix functionally different from V_1^*) accept with probability 1 by finding the witness for $x \in L$ and acting as an honest prover in the second stage. In this case, the simulator S, without being given the code V_1^*, can also act as an honest prover in the first prover step and generate a random first stage transcript (which does not form a trapdoor), and based on this transcript, an unbounded prover can always find a witness for $x \in L$ to make a random verifier accept with probability 1.
- When $x \notin L$, the "if" clause does not hold: For every two different first verifier steps V_1^1, V_1^2, and every two different second verifier steps V_2^1, V_2^2 (that will output different challenges in Blum's protocol), where V_1^t ($t \in \{1, 2\}$) commits to r_t and then opens the commitment, and V_2^b ($b \in \{1, 2\}$) simply sends challenge e_b, the simulator S, given a code V_1^* that is functionally equivalent to V_1^i ($i \in \{1, 2\}$) as input, will generate a first stage transcript for which an unbounded prover *cannot* make a random verifier from the set of four verifiers $\{V_1^t, V_2^b\}$ accept with probability 1, since for verifier prefix V_1^j different from V_1^i, the first stage transcript output by $S(V_1^*)$ will not form a valid trapdoor for the prover, and thus, if the random verifier is chosen from the verifier set $\{V_1^j, V_2^b\}$, based on this first stage transcript, an unbounded prover cannot make the random verifier accept with probability greater than $\frac{1}{2}$.

In sum, FLS-type ZK proofs are also canonical according to Definition 6: Whenever Definition 6's "if" clause holds for a verifier step k, the simulator can generate a good prefix prior to the k-th verifier step without being given the code of this verifier step.

Barak's argument system. Finally, one may wonder where Barak's argument system (not known to be a proof system) fits in all this. We view the first three messages in the system (the hash function selected by the verifier, the commitment computed by the prover, and the verifier's random challenge—recall the description in Section 1) as the first stage, and the remaining WIUA (Witness Indistinguishable Universal Argument) as the second stage. Thus, following the same reasoning as above, for $k > 2$, the canonical property is satisfied at the k-th verifier step. However, for the second verifier step (at which the verifier outputs a random challenge), when $x \notin L$, we do not know if Definition 6's "if" clause holds.

Canonical ZK proofs are used in the next section to formulate the "verifier-distinguishing problem," to which the existence of constant-round public-coin ZK proofs is reduced.

[7] Note that the second verifier message is bound to the first verifier message c_1, and that, when considering the canonical property at the second verifier step, the "if" condition of the definition 6 requires that, after for a single verifier's first step, there are many possible distinct verifier's second steps (i.e., $t \geq 2$).

3.3 The *verifier-distinguishing* problem

In a nutshell, given a set of distinct verifier k-th next-message functions, the problem resides in constructing a distinguishing algorithm U which, given a session prefix (prior to the k-th verifier step) output by simulator S, such that for any polynomial-time constructible program V_k^* that is promised to be functionally equivalent to one of the next-message functions, is able to discern one from the set that is functionally *different* from V_k^*. Formally:

Definition 7 (The *Verifier-Distinguishing* Problem). *Let $\langle P, V\rangle$ be a $2m$-round canonical ZK proof system for a language L (Definition 6), S be its simulator, p the length of each prover's message, and t a polynomial in the security parameter n. Given are a set $\mathcal{V}_{[1,k-1]}$ of deterministic honest verifiers that share the same prefix verifier $V_{[1,k-1]}$, but have t* distinct *k-th next-message functions $V_k^1, V_k^2, ..., V_k^t$, denoted by set $\mathcal{V}_k$, and an auxiliary input aux[8]. The* verifier-distinguishing problem *is to find a non-uniform algorithm U, running in time $2^{O(p)}$, such that for every polynomial-time algorithm C, the following holds:*

- *First, C picks a machine $V_k^i \in \mathcal{V}_k$ at random and outputs a polynomial-time Turing machine V_k^* such that $V_k^* \overset{\mathrm{f}}{=} V_k^i$.*
- *Next, U, taking $(\mathcal{V}_{[1,k-1]}, \mathcal{V}_k)$ and a session prefix $(r_1, p_1, ..., p_{k-1})$ output by $S(\mathsf{aux}, V_k^*)$, outputs $V_k^j \in \mathcal{V}_k$ such that $V_k^j \overset{\mathrm{f}}{\neq} V_k^*$ with probability negligibly close to 1. I.e.,*

$$\Pr\left[\begin{matrix} V_k^* \leftarrow C(\mathcal{V}_k, i);\ (r_1, p_1, ..., p_{k-1}) \leftarrow S(\mathsf{aux}, V_k^*); \\ j \leftarrow U(\mathcal{V}_{[1,k-1]}, \mathcal{V}_k, r_1, p_1, ..., p_{k-1}) \end{matrix} : V_k^* \overset{\mathrm{f}}{\neq} V_k^j\right] > 1 - \mathsf{neg}(n),$$

where the probability is taken over the random choice i and the randomness used by C,U and S.

Remark 3. We note that in the definition, algorithm U is not given V^*'s code as input. This means that if U is able to carry out its task, then the simulator must encode some non-trivial functionality of V_k^* in such a session prefix. As mentioned before, this is in sharp contrast with known straight-line simulators such as Barak's, which are oblivious to the verifier's functionality in computing a session prefix. We elaborated on some of the difficulties in solving this problem in Section 1, overcoming which (if at all possible) would require a technical breakthrough in simulation techniques.

4 An Improved Derandomization Lemma for Interactive Proofs

In this section we prove a structure-preserving version of the well-known Babai-Moran "Speedup Theorem" [1,10] with improved parameters for our application, which we will then use in the proof of our main result (Theorem 2). Essentially, the result says that for any constant-round public-coin interactive proof system with small soundness error, there exists a polynomial set of random verifier tapes such that the cheating probability for the unbounded prover over these verifier tapes is bounded away from 1—and this holds even when the prover knows this small set of random tapes in advance.

We first recall the Babai-Moran theorem. Let $\mathrm{AM}[k]$ denote the set of languages whose membership can be proved via a k-round public-coin proof system.

Theorem 1 ([10]). *For any polynomial $t(n)$, $\mathrm{AM}[t+1] = \mathrm{AM}[t]$. In particular, for any constant k, $\mathrm{AM}[k] = \mathrm{AM}[2]$.*

For our application, we wish to de-randomize the verifier while keeping the original proof system structure intact (that is, without "collapsing" the round complexity). The $\mathrm{AM}[k] = \mathrm{AM}[2]$ proof—and its randomness-efficient variant in [11][9]—actually yield such a result: for any $2m$-round public-coin proof system with small

[8] This auxiliary input is given to S; in our main theorem (Theorem 2) it will be the code of some verifier prefix strategy.

[9] In [11], Bellare and Rompel present a randomness-efficient approach to transform $\mathrm{AM}[k]$ into $\mathrm{AM}[2]$: to halve the number of rounds of an Arthur-Merlin proof system, they introduce a so-called "oblivious sampler" and use a small amount of randomness

soundness error ϵ, there exist $(O(p))^m$ verifier random tapes over which the cheating probability of an unbounded prover is still bounded away from 1, where p is the length of the prover's messages.

Next, we present an improvement to this result, in which the number of such verifier random tapes reduces to $(p/O(\log \frac{1}{\epsilon}))^m$. In addition, we show that this de-randomization lemma is essentially tight with respect to the round complexity, as there are super-constant-round public-coin proof systems for which the prover's cheating probability is 1, over any polynomial number of verifier random tapes.

Before stating the lemma, we introduce some additional notation:

- $V_{|(\mathbf{r}^1,\mathbf{r}^2...,\mathbf{r}^t)}$ denotes the honest verifier that is restricted to choose *uniformly at random* one of $\mathbf{r}^1, \mathbf{r}^2..., \mathbf{r}^t$ as its random tape, where t is a polynomial; we use $V_{|(\mathbf{r}^1,\mathbf{r}^2...,\mathbf{r}^t)}(\mathbf{r}^i)$ to denote the verifier that takes $\mathbf{r}^i$, $1 \leq i \leq t$, as its random tape.
- $P^*(\mathbf{r}^1, \mathbf{r}^2..., \mathbf{r}^t)$ denotes the unbounded *cheating* prover with auxiliary input $(\mathbf{r}^1, \mathbf{r}^2..., \mathbf{r}^t)$, indicating that it will interact with $V_{|(\mathbf{r}^1,\mathbf{r}^2...,\mathbf{r}^t)}$.

