A Zero-One Law for Secure Multi-Party Computation with Ternary Outputs (full version)

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Abstract. There are protocols to privately evaluate any function in the passive (honest-but-curious) setting assuming that the honest nodes are in majority. For some specific functions, protocols are known which remain secure even without an honest majority. The seminal work by Chor and Kushilevitz [7] gave a complete characterization of Boolean functions, showing that each Boolean function either requires an honest majority, or is such that it can be privately evaluated regardless of the number of colluding nodes.
The problem of discovering the threshold for secure evaluation of more general functions remains an open problem. Towards a resolution, we provide a complete characterization of the security threshold for functions with three different outputs. Surprisingly, the zero-one law for Boolean functions extends to $\mathbb{Z}_3$, meaning that each function with range $\mathbb{Z}_3$ either requires honest majority or tolerates up to n colluding nodes.

1 Introduction

Multi-party secure function evaluation (SFE) is a cornerstone of modern cryptography, and has been extensively studied since it was introduced by Yao [14]. In this work we consider the joint evaluation by n parties of a public n-ary function f in such a way that no collusion of parties learns anything more than what they do by knowing their own inputs and seeing the output. We consider the *symmetric* case where all participants receive the same output.

Several models of adversaries occur in the SFE literature. A first distinction is whether the adversary has limited computational power (*computational security*) or not (*information-theoretic security*). A second important distinction is whether the parties corrupted by the adversary must still follow the protocol (*passive*) or not (*active*). In the present work, we are concerned with information-theoretic security and all adversaries considered are passive. We assume that the parties communicate over a complete network with *private channels*, meaning that the adversary cannot see messages sent between two honest parties.

Another important limitation put upon the adversary is which parties she can corrupt. The most common adversary is allowed to corrupt up to a threshold $t \leq n$ participants for some t which is typically a function of n. We say that a

function for which there is a protocol tolerating up to t corruptions is t-private. In this paper, we will only consider threshold adversaries. More general adversarial models have also been studied, both in terms of a more general specification of the parties the adversary can corrupt by Hirt and Maurer [10] and considering a mix active and passive adversarial corruptions by Beerliová-Trubíniová *et al.* [2].

There exist protocols to securely evaluate any function $\lfloor (n-1)/2 \rfloor$-privately in our setting by Ben-Or, Goldwasser, and Wigderson [3], and Chaum, Crépeau, and Damgård [4]. For some functions, in particular Boolean disjunction, this has been proved to be an upper bound meaning that there are no protocols to evaluate them which remain secure against more than $\lfloor (n-1)/2 \rfloor$ colluding parties. For other functions, in particular summation over a finite Abelian group, there are n-private protocols. This raises the question of determining the privacy threshold of functions.

Chor and Kushilevitz [7] completely answered the question for Boolean functions. They proved a zero-one law showing that each Boolean function is either $\lfloor (n-1)/2 \rfloor$-private (and not $\lceil n/2 \rceil$-private) or n-private. Their work presents a proof that a function containing an OR-like substructure (an *embedded* OR) is $\lfloor (n-1)/2 \rfloor$-private and that all Boolean functions without such a substructure can be computed by a single Boolean summation.

Proving that a function f cannot be t-privately computed is often done by a partition argument, reducing to the two-party case. In these proofs, the parties are partitioned into two parts of size $\leq t$ and we think of f as a two-party function with each party supplying all inputs for one set of the partition. If the two-party function is not 1-private, then f is not t-private. Chor and Ishai [6] analyzed partition arguments and gave a generalization partitioning the parties into $k > 2$ sets which increases the power of the framework. However, in this paper, we will only need partitioning arguments with two sets.

Chor, Geréb-Graus, and Kushilevitz [5] showed that for every t, $\lceil n/2 \rceil \leq t \leq n-2$ there exists a function such that it is t-private but not $(t+1)$-private. We remark that the functions they construct in their proofs have very large ranges which grow exponentially with t.

The privacy of symmetric[1] functions with Boolean arguments has been studied by Chor and Shani [9]. For such functions, they prove a necessary condition on the preimages of outputs for the function to be $\lceil n/2 \rceil$-private. They also define a class called dense symmetric functions where this necessary condition is also sufficient for n-privacy. Thus, they also prove a zero-one law where for a class of functions, where each function in the class is either n-private or not $\lceil n/2 \rceil$-private.

For two-party computation, a complete characterization of the 1-private functions was made independently by Beaver [1] and Kushilevitz [13]. They both show that a function f is 1-private if and only if it is decomposable, and for decomposable functions, there is a straightforward 1-private protocol. One of

[1] Here, symmetric means the standard notion of a symmetric function, not the SFE-specific notion that all parties receive the same output.

our protocols, Protocol 3, can be viewed as a generalization of the protocol for decomposable functions to the multi-party case.