We now state the result formally. For simplicity, we assume that all the prover messages are of equal length.

Lemma 1. *Let m be a constant and $\langle P, V\rangle$ be a $2m$-round public-coin interactive proof system for language L with negligible soundness error ϵ. Let p denote the length of the prover's messages. Then for every $x \notin L$, there exist $q = (p/O(\log \frac{1}{\epsilon}))^m$ different random tapes, $\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q$, such that for every unbounded prover P,*

$$\Pr[\langle P(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q), V_{|(\mathbf{r}^1,\mathbf{r}^2,...\mathbf{r}^q)}\rangle(x) = 1] \leq 1 - \frac{1}{q} \; .$$

Here we present the intuition and basic inequalities that yield the proof for the case of a 3-round proof system[10] (similar ideas also appeared in [1,10]), and defer the full proof of the lemma to Appendix B.1.

Let us consider a 3-round public-coin proof system $\langle P, V\rangle$ with negligible soundness error for some language L[11], in which the prover sends the first message p_1 and the last message p_2, and the verifier sends the second message $\mathbf{r}$ (its public coins). Without loss of generality, we assume $|p_1| = |p_2| = p$, and $|\mathbf{r}| = n$. We now prove that there exists a number p of verifier random tapes[12] $(\mathbf{r}^1, \mathbf{r}^2, ..., \mathbf{r}^p)$ over which the cheating probability is at most $1 - 1/p$.

For the sake of contradiction, assume that for some false statement $x \notin L$ there is an unbounded prover $P^\diamond$ such that for any p-tuple $(\mathbf{r}^1, \mathbf{r}^2, ..., \mathbf{r}^p)$, $P^\diamond(\mathbf{r}^1, \mathbf{r}^2, ..., \mathbf{r}^p)$ can cheat $V_{|(\mathbf{r}^1,\mathbf{r}^2,...\mathbf{r}^p)}$ with probability 1. Now note that the number of such successful cheating provers is $\binom{2^n}{p}$, and that there are at most 2^p different first prover messages p_1. Thus, there is a number of at least $\binom{2^n}{p}/2^p$ $P^\diamond(\mathbf{r}^1, \mathbf{r}^2, ..., \mathbf{r}^p)$'s that produce the same first prover message, denote it p_1^*, for which if the verifier is using a random tape in any of the p-tuples

$$\{(\mathbf{r}^1, \mathbf{r}^2, ..., \mathbf{r}^p) : p_1^* \leftarrow P^\diamond(\mathbf{r}^1, \mathbf{r}^2, ..., \mathbf{r}^p)\},$$

we have an unbounded prover that can produce a second prover message p_2^* to make the verifier accept.

On the other hand, the number of p-tuple choices $(\mathbf{r}^1, \mathbf{r}^2, ..., \mathbf{r}^p)$ out of a $1/2e$ fraction of all possible verifier random tapes is at most $\binom{\frac{2^n}{2e}}{p}$. Since

$$\binom{\frac{2^n}{2e}}{p} < (\frac{2^n}{2p})^p < \frac{\binom{2^n}{p}}{2^p},$$

to specify roughly $O(p)$ verifier messages in the original proof system. Their proof, however, yields almost the same result as the Speedup Theorem in our setting where we want to maintain the structure of the original proof system, and only care about the number of original verifier random tapes that are needed to make sure the resulting protocol after derandomization is still a proof system.

[10] The basic reasoning here applies to a proof system of even number (4) of rounds as well, by having the verifier send a dummy message first.

[11] For example, the n-folded parallel version of Blum's 3-round proof for Graph Hamiltonicity [8], or the 3-round proof for Graph Isomorphism [23].

[12] For simplicity's sake, we do not optimize this parameter here.

we have that the set $\{(\mathbf{r}^1, \mathbf{r}^2, ..., \mathbf{r}^p) : p_1^* \leftarrow P^\diamond(\mathbf{r}^1, \mathbf{r}^2, ..., \mathbf{r}^p)\}$ covers at least a $1/2e$ fraction of all possible verifier random tapes.

In sum, we are able conclude that there is an unbounded prover, which sends p_1^* as its first message, that can make the verifier accept the false statement with probability at least $1/2e$. This contradicts the negligible soundness error of $\langle P, V \rangle$.

The proof of the lemma for the general (arbitrary constant rounds) case can be found in Appendix B.1, and the tightness result, i.e., the counterexample for superconstant-round proof systems, in Appendix C.

5 Constant-Round Public-Coin Zero-Knowledge Proofs Imply Distinguishing Verifiers' Programs

We are now ready to present our main result, which exhibits a reduction from constant-round public-coin canonical ZK proofs to the functionality-distinguishing problem (Definition 7), a problem seemingly quite different in nature. We first fix some parameters and revisit notation:

- $\langle P, V \rangle$: A $2m$-round public-coin canonical ZK proof sytem for some constant m. We let n be the security parameter and p be the length of each prover's message.
- $\mathcal{V}_{[1,k-1]}$: A set of deterministic honest verifiers that share the same (honest) prefix verifier $V_{[1,k-1]}$, but have t *distinct* k-th step functions $V_k^1, V_k^2, ..., V_k^t$; $|\mathcal{V}_{[1,k-1]}| \leq q$, where t and q are polynomials (defined in Lemma 1)[13].
- $\mathcal{V}_k$: The set $\{V_k^1, V_k^2, ..., V_k^t\}$, as above.
- $V'_{[1,k-1]}$: The auxiliary input to S, which is the code of a prefix verifier such that $V'_{[1,k-1]} \overset{\mathrm{f}}{=} V_{[1,k-1]}$. (When $k = 1$, it is set to the empty string.)

We now show that if $\langle P, V \rangle$ admits a universal simulator S, then there is an algorithm U, taking $\mathcal{V}_{[1,k-1]}$, $\mathcal{V}_k$ and a session prefix as inputs, which can solve the functionality-distinguishing problem (cf. Definition 7) with respect to verifier set $\mathcal{V}_k$. Formally:

Theorem 2. *Let $\langle P, V \rangle$ be a $2m$-round, public-coin canonical ZK proof system with negligible soundness error for a non-trivial language $L \notin \mathcal{BPP}$, and S be its universal simulator. Then, there exist an infinite set I, a sequence of false statements $x \notin L$ of length n for each $n \in I$, a constant k, $2 \leq k \leq m$, sets $\mathcal{V}_{[1,k-1]}$ and $\mathcal{V}_k$, a verifier code $V'_{[1,k-1]}$ as above, and an algorithm U, running in time $2^{O(p)}$, such that, for any polynomial-time algorithm C that on input $(\mathcal{V}_k, i)$, $1 \leq i \leq t$ outputs V_k^* satisfying $V_k^* \overset{\mathrm{f}}{=} V_k^i \in \mathcal{V}_k$, the following holds:*

$$\Pr\left[\begin{array}{c} V_k^* \leftarrow C(\mathcal{V}_k, i); (r_1, p_1, ..., p_{k-1}) \leftarrow (S([V'_{[1,k-1]}, V_k^*]) \Leftrightarrow V'_{[1,k-1]}) \\ j \leftarrow U(\mathcal{V}_{[1,k-1]}, \mathcal{V}_k, r_1, p_1, ..., p_{k-1}) \end{array} : V_k^* \overset{\mathrm{f}}{\neq} V_k^j \in \mathcal{V}_k\right] > 1 - \mathsf{neg}(n),$$

where the probability is taken over the random choice i and the randomness used by C, U and S.

We now give a high-level sketch of proof of the theorem, which mainly consists of three steps. (Refer to Figure 2.)