Künzler, Müller-Quade, and Raub [12] give a combinatorial classification of functions computable in several different adversarial models, including the information-theoretic passive model which we work with in this paper. However, in this setting, they consider the broadcast model of communication which gives different results from private channels. For instance, summation is not n-private in the broadcast channel model.

1.1 Our Contribution

In this work, we extend the zero-one law of Boolean privacy to functions with three outputs. For notational convenience, we talk about functions with range $\mathbb{Z}_3$, but we would like to emphasize our results do not depend on any algebraic structure over the range of the function. More formally, we prove the following statement:

Theorem 1 (Main theorem). *For every n-argument function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$, f is either n-private, or it is $\lfloor (n-1)/2 \rfloor$-private and not $\lceil n/2 \rceil$-private.*

The core part of our proof is a structure lemma (Lemma 8) showing that every function f with range $\mathbb{Z}_3$ must have at least one of three properties (which we define more formally later):

- f has an embedded OR
- f is a permuted sum
- f is collapsible.

We provide protocols for n-privately evaluating those functions of the two latter types which do not contain an embedded OR.

Our definition of an embedded OR is a generalization of the one commonly found in the literature, but the presence of one implies that there is no protocol which can securely evaluate f and tolerate more than t colluding parties for some t (but potentially for a $t > \lceil n/2 \rceil$).

Finally, we prove (Theorem 22) that the existence of an embedded OR (in our generalized sense) also implies the existence of a "small" embedded OR, giving $t = \lceil n/2 \rceil$. By combining this result with our structure lemma and the result from [7] that a function with an embedded OR of size at most $\lceil n/2 \rceil$ cannot be $\lceil n/2 \rceil$-privately computed, our main theorem follows. We state the proof more formally in Section 6.

We remark that while our statements are true for $n = 2$, there are complete classifications [1,13] for the 2-party case which are simpler than ours (for $n = 2$, our protocols reduce to decomposition) and not limited to functions with range $\mathbb{Z}_3$. Our contribution lies in the case when $n \geq 3$.

The proof of our theorems are significantly more involved than the analogous proofs for Boolean functions. In several of our proofs we need to apply a fairly extensive case analysis.

Our result answers in part a question raised by Chor and Ishai [6] by showing that partition reductions (with only two sets) are universal for proving non-privacy of functions mapping to $\mathbb{Z}_3$.

2 Notation and Preliminary Theorems

We use boldface letters to refer to vectors, like: $\boldsymbol{x}$, $\boldsymbol{y}$. We work with functions with range $\mathbb{Z}_3$, and use the three Greek letters α, β, and γ to denote the three different outputs of the function. We take as convention that the three represent distinct outputs (so $\alpha \neq \beta \neq \gamma$). Sometimes we need to discuss an output as being not α, which we denote by $\not\alpha$.

In the proceeding discussion, we often need to discuss the behavior of a subfunction when keeping some subset of its arguments fixed. To simplify this discussion, we introduce some notation. For disjoint $S_1, S_2, S_3 \subseteq [n]$ we define

$$\begin{aligned} f^{\boldsymbol{a}}_{\{S_1\}}(\boldsymbol{x}) &\stackrel{\text{def}}{=} f(\{x_i\}_{i\in S_1}, \{a_i\}_{i\in S_1^C}) \\ f^{\boldsymbol{a}}_{\{S_1,S_2\}}(\boldsymbol{x},\boldsymbol{y}) &\stackrel{\text{def}}{=} f(\{x_i\}_{i\in S_1}, \{y_i\}_{i\in S_2}, \{a_i\}_{i\in (S_1\cup S_2)^C}) \\ f^{\boldsymbol{a}}_{\{S_1,S_2,S_3\}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) &\stackrel{\text{def}}{=} f(\{x_i\}_{i\in S_1}, \{y_i\}_{i\in S_2}, \{z_i\}_{i\in S_3}, \{a_i\}_{i\in (S_1\cup S_2\cup S_3)^C}) \,. \end{aligned}$$

We sometimes consider singleton sets S_1, S_2, S_3 and then denote them simply by their only element, with some abuse of notation. That is,

$$\begin{aligned} f^{\boldsymbol{a}}_{\{i\}}(x) &\stackrel{\text{def}}{=} f(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n) \\ f^{\boldsymbol{a}}_{\{i,j\}}(x,y) &\stackrel{\text{def}}{=} f(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_{j-1},y,a_{j+1},\ldots,a_n) \,, \end{aligned}$$

and analogously for $f^{\boldsymbol{a}}_{\{i,j,k\}}(x,y,z)$ and $f^{\boldsymbol{a}}_{\{i,j,k,l\}}(x,y,z,w)$.