1. The first step is Lemma 1 from the previous section. Let $V^1, V^2, ..., V^q$ denote the q deterministic verifiers given by the lemma. The various trees in Figure 2(a) correspond to these q verifiers.
2. Next, we show that there exists a sequence of infinitely many false statements x such that for every verifier V^i, $1 \leq i \leq q$, and any polynomial-time constructible code V^* which is functionally equivalent to V^i, the session $(S(V^*) \Leftrightarrow V^*)$ (which, by Fact 1 is identical to $(S(V^*) \Leftrightarrow V^i)$) is accepting except with negligible probability. This is shown in Figure 2(a).

[13] At the k-th verifier step, the number of distinct next-message functions should in fact be t_k. For simplicity, we assume $t = t_k$ for all $1 \leq k \leq m$.

3. For any false statement x in the above sequence, we prove that among these q verifiers, we can find a (sub)tree $\mathcal{V}_{[1,k-1]}$ that has the same prefix strategy $[V_1, V_2, ..., V_{k-1}]$ up to the $(k-1)$-th verifier step but "splits" at the k-th verifier step, and a code $V'_{[1,k-1]}$ that is functionally equivalent to $V_{[1,k-1]} = [V_1, V_2, ..., V_{k-1}]$, such that, for any polynomial-time constructible code V_k^* that is promised to be functionally equivalent to one of those V_k^i's (nodes) at level k, the following two conditions hold:

   – The session prefix $(r_1, p_1, ..., p_{k-1})$ produced by $(S([V'_{[1,k-1]}, V_k^*]) \Leftrightarrow V'_{[1,k-1]})$ (or equivalently, by $(S([V'_{[1,k-1]}, V_k^*]) \Leftrightarrow V_{[1,k-1]}))$ is *bad* with respect to $\mathcal{V}_{[1,k-1]}$ (cf. Definition 5). This implies that there is a subtree (refer to Figure 2(b)) in $\mathcal{V}_{[1,k-1]}$, with respect to which $(r_1, p_1, ..., p_{k-1})$ is bad.

   – However, the session prefix $(r_1, p_1, ..., p_{k-1})$ is *good* with respect to the subtree that shares the same prefix strategy $[V_{[1,k-1]}, V_k^i]$ (refer to Figure 2(b)).

   This enables us to construct an algorithm that is able to "understand" the code V_k^*, by pin-pointing another verifier code, say, V_k^j, such that $V_k^j \overset{\mathrm{f}}{\neq} V_k^*$.

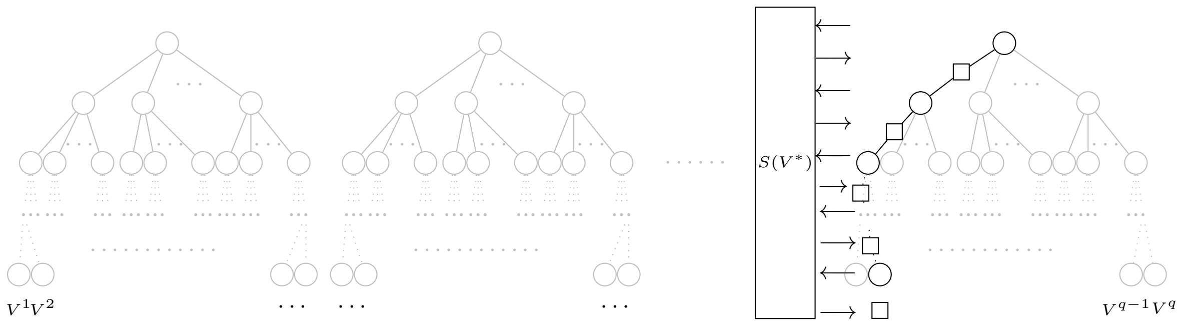
The proof in detail. Given Lemma 1, we now present the remaining details of the proof of Theorem 2. Again, let $\langle P, V\rangle$ be a $2m$-round public-coin canonical ZK proof for some non-trivial (outside $\mathcal{BPP}$) language L, and S be its associated simulator. We first prove the following lemma, where Lemma 1 is used.

Lemma 2. *Let $\langle P, V\rangle$ be as above. Then there exist an infinite set I, a sequence of false statements $x \notin L$ of length n for each $n \in I$, and q honest verifiers $V^1, V^2, ..., V^q$ (recall that we use V^i as a shorthand for $V(\mathbf{r}^i)$, $1 \le i \le q$), such that given the description of any polynomial-time constructible $V^* \overset{\mathrm{f}}{=} V^i$ for a random i as input, the interaction $(S(V^*) \Leftrightarrow V^*)$[14] will produce an accepting transcript with probability negligibly close to 1, while the unbounded prover can cheat only with probability at most $1 - 1/q$.*

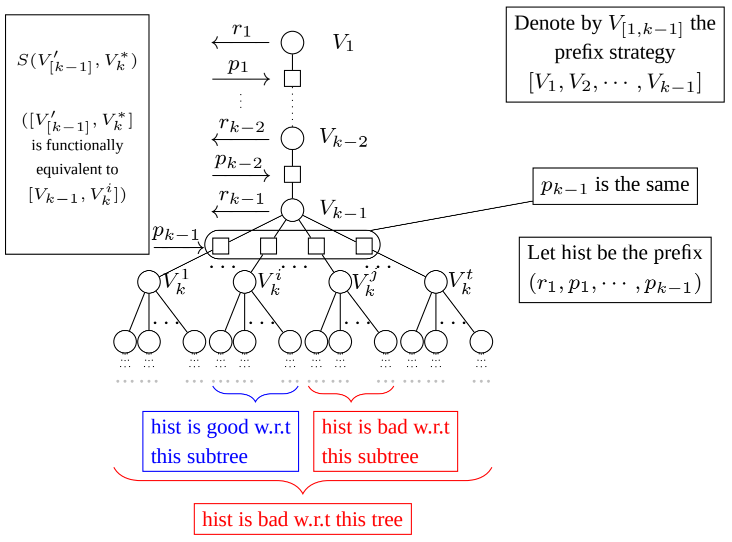
Proof. We first prove that there is an infinite set I of security parameters and a sequence of false statements $x \notin L$ of length n for each $n \in I$ so that for every PPT algorithm C which takes picks a random V from the set of verifiers and outputs V^* such that $V^* \overset{\mathrm{f}}{=} V$, the simulation $(S(V^*) \Leftrightarrow V^*)$ will generate an accepting transcript with probability negligibly close to 1 (over the randomness used by S and the random choice of verifier). Suppose otherwise, for sufficiently large n, there is no false statement x of length n and a code V^* output by some PPT algorithm C on which $(S(V^*) \Leftrightarrow V^*)$ will generate an accepting transcript with probability less than $1 - 1/poly(n)$ for some polynomial $poly$, then the following simple algorithm could be used to decide membership in L efficiently[15]: Pick a verifier at random and run C to construct $V^* \overset{\mathrm{f}}{=} V$ as above, and then have S on input x and V^* interact with V^*; if V^* accepts, output "$x \in L$," otherwise output "$x \notin L$."

Now fix a false statement $x \notin L$ in the above sequence, and set Q to be the set of verifier random tapes such that for any $\mathbf{r} \in Q$ and any polynomial-time constructible $V^* \overset{\mathrm{f}}{=} V(\mathbf{r})$, $(S(V^*) \Leftrightarrow V^*)$ will generate an accepting transcript with probability negligibly close to 1. We now show that the size of Q is larger than a $(1 - \mathsf{neg}(n))$ fraction of all possible random tapes. Assume the verifier's random tape $\mathbf{r}$ and S's random tape R are uniformly distributed over $\{0,1\}^l$ and $\{0,1\}^s$, respectively, where l and s are some polynomials, and denote by E the event that the simulation $(S(V^*) \Leftrightarrow V^*)$ generates an accepting transcript. We have

[14] Recall that this interaction is identical to $(S(V^*) \Leftrightarrow V^i)$ (Fact 1).
[15] Although the error probability here may be high, it can be reduced by standard parallel repetition.



(a)



$S(V'_{[k-1]}, V^*_k)$
$([V'_{[k-1]}, V^*_k]$ is functionally equivalent to $[V_{k-1}, V^i_k])$
r_1
p_1
V_1
r_{k-2}
V_{k-2}
p_{k-2}
r_{k-1}
V_{k-1}
p_{k-1}
V^1_k
V^i_k
V^j_k
V^t_k
Denote by $V_{[1,k-1]}$ the prefix strategy $[V_1, V_2, \cdots, V_{k-1}]$
p_{k-1} is the same
Let hist be the prefix $(r_1, p_1, \cdots, p_{k-1})$
hist is good w.r.t this subtree
hist is bad w.r.t this subtree
hist is bad w.r.t this tree

(b)

Fig. 2. Pictorial depiction of the proof of Theorem 2. Figures (a) and (b) correspond to Lemma 2 and Lemma 3, respectively. In Figure (b), the prefix $(r_1, p_1, ..., p_{k-1})$ is bad w.r.t. the entire tree, which implies that there is a subtree for which this session prefix is bad; however, the prefix is good w.r.t. the subtree that shares the same prefix strategy $[V_{[1,k-1]}, V^i_k]$ for which $V^*_k \overset{\mathrm{f}}{=} V^i_k$.

$$\begin{aligned}
&\Pr_{\substack{\mathbf{r}\leftarrow\{0,1\}^l\\ R\leftarrow\{0,1\}^r}}[V^*\leftarrow C(\mathbf{r}):E] \qquad (1)\\
=\ &\Pr_{\substack{\mathbf{r}\leftarrow\{0,1\}^l\\ R\leftarrow\{0,1\}^r}}[V^*\leftarrow C(\mathbf{r}):E|\mathbf{r}\in Q]\Pr[\mathbf{r}\in Q]\\
&+\Pr_{\substack{\mathbf{r}\leftarrow\{0,1\}^l\\ R\leftarrow\{0,1\}^r}}[V^*\leftarrow C(\mathbf{r}):E|\mathbf{r}\notin Q]\Pr[\mathbf{r}\notin Q]\\
\le\ &\Pr[\mathbf{r}\in Q]+(1-\frac{1}{\mathsf{poly}(n)})\Pr[\mathbf{r}\notin Q]\\
=\ &\frac{|Q|}{2^l}+(1-\frac{1}{\mathsf{poly}(n)})(1-\frac{|Q|}{2^l})\\
=\ &1-\frac{1}{\mathsf{poly}(n)}(1-\frac{|Q|}{2^l}).
\end{aligned}$$

Given that the probability in expression (1) is greater than $1-\mathsf{neg}(n)$, so is the quantity $\frac{|Q|}{2^l}$.

Thus, given $x\notin L$, for any unbounded prover, the cheating probability, taken over the choices of verifier random tapes in Q, is still negligible. Applying now Lemma 1, we can find q random tapes $\mathbf{r}_i\in Q$, $1\le i\le q$, such that the probability, taken over these q random tapes, that the unbounded prover makes the verifier accept is at most $1-1/q$. This completes the proof of the lemma. □

The next lemma, where Lemma 2 is used, is the key step in establishing our main theorem.

Lemma 3. *Let $\langle P,V\rangle$ be as above. Fix the infinite set I and the sequence of false statements $x\notin L$ guaranteed by lemma 2. Then there exists a triplet $(k,\mathcal{V}_{[1,k-1]},V'_{[1,k-1]})$, where:*

- $2\le k\le m$;
- *$\mathcal{V}_{[1,k-1]}$ is a subset of verifiers that share the same prefix strategy $V_{[1,k-1]}$ but have t distinct k-th step strategies $V_k^1,V_k^2,...,V_k^t$, denoted by $\mathcal{V}_k$ (we let $\mathcal{V}^i_{[1,k]}$ denote the subset of verifiers in $\mathcal{V}_{[1,k-1]}$ that share the same prefix strategy $[V_{[1,k-1]},V_k^i]$); and*
- *$V'_{[1,k-1]}$ is a prefix verifier code functionally equivalent to $V_{[1,k-1]}$,*

such that, for any $1\le i\le t$ and any polynomial-time constructible code V_k^* satisfying $V_k^*\overset{\mathrm{f}}{=}V_k^i$, $(S([V'_{[1,k-1]},V_k^*])$ $\Leftrightarrow[V'_{[1,k-1]},V_k^*])$ will generate a session prefix $(r_1,p_1,...,p_{k-1})$ satisfying the following two conditions:

1. $(r_1,p_1,...,p_{k-1})$ *is* bad *with respect to $\mathcal{V}_{[1,k-1]}$;*
2. $(r_1,p_1,...,p_{k-1})$ *is* good *with respect to $\mathcal{V}^i_{[1,k]}$.*

Fix a security parameter n and a false statement $x\notin L$ of length n in the sequence guaranteed by Lemma 2. We prove the lemma by examining the next-message functions of the q honest verifiers $V^1,V^2,...V^q$ guaranteed by Lemma 2, step by step. At a high level, the structure of the proof is as follows:

1. First, show that there exists a triplet $(2,\mathcal{V}_{[1]},V'_{[1]})$ satisfying condition 1.
2. Show that any $(m-1,\mathcal{V}_{[1,m-1]},V'_{[1,m-1]})$ satisfies condition 2.
3. Show that, for any $2\le k\le m-1$, if a given $(k,\mathcal{V}_{[1,k-1]},V'_{[1,k-1]})$ satisfes condition 1, but not condition 2, then we have a triplet $(k+1,\mathcal{V}_{[1,k]},V'_{[1,k]})$ that satisfies condition 1.

This reasoning guarantees that we can find a triplet $(k,\mathcal{V}_{[1,k-1]},V'_{[1,k-1]})$, for some $2\le k\le m$, which satisfies both conditions. The detailed proof of the above three steps is presented in Appendix B.2.

We are now ready to construct the distinguishing algorithm U, yielding the proof of the theorem. Fix a false statement x in the sequence guaranteed by Lemma 2, and k, $\mathcal{V}_{[1,k-1]}$, $\mathcal{V}_k$, and $V'_{[1,k-1]}$ as in Lemma 3.

Let the output of an arbitrary PPT algorithm C on input $(\mathcal{V}_k, i)$ for random i be V_k^* such that $V_k^* \overset{\text{f}}{=} V_k^i \in \mathcal{V}_k$. Algorithm U works as follows.[16]

The *distinguishing* algorithm U.

Input to U: $\mathcal{V}_{[1,k-1]}$, $\mathcal{V}_k$, $(r_1, p_1, ..., p_{k-1}) \leftarrow (S([V'_{[1,k-1]}, V_k^*]) \Leftrightarrow V'_{[1,k-1]})$ and an initially empty set T.

1. For each j, $1 \leq j \leq t$, exhaust all possible prover messages after the k-th verifier step, checking if the session prefix $(r_1, p_1, ..., p_{k-1})$ is good with respect to $\mathcal{V}_k^j$. If not, add j to set T.
2. Output an arbitrary j in T.

As mentioned before, in its second step, U can check whether the given session prefix is good in time $2^{O(p)}$, which overwhelmingly dominates its the running time.

Condition 1 of Lemma 3 guarantees that there exists j such that the session prefix $(r_1, p_1, ..., p_{k-1})$ produced in U's step 1 is *bad* with respect to $\mathcal{V}_{[1,k-1]}^j$, which implies that T is not empty. Condition 2 of Lemma 3 guarantees that if $(r_1, p_1, ..., p_{k-1})$ is bad with respect to $\mathcal{V}_{[1,k-1]}^j$, then $V_k^* \overset{\text{f}}{\neq} V_k^j$. In other words, algorithm U was able to pin-point a program (V_k^j) functionally *different* from V_k^*. This concludes the proof of Theorem 2.

6 Conclusions

A natural question which arises from our reduction is: How hard is the functionality-predicting problem (Definition 7)? As mentioned before, since our predicting algorithm U does not take the target code V_k^* as input, the simulator must encode some non-trivial functionality of V_k^* in the session prefix $(r_1, p_1, ..., p_{k-1})$. However, if the simulator runs in a straight-line manner such as Barak's [2], it does not execute V_k^* in computing the history prefix prior to the verifier's k-th step, and this means it is able to discern some non-trivial property of V_k^*'s functionality and encode it in the session prefix $(r_1, p_1, ..., p_{k-1})$ without executing V_k^*, which seems to be highly unlikely. We leave the exact characterization of this problem's hardness as an interesting research question.