We need to describe details of functions' behaviors, and adopt a geometric viewpoint. In the proofs, we speak of inputs as being neighbors and of rows, diagonals, and rectangles and induced rectangles in the function table. By neighbors we mean points at Hamming distance 1. By a row, we mean the values taken by the function fixing all but one values, i.e. the values $f^{\boldsymbol{a}}_{\{i\}}(x)$ for all $x \in A_1$ with a fixed i and $\boldsymbol{a}$ which are clear from the context. By a rectangle, we mean the values $f^{\boldsymbol{e}}_{\{S_1,S_2\}}(\boldsymbol{a},\boldsymbol{c}), f^{\boldsymbol{e}}_{\{S_1,S_2\}}(\boldsymbol{a},\boldsymbol{d}), f^{\boldsymbol{e}}_{\{S_1,S_2\}}(\boldsymbol{b},\boldsymbol{c}), f^{\boldsymbol{e}}_{\{S_1,S_2\}}(\boldsymbol{b},\boldsymbol{d})$. Note that a rectangle by this definition is a high-dimensional structure. By induced rectangle, we mean a rectangle as before but where $|S_1| = |S_2| = 1$, thus looking like a rectangle in the function table. We only use the concept of a diagonal of a 2×2 induced rectangle. For fixed inputs $\boldsymbol{a}$ and dimensions i, j we say that $f^{\boldsymbol{a}}_{\{i,j\}}(x_1,y_1), f^{\boldsymbol{a}}_{\{i,j\}}(x_2,y_2)$ is a diagonal for $x_1 \neq x_2$ and $y_1 \neq y_2$.

Definition 1 (Redundant inputs). *For an n-argument function f, we say that inputs $x, y, x \neq y$ are* redundant for player k *if for all $\boldsymbol{a}$ it holds that $f^{\boldsymbol{a}}_{\{k\}}(x) = f^{\boldsymbol{a}}_{\{k\}}(y)$.*

Definition 2 (Normalized function). *An n-argument function f with no redundant inputs for any player is said to be* normalized.

We take as convention that all functions are normalized. This assumption is without loss of generality as a function can easily be normalized by for each set of redundant inputs removing all but one. A protocol for evaluating the normalized function can be used to evaluate the original function as well by performing the same procedure.

To prove Theorem 1, we make use of a theorem by Chor and Kushilevitz [7] which states that there is no 1-private protocol for a 2-party computation of disjunction. Through standard simulation techniques, this gives impossibility results for multi-party protocols of functions containing an OR-like substructure. This is commonly referred to as an embedded OR, or a corner. We formally define an embedded OR and then restate their result. For a two-party function, the definition is straightforward:

Definition 3 (Embedded OR (2 parties)). *We say that a two-argument function f contains an* embedded OR *if there exists inputs x_1, x_2, y_1, y_2 ($x_1 \neq x_2, y_1 \neq y_2$) such that $f(x_1, y_1) = f(x_1, y_2) = f(x_2, y_1) \neq f(x_2, y_2)$.*

However, when considering the n-party case, the definition of an embedded OR becomes slightly more complex. In particular, we need our definition to capture the size of the collusion required to realize an embedded OR, as that size also limits the impossibility result that follows from the existence of such an embedded OR. To this end, we define an embedded OR as having a degree k. We remark that Kilian *et al.* [11] define an embedded OR as one of degree 1. Much of the previous literature has mostly been concerned with Boolean functions, and then, the existence of an embedded OR (of any degree) implies the existence of one of degree 1, as proved in [11]. However, for functions with larger ranges, the situation is more complex, as shown by our Theorem 22.

Definition 4 (Embedded OR (n parties, induced, generalized), corner-free). *We say that an n-argument function f contains an* embedded OR *of degree k if there exists disjoint subsets $S_1, S_2 \subset [n]$ where $|S_1|, |S_2| \leq k$, and values $\boldsymbol{a}$ such that the two-argument function $f'(\boldsymbol{x}, \boldsymbol{y}) = f^{\boldsymbol{a}}_{\{S_1,S_2\}}(\boldsymbol{x}, \boldsymbol{y})$ contains an embedded* OR*. We refer to an embedded* OR *of degree 1 as an* induced *embedded* OR*, and one of degree greater than 1 as a* generalized *embedded* OR*. A function without an embedded* OR *(of any degree) is said to be* corner-free.

With the definitions in place, we are ready to restate a result by Chor and Kushilevitz [7]. The result we need was not presented as a separate lemma in their paper, but instead follows as a corollary from two of their lemmas which we restate in simplified form.