Since, as also argued in the introduction, rewinding seems to be out of the picture, this leads us to think of our main theorem as strong evidence against the existence of such proof systems, and safely conclude that constructing non-trivial constant-round public-coin ZK proofs (if they exist) requires a paradigm-shifting simulation technique.

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References

[1] L. Babai: Trading Group Theory for Randomness. STOC 1985, pp. 421-429, 1985.

[2] B. Barak: How to go beyond the black-box simulation barrier. FOCS 2001, pp.106-115.

[3] G. Brassard, D. Chaum, and C. Crépeau: Minimum disclosure proofs of knowledge. J. Comput. Syst. Sci., 37(2):156-189, 1988.

[4] N. Bitansky, D. Dachman-Soled, S. Garg, A. Jain, Y. Kalai, A. Lpez-Alt, D. Wichs: Why "Fiat-Shamir for Proofs" Lacks a Proof. TCC 2013: 182-201.

[5] B. Barak, O. Goldreich, S. Goldwasser, Y. Lindell: Resettably sound ZK and its Applications. FOCS 2001, pp. 116-125, 2001.

[6] B. Barak, O. Goldreich, R. Impagliazzo, S. Rudich, A. Sahai, S. P. Vadhan, K. Yang: On the (Im)possibility of Obfuscating Programs. CRYPTO 2001, pp.1-18, 2001.

[7] B. Barak, Y. Lindell: Strict polynomial-time in simulation and extraction. STOC 2002, pp.484-493. 2002.

[8] M. Blum: How to prove a theorem so no one else can claim it. Proceedings of theInternational Congress of Mathematicians, pp.444-451, 1986.

[16] Keep in mind that we omit inputs x and randomness to S and U for simplicity.

[9] B. Barak, Y. Lindell, S. P. Vadhan: Lower Bounds for Non-Black-Box Zero Knowledge. FOCS 2003, pp.384-393,2003.

[10] L. Babai, S. Moran: Arthur-Merlin Games: A Randomized Proof System, and a Hierarchy of Complexity Classes. J. Comput. Syst. Sci. 36(2): 254-276, 1988.

[11] M. Bellare, J. Rompel: Randomness-Efficient Oblivious Sampling. FOCS 1994, pp.276-287.

[12] R. Canetii, Y. Chen and L. Reyzin: On the Correlation Intractability of Obfuscated Pseudorandom Functions. TCC(A) 2016, pp.389-415.

[13] R. Canetti, O. Goldreich, S. Goldwasser, S. Micali. Resettable Zero Knowledge. STOC 2000, pp.235-244, 2000.

[14] R. Canetti, J. Kilian, E. Petrank and A. Rosen. Concurrent Zero-Knowledge requires $\Omega(logn)$ rounds. STOC 2001, pp.570-579, 2001.

[15] I. Damgaard. Efficient Concurrent Zero-Knowledge in the Auxiliary String Model. EUROCYPT 2000, pp.174-187, 2000.

[16] Y. Deng, V. Goyal, A. Sahai: Resolving the Simultaneous Resettability Conjecture and a New Non-Black-Box Simulation Strategy. FOCS 2009, pp.251-260.

[17] G. Di Crescenzo, Ivan Visconti. Concurrent ZK in the Public-Key Model. ICALP 2005, pp.816-827, 2005.

[18] C. Dwork, M. Naor and A. Sahai. Concurrent Zero-Knowledge. STOC 1998, pp.409-418, 1998.

[19] U. Feige, D. Lapidot and A. Shamir: Multiple Non-Interactive Zero Knowledge Proofs Under General Assumptions. SIAM J. on Computing 29 (1999) 1C28.

[20] O. Goldreich: The Foundations of Cryptography - Volume 1, Basic Techniques. Cambridge University Press 2001.

[21] Sanjam Garg, Craig Gentry, Shai Halevi, Mariana Raykova, Amit Sahai and Brent Waters: Candidate Indistinguishability Obfuscation and Functional Encryption for all Circuits. FOCS 2013, pp.40-49.

[22] O. Goldreich and H. Krawczyk: On the Composition of Zero-Knowledge Proof Systems. SIAM J. Comput. 25(1), pp.169-192, 1996.

[23] O. Goldreich, S. Micali and A. Wigderson. Proofs that yield nothing but their validity or All languages in NP have zero-knowledge proof systems. J. ACM, 38(3), pp.691-729, 1991.

[24] S. Goldwasser, S. Micali, and C. Rackoff. The knowledge complexity of interactive proof systems. SIAM. J. Computing, 18(1):186-208, February 1989.

[25] O. Goldreich, S. Vadhan and A. Wigderson. On Interactive Proofs with a Laconic Prover. ICALP 2001, pp. 334-345.

[26] S. Hada: Zero-Knowledge and Code Obfuscation. ASIACRYPT 2000, pp.443-457, 2000.

[27] R. Pass, A. Rosen: New and improved constructions of non-malleable cryptographic protocols. STOC 2005: 533-542.

[28] M. Prabhakaran, A. Rosen, A. Sahai: Concurrent Zero Knowledge with Logarithmic Round-Complexity. FOCS 2002, pp.366-375, 2002.

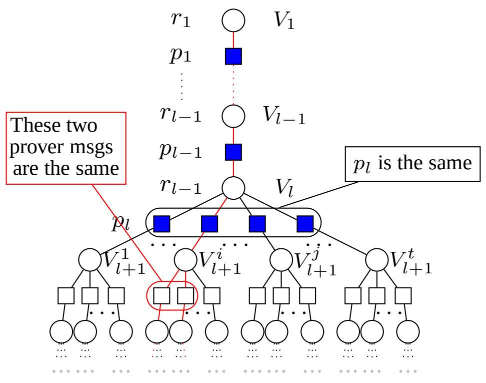
A Good/Bad/Session Prefixes (cont'd)

In this section we provide a more detailed formulation of the notion, as well as an illustration of good/bad prefixes. Recall Definition 5. Equivalently, we call a session prefix $(r_1, p_1, ..., p_\ell)$ "good" with respect to $\mathcal{V}_{[1,\ell]}$ if the following holds, which can be decided in time exponential in the length of the prover's messages. Let poly be the size of $\mathcal{V}_{[1,\ell]}$. Then there are poly number of session continuations of the form $(r_{\ell+1}, ..., p_m)$, each assigned to a verifier in $\mathcal{V}_{[1,\ell]}$, such that the following conditions hold:

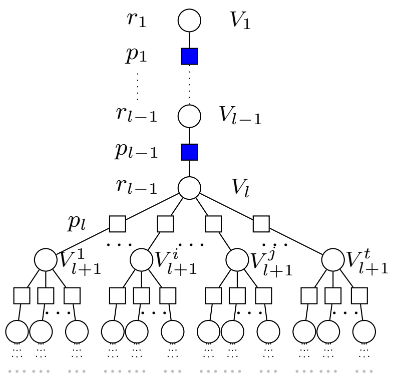
1. Every verifier in $\mathcal{V}_{[1,\ell]}$ will accept the transcript $(r_1, p_1..., p_\ell, r_{\ell+1}, ..., p_m)$ assigned to it.
2. If two verifiers in $\mathcal{V}_{[1,\ell]}$ share the same prefix strategy up to the ℓ'-th step, $\ell \le \ell' \le m$, then the two transcripts assigned to them share the same session prefix $(r_1, p_1..., r_{\ell'}, p_{\ell'})$.

A good session prefix is pictorially depicted in Figure 1(a). In the figure, if $(r_1, p_1, ...p_\ell)$ is good with respect to the tree, then for every edge below V_ℓ, we can assign a prover message to it such that: (1) each path is accepting, and (2) for every two paths that share the same prefix strategy up to the ℓ'-th verifier step, $\ell \le \ell' \le m$ (e.g., the red paths), the session prefixes of these two paths up to the ℓ'-th prover step are the same.

In addition, one can easily verify the following "robustness" fact about a good session prefix: if a session prefix $(r_1, p_1, ..., p_\ell)$ is good with respect to $\mathcal{V}_{[1,\ell]}$, then for any $1 \le i \le \ell$, the session prefix $(r_1, p_1, ..., p_i)$ is also good with respect to $\mathcal{V}_{[1,\ell]}$. See Figure 1(b). The figure illustrates the fact that if $(r_1, p_1, ..., p_{\ell-1}, r_\ell, p_\ell)$ is good, so is the (sub)prefix $(r_1, p_1, ..., p_{\ell-1})$ with respect to the same tree. This is because all prover messages on edges below V_ℓ (including p_ℓ) simply satisfy the two conditions that make a session prefix good.



(a)



(b)

Fig. 3. A *good* session prefix (a) and its robustness (b). Each node (circle) represents a verifier next-message function, or equivalently (in our case of public-coin proof systems), a random string that is used in this step. Each path represents a (complete) interaction with an honest verifier.

B Proofs

B.1 Proof of Lemma 1

We first introduce some definitions and additional notation that will be used in the proof.

We assume that the length of each prover message is greater than any constant, in particular, $p > 10$. Note that this assumption is without loss of generality because if the length of the prover message in a constant-round interactive proof for a language L is constant, then L is trivial (see [25]), which in turn implies our lemma immediately.

Throughout this subsection, we consider only *structured* q-tuples of verifier's random tapes, which are selected in the following way:

1. For each verifier step i, $1 \le i \le m$, if $|\{0,1\}^{l_i}| > \frac{m^2 p}{2\log\frac{1}{\epsilon}}$, set $t_i = \frac{m^2 p}{2\log\frac{1}{\epsilon}} \in p/O(\log\frac{1}{\epsilon})$; otherwise, set $t_i = 2^{l_i}$, where l_i is the length of the i-th verifier message;
2. Choose t_i *distinct* strings $r_{1i}, r_{2i}, ..., r_{t_i i}$ from $\{0,1\}^{l_i}$;
3. Choose an i-th verifier message $r_{j_i i} \in (r_{1_i i}, r_{2i}, ..., r_{t_i i})$, $1 \le j_i \le t_i$ for each step i, and set random tape $\mathbf{r}^j = [r_{j_1 1}, r_{j_2 2}, ..., r_{j_m m}]$.
4. A q-tuple of random tapes is now the set of all possible random tapes set in step 3, $(\mathbf{r}^1, \mathbf{r}^2, ..., \mathbf{r}^q)$. Note that the size q of this set is $\prod_{i=1}^m t_i$, which is determined by Step 2.

We identify $(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q)$ with $(\mathbf{r}^{\pi(1)}, \mathbf{r}^{\pi(2)}, ...\mathbf{r}^{\pi(q)})$ for any permutation π on $\{1, 2, ..., q\}$. Two q-tuples, $(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q)$ and $(\mathbf{r'}^1, \mathbf{r'}^2, ...\mathbf{r'}^q)$, are said to be *distinct* if there exists at least one $\mathbf{r}^i$ such that $\mathbf{r}^i \in (\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q)$ but $\mathbf{r}^i \notin (\mathbf{r'}^1, \mathbf{r'}^2, ...\mathbf{r'}^q)$, or vice-versa. Thus the number of all possible distinct such structured q-tuples is

$$\prod_{i=1}^{m} \binom{2^{l_i}}{t_i}.$$

Some more basic notation before the proof:

- $\text{prefix}_i(\mathbf{r}^j)$: the first i messages from the verifier using random tape $\mathbf{r}^j$, that is, for $\mathbf{r}^j = [r_1^j, r_2^j, ..., r_m^j]$, $\text{prefix}_i(\mathbf{r}^j) = [r_1^j, r_2^j, ..., r_i^j]$.
- $\overrightarrow{T}$ and its size $|\overrightarrow{T}|$: $\overrightarrow{T}$ is a *set* of structured q-tuples of verifier's random tapes, The size of $\overrightarrow{T}$, denoted by $|\overrightarrow{T}|$, is simply defined to be the number of *distinct* q-tuples in $\overrightarrow{T}$.

– $p_k \leftarrow \langle P(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q), V\rangle_{|\mathsf{hist}}$ denotes the k-th prover message produced by the prover $P(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q)$ (the prover strategy taking q-tuple $(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q)$ as auxiliary input), conditioned on hist being the current history so far.

The proof of the lemma is by contradiction. Assume that there exists an unbounded prover, call it $P^\diamond$, and $x \notin L$, such that for any q-tuple $(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q)$, $r_i \neq r_j$ for $i \neq j$:

$$\Pr[\langle P^\diamond(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q), V_{|(\mathbf{r}^1,\mathbf{r}^2,...\mathbf{r}^q)}\rangle(x) = 1] > 1 - \frac{1}{q} \; . \tag{2}$$

First note that $V_{|(\mathbf{r}^1,\mathbf{r}^2,...,\mathbf{r}^t)}(\mathbf{r}^i)$ acts exactly the same as $V(\mathbf{r}^i)$. Therefore

$$\Pr[\langle P^\diamond(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q), V_{|(\mathbf{r}^1,\mathbf{r}^2,...\mathbf{r}^q)}\rangle(x) = 1] \tag{3}$$

$$= \sum_i \Pr[\langle P^\diamond(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q), V_{|(\mathbf{r}^1,\mathbf{r}^2,...\mathbf{r}^q)}(\mathbf{r}^i)\rangle(x) = 1]\frac{1}{q} \tag{4}$$

$$= \sum_i \Pr[\langle P^\diamond(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q), V(\mathbf{r}^i)\rangle(x) = 1]\frac{1}{q} \; . \tag{5}$$

Further, observe that the probability $\Pr[\langle P^\diamond(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q), V(\mathbf{r}^i)\rangle(x) = 1]$ is either 0 or 1 because in this interaction the tapes are fixed and both prover and verifier are deterministic. Thus, if inequality (2) holds, we have

$$\Pr[\langle P^\diamond(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q), V_{|(\mathbf{r}^1,\mathbf{r}^2,...\mathbf{r}^q)}\rangle(x) = 1] = 1 \; , \tag{6}$$

and, by (5),

$$\Pr[\langle P^\diamond(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q), V(\mathbf{r}^i)\rangle(x) = 1] = 1 \; . \tag{7}$$

Now, given prover $P^\diamond$ such that (7) holds for any q-tuple $(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q)$, we describe a prover P^* that will cheat V with probability greater than ϵ.

The Cheating Prover P^*.
Input: x, as in inequality (2).

1. Set $\overrightarrow{T^0}$ to be the set of all possible distinct structured q-tuples over $\{0,1\}^{l_1+l_2+...+l_m}$, and G_1 the set of all possible first verifier's messages (i.e., the set $\{0,1\}^{l_1}$).
2. For $k = 1$ to m, do
   2.1. Upon receiving the k-th verifier message r_k, set hist to be the current history $(r_1, p_1^*, ..., r_k)$.
   Check if $r_k \in G_k$. If $r_k \notin G_k$, abort and output "$\perp$". Otherwise, for every q-tuple $(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q) \in \overrightarrow{T^0}$ such that: a) it contains some $\mathbf{r}^i$ such that $\text{prefix}_{k-1}(\mathbf{r}^i) = [r_1, r_2, ..., r_k]$, and, b) the current hist is consistent with the interaction between $P^\diamond(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q)$ and V, set $t' = \prod_{k+1}^m t_i$, compute the k-th prover message by running $P^\diamond(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q)$, and obtain the set of k-th prover messages

$$\{p_k \leftarrow \langle P^\diamond(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q), V\rangle_{|\mathsf{hist}} :$$
$$(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q) \in \overrightarrow{T^0} \text{ and } \exists(\mathbf{r}^{i_1}, \mathbf{r}^{i_2}, ...\mathbf{r}^{i_{t'}}) \in (\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q) \text{ s.t.}$$
$$\text{prefix}_k(\mathbf{r}^{i_j}) = [r_1, r_2, ..., r_k] \text{ for all } 1 \leq j \leq t' = \prod_{k+1}^{m} t_i\}^{17} \; .$$

[17] Observe that, by the structure of q-tuple, if there exists a $\mathbf{r}^i \in (\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q)$ such that $\text{prefix}_k(\mathbf{r}^i) = [r_1, r_2, ..., r_k]$, then there exist $t' = \prod_{k+1}^m t_i$ many such random tapes.

Set p_k^* to be the p_k that maximizes the size of the set

$$\{(\mathbf{r}^{i_1}, \mathbf{r}^{i_2}, ...\mathbf{r}^{i_{t'}}) : p_k \leftarrow \langle P^{\diamond}(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q), V\rangle_{|\mathsf{hist}}, \text{ and}$$

$$(\mathbf{r}^{i_1}, \mathbf{r}^{i_2}, ...\mathbf{r}^{i_{t'}}) \in (\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q), \text{ and}$$

$$\text{prefix}_k(\mathbf{r}^{i_j}) = [r_1, r_2, ..., r_k] \text{ for all } 1 \leq j \leq t' = \prod_{k+1}^{m} t_i\}$$

2.2. If $k < m$, denote by $\overrightarrow{T^k}$ the above set that achieves its maximum size, and set (guessing the next verifier messages)

$$G_{k+1} \leftarrow \{r_{k+1} \in \{0,1\}^{l_{k+1}} : |\{(\mathbf{r}^{i_1}, \mathbf{r}^{i_2}, ...\mathbf{r}^{i_{t'}}) \in \overrightarrow{T^k} :$$

$$\text{prefix}_{k+1}(\mathbf{r}^{i_j}) = [r_1, r_2, ..., r_k, r_{k+1}] \text{ for all } 1 \leq j \leq t' = \prod_{k+1}^{m} t_i\}| \geq \frac{\prod_{i=k+1}^{m} \binom{2^{l_i}}{t_i}}{2^{1.1kp}}\}.$$

In a nutshell, the above algorithm just tries many different cheating provers $P^{\diamond}(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q)$ to make the current history accepted by as many verifiers as possible.

Analysis of algorithm P^*. Let us now analyze the success probability of the prover's strategy outlined above. We first show that the size of G_k is large enough for every k.

Claim. For every $1 \leq k \leq m$, *conditioned on* P^* *not outputting* $\perp$, $|G_k| \geq \frac{2^{l_k}}{2^{1.1kp/t_k}e}$.

Proof. When $k = 1$, $|G_1| = |\{0,1\}^{l_1}| > \frac{2^{l_k}}{2^{1.1kp/t_k}e}$.

When $k \geq 2$, the condition of P^* not outputting "$\perp$" implies that, for $j \leq k$, r_j is in G_j, and that

$$|\{(\mathbf{r}^{i_1}, \mathbf{r}^{i_2}, ...\mathbf{r}^{i_{t''}}) \in \overrightarrow{T^{k-1}} : \text{prefix}_k(\mathbf{r}^{i_j}) = [r_1, r_2, ..., r_k] \text{ for all } 1 \leq j \leq t'' = \prod_{k}^{m} t_i\}| \geq$$

$$\frac{\prod_{i=k}^{m} \binom{2^{l_i}}{t_i}}{2^{1.1(k-1)p}}.$$

which in turn leads to (recall that the length of prover messages is p), for $k \geq 2$,

$$|\overrightarrow{T^k}| \geq \frac{\prod_{i=k}^{m} \binom{2^{l_i}}{t_i}}{2^{1.1(k-1)p+p}} = \frac{\prod_{i=k}^{m} \binom{2^{l_i}}{t_i}}{2^{1.1kp-0.1p}}. \tag{8}$$

Now assume that, for $k \geq 2$, conditioned on P^* not outputting "$\perp$" (i.e., for $j \leq k$, r_j is in G_j), $|G_k| < \frac{2^{l_k}}{2^{1.1kp/t_k}e}$.

Set $t' = \prod_{k+1}^{m} t_i$. Recall that all t'-tuples $(\mathbf{r}^{i_1}, \mathbf{r}^{i_2}, ..., \mathbf{r}^{i_{t'}}) \in \overrightarrow{T^k}$ share the same prefix $[r_1, r_2, ..., r_k]$, and that, by the structure of q-tuple of random tapes, within a t'-tuple $(\mathbf{r}^{i_1}, \mathbf{r}^{i_2}, ..., \mathbf{r}^{i_{t'}}) \in \overrightarrow{T^k}$, there are only t_k distinct k-th verifier messages, say $(r_k^1, r_k^2, ..., r_k^{t_k})$. We partition these t'-tuples in $\overrightarrow{T^k}$ in two classes by the property of $(r_k^1, r_k^2, ..., r_k^{t_k})$:

1. Every $r_k^i \in (r_k^1, r_k^2, ..., r_k^{t_k})$ is in G_k (which implies $t_k \leq |G_k|$). The number of t'-tuples in $\overrightarrow{T^k}$ satisfying this condition is at most

$$\binom{|G_k|}{t_k} \prod_{i=k+1}^{m} \binom{2^{l_i}}{t_i}.$$

2. There is at least one $r_k^i \in (r_k^1, r_k^2, ..., r_k^{t_k})$ that is *not* in G_k. Then by the definition of G_k, and by the fact that, within a t'-tuple $(\mathbf{r}^{i_1}, \mathbf{r}^{i_2}, ..., \mathbf{r}^{i_{t'}}) \in \overrightarrow{T^k}$, for every i, the number of random tapes in this t'-tuple with each prefix $[r_1, r_2, ..., r_{k-1}, r_k^i]$ is the same (equal to $\prod_{k+1}^m t_i$), then the number of t'-tuples in $\overrightarrow{T^k}$ satisfying this condition is at most
$$\binom{2^{l_k}}{t_k}\frac{\prod_{i=k+1}^m \binom{2^{l_i}}{t_i}}{2^{1.1kp}}.$$
Thus, we have
$$\begin{aligned}|\overrightarrow{T^k}| &\leq \binom{|G_k|}{t_k}\prod_{i=k+1}^m \binom{2^{l_i}}{t_i} + \binom{2^{l_k}}{t_k}\frac{\prod_{i=k+1}^m \binom{2^{l_i}}{t_i}}{2^{1.1kp}} \\ &< \binom{\frac{2^{l_k}}{2^{1.1kp/t_k}e}}{t_k}\prod_{i=k+1}^m \binom{2^{l_i}}{t_i} + \binom{2^{l_k}}{t_k}\frac{\prod_{i=k+1}^m \binom{2^{l_i}}{t_i}}{2^{1.1kp}} \\ &< (\frac{2^{l_k}}{2^{1.1kp/t_k}t_k})^{t_k}\prod_{i=k+1}^m \binom{2^{l_i}}{t_i} + \binom{2^{l_k}}{t_k}\frac{\prod_{i=k+1}^m \binom{2^{l_i}}{t_i}}{2^{1.1kp}} \\ &< \frac{(\frac{2^{l_k}}{t_k})^{t_k}}{2^{1.1kp}}\prod_{i=k+1}^m \binom{2^{l_i}}{t_i} + \binom{2^{l_k}}{t_k}\frac{\prod_{i=k+1}^m \binom{2^{l_i}}{t_i}}{2^{1.1kp}} \\ &< \frac{\prod_{i=k}^m \binom{2^{l_i}}{t_i}}{2^{1.1kp-1}},\end{aligned}$$

which contradicts (8) when $p > 10$, which we can always assume without loss of generality (otherwise our lemma holds trivially; see [25]). □

Now observe that, for every prover step $k \leq m$, if $G_k \geq \frac{2^{l_k}}{2^{1.1kp/t_k}e}$, then the probability that P^* guesses the next verifier message correctly, i.e., the probability that $r_k \in G_k$, is $|G_k|/2^{l_k} = \frac{1}{2^{1.1kp/t_k}e}$. Therefore P^* guesses all the next verifier messages correctly with probability at least
$$\prod_{k=1}^m \frac{|G_k|}{2^{l_k}} = \prod_{k=1}^m \frac{1}{2^{1.1kp/t_k}e},$$
which is greater than ϵ for $t_k \leq \frac{m^2p}{2\log\frac{1}{\epsilon}}$. (Recall that either $t_k = \frac{m^2p}{2\log\frac{1}{\epsilon}}$, or $t_k = 2^{l_k}$ when $2^{l_k} \leq \frac{m^2p}{2\log\frac{1}{\epsilon}}$.)

Notice also that, in case that all guesses of the next verifier messages are correct, there exists at least one q-tuple $(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q)$ such that the complete transcript $(r_1, p_1^*...r_m, p_m^*)$ is generated in the interaction between $P^\diamond(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q)$ and $V(\mathbf{r}^i)$, $\mathbf{r}^i = [r_1, r_2...r_m] \in (\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q)$, which is guaranteed by our assumption to be accepting.

In sum, our cheating prover P^* will cheat with probability greater than ϵ, which breaks the soundness of the proof system $\langle P, V\rangle$, thus yielding the lemma.

B.2 Proof of Lemma 3

Fix a security parameter n and a false statement $x \notin L$ of length n in the sequence guaranteed by lemma 2. We prove the lemma by examining the next-message functions of the q honest verifiers $V^1, V^2, ...V^q$ guaranteed by Lemma 2, step by step. Recall that the structure of the proof is as follows:

1. First, show that there exists a triplet $(2, \mathcal{V}_{[1]}, V'_{[1]})$ satisfying condition 1.

2. Show that any $(m-1, \mathcal{V}_{[1,m-1]}, V'_{[1,m-1]})$ satisfies condition 2.
3. Show that, for any $2 \leq k \leq m-1$, if a given $(k, \mathcal{V}_{[1,k-1]}, V'_{[1,k-1]})$ satisfes condition 1, but not condition 2, then we have a triplet $(k+1, \mathcal{V}_{[1,k]}, V'_{[1,k]})$ that satisfies condition 1.

This reasoning guarantees that we can find a triplet $(k, \mathcal{V}_{[1,k-1]}, V'_{[1,k-1]})$, for some $2 \leq k \leq m$, which satisfies both conditions. We now turn to proving the above three steps.

The proof of step 1 is as follows. By Lemma 2, no unbounded prover can cheat a random verifier from set $\{V^1, V^2, ...V^q\}$ with probability 1. This immediately means (recall that we assume that verifier sends the first message in a session) that there exists V_1 such that no unbounded prover can cheat a random verifier having the same prefix strategy V_1 chosen from $\{V^1, V^2, ...V^q\}$ with probability 1.

Thus, we can have $(2, \mathcal{V}_{[1]}, V'_{[1]})$, where $\mathcal{V}_{[1]}$ is the set of verifiers in $\{V^1, V^2, ...V^q\}$ having the same prefix strategy V_1 and the code V'_1 is V_1. By the structure of these q verifiers, we have that $\mathcal{V}_{[1]}$ has a set $\mathcal{V}_2$ of t distinct second-step strategies $V_2^1, V_2^2, ..., V_2^t$. It is easy to see that the first condition of the lemma now holds, as otherwise, if there exists an i and code $V_k^* \overset{\mathrm{f}}{=} V_2^i$, such that the session prefix $(r_1, p_1) \leftarrow (S([V'_1, V_2^*]) \Leftrightarrow [V'_1, V_2^*])$ is good with respect to $\mathcal{V}_{[1]}$, then the following unbounded prover with auxiliary input $[V'_1, V_2^*]$ and $\mathcal{V}_{[1]}$ will cheat a random verifier V in $\mathcal{V}_{[1]}$ with probability 1: Upon receiving the first verifier message (produced by V_1), it runs $S([V'_1, V_2^*])$, obtains p_1, and then runs the residual prover strategy guaranteed to exist by the definition of a good session prefix (Definition 5) to complete the interaction with V.

Step 2 is guaranteed by Lemma 2. Given any $(m-1, \mathcal{V}_{[1,m-1]}, V'_{[1,m-1]})$, where $\mathcal{V}_{[1,m-1]}$ shares the same prefix strategy $V_{[1,m-1]}$ but has t distinct m-th step strategies $V_m^1, V_m^2, ..., V_m^t$, and $V'_{[1,m-1]} \overset{\mathrm{f}}{=} V_{[1,m-1]}$, the reason for this triplet satisfying condition 2 is that, for any i and any polynomial-time constructible code V_m^* satisfying $V_m^* \overset{\mathrm{f}}{=} V_m^i$, the session prefix $(r_1, p_1, ..., p_m) \leftarrow (S([V'_{[1,m-1]}, V_m^*]) \Leftrightarrow [V'_{[1,m-1]}, V_m^*])$ must be good since $(r_1, p_1, ..., p_m)$ is, by the property of the simulator guaranteed by Lemma 2, an accepting and complete transcript.

We now prove step 3 using the canonical property of ZK proofs (Definition 6). Assume there is a triplet $(k, \mathcal{V}_{[1,k-1]}, V'_{[1,k-1]})$, $2 \leq k \leq m-1$ (where again $\mathcal{V}_{[1,k-1]}$ shares the same prefix strategy $V_{[1,k-1]}$ but has t distinct k-th step strategies $V_k^1, V_k^2, ..., V_k^t$, and $V'_{[1,k-1]} \overset{\mathrm{f}}{=} V_{[1,k-1]}$), which satisfies condition 1, but not condition 2. Note that conditioned on not satisfying condition 2, we have an i such that for any code $V_k^* \overset{\mathrm{f}}{=} V_k^i$, the session prefix $(r_1, p_1, ..., p_k) \leftarrow (S([V'_{[1,k-1]}, V_k^*]) \Leftrightarrow [V'_{[1,k-1]}, V_k^*])$ is *bad* with respect to the set of verifiers in $\mathcal{V}_{[1,k]}$ having the same prefix strategy $[V_{[1,k-1]}, V_k^i]$ (again, $\mathcal{V}_{[1,k]}$ has t distinct $(k+1)$-th step strategies $V_{k+1}^1, V_{k+1}^2, ..., V_{k+1}^t$).

By setting $V'_{[1,k]}$ to be $[V'_{[1,k-1]} and V_k^*]$, $\mathcal{V}_{[1,k]}$ as above, we now have a triplet $(k+1, \mathcal{V}_{[1,k]}, V'_{[1,k]})$ for which the condition 1 holds, for the following reason: Assume otherwise, i.e., that there exist V_{k+1}^i, and a code $V_{k+1}^* \overset{\mathrm{f}}{=} V_{k+1}^i$ such that $(r_1, p'_1, ..., p'_k) \leftarrow (S([V'_{[1,k]}, V_{k+1}^*]) \Leftrightarrow [V'_{[1,k]}, V_{k+1}^*])$ is good with respect to $\mathcal{V}_{[1,k]}$. Then, by the canonical property (Definition 6), $(r_1, p_1, ..., p_k) \leftarrow (S(V'_{[1,k]}) \Leftrightarrow V'_{[1,k]})$ is also good with respect to $\mathcal{V}_{[1,k]}$, which contradicts the assumption that $(k, \mathcal{V}_{[1,k-1]}, V'_{[1,k-1]})$ does not satisfy condition 2.

C Interactive Proof Systems with Super-Constant Rounds

In this section we give a simple super-constant-round public-coin interactive proof system for which Lemma 1 does not hold.

Preamble: For $1 \leq k \leq s$, do:
$P \rightarrow V$: Send n random strings p_1^k,...,p_n^k of length n each.
$V \rightarrow P$: Send a random string r_k of length n.
Main proof: If there is some $p_i^k = r_k$, V accepts; otherwise execute a 3-round Blum protocol [8] with negligible soundness error.

Observe that for any q, if q different verifier random tapes $(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q)$ are fixed in advance and known to an all-powerful prover, then for the cheating probability to be strictly less than 1, at any verifier step $k \leq s$, given $(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q)$ and current history hist), there must be at least $n + 1$ possible different verifier next messages (i.e., the entropy $H(r_k|(\mathbf{r}^1, \mathbf{r}^2, ...\mathbf{r}^q), \mathsf{hist})$ is greater than $\log n$), which leads to $q \geq (n + 1)^s$. That is, if s is super-constant, for any polynomial number of verifier's random tapes that are fixed in advance we have a prover with cheating probability 1.