Lemma 2 (Partition lemma, [7]). *Let $f : A_1 \times \ldots \times A_n \to R$ be $\lceil n/2 \rceil$-private. Then for every subset S_1 of size $\lceil n/2 \rceil$, the two-argument function $f'(\boldsymbol{x}, \boldsymbol{y}) = f_{\{S_1,S_1^C\}}(\boldsymbol{x}, \boldsymbol{y})$ is 1-private.*

Lemma 3 (Corners lemma, [7]). *A two-argument function is not 1-private if it contains an embedded* OR.

Corollary 4. *A function containing an embedded* OR *of degree at most* $\lceil n/2 \rceil$ *is not* $\lceil n/2 \rceil$*-private.*

We also make use of [7, Theorem 4] which states that a corner-free Boolean function can be expressed as a Boolean sum:

Theorem 5 ([7]). *For a corner-free Boolean function* f *there are functions* f_i *such that* $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} f_i(x_i)$ *where the sum is computed modulo 2.*

We formally restate the theorem from [11] showing that a generalized embedded OR in a Boolean function implies an induced embedded OR. In our terminology:

Theorem 6 ([11]). *A Boolean function* f *containing an embedded* OR *contains an embedded* OR *of degree 1.*

We show functions and subfunctions which depend on up to 4 arguments. To be able to draw them, we show 2-dimensional projections separated by lines with vertical lines indicating a 3^{rd} dimension and horizontal lines indicating a 4^{th} dimension. We present sample function in Figure 1, showing a function which contains an embedded OR of degree 2 but does not contain an embedded OR of degree 1. The highlighted embedded OR occurs with the subsets $S_1 = \{P_1, P_3\}$ and $S_2 = \{P_2, P_4\}$ with inputs $(2, 1)$ and $(1, 2)$ for S_1 and $(1, 1)$ and $(2, 2)$ for S_2. As the function is drawn, the coalition S_1 in the embedded OR controls the horizontal position, and S_2 controls the vertical position.

0	**1**	**1**	2
1	0	2	1
2	0	0	1
0	**2**	**1**	0

Fig. 1. An example function containing an embedded OR of degree 2 (highlighted).

We use the following lemma which we believe is well-known. For completeness, we include a proof in the appendix.

Lemma 7. *If an* n*-argument function* $f : A_1 \times \ldots \times A_n \to G$*, where* G *is an Abelian group, has the property that for every pair of dimensions* j, k *and inputs* $x_1, x_2, y_1, y_2, \boldsymbol{a}$ *the following equality holds:*

$$f^{\boldsymbol{a}}_{\{j,k\}}(x_1, y_1) + f^{\boldsymbol{a}}_{\{j,k\}}(x_2, y_2) = f^{\boldsymbol{a}}_{\{j,k\}}(x_1, y_2) + f^{\boldsymbol{a}}_{\{j,k\}}(x_2, y_1), \tag{1}$$

then f *can be rewritten as* $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} f_i(x_i)$.

3 A Structure Lemma

The main step towards proving Theorem 1 is the establishment of a structure lemma for functions with range $\mathbb{Z}_3$. Thus, we turn toward some global properties of functions (as opposed to the comparatively local property of the existence of an embedded OR). The first such property captures the case when we can split the range of a function into two parts, and compute a Boolean sum to discover which part the output lies in. If we can then proceed with further such subdivisions until we arrive at a single possible output, this immediately gives a protocol to compute f. We prove that this further subdivision is always possible for corner-free f with range $\mathbb{Z}_3$ in Lemma 21. We remark that this is a further generalization of the multi-party decomposability defined in [12], which in turn was a generalization of 2-party decomposability defined in [13]. We show a collapsible function and the generalized decomposition of it in Figure 2.

Definition 5 (Collapsible). *We say that a function $f : A_1 \times \ldots A_n \to R$ is* collapsible *if there is a subset $R', \emptyset \subset R' \subset R$ such that the Boolean function*

$$f'(\boldsymbol{x}) = \begin{cases} 1 & \text{if } f(\boldsymbol{x}) \in R' \\ 0 & \text{otherwise} \end{cases}$$

does not contain an embedded OR *and can thus be n-privately computed. We refer to f' as being collapsed.*

For a collapsible function f with range $\mathbb{Z}_3$ if f is collapsible we can choose R' with two elements α, β and say that f is collapsible by collapsing α and β.

0 1 2	2 2 0	
1 0 2	2 2 1	
2 2 0	1 0 2	

(a) Collapsible f

1 1 0	0 0 1	
1 1 0	0 0 1	
0 0 1	1 1 0	

(b) f collapsed

Fig. 2. An example collapsible function and the collapsed function.

Summation in a finite Abelian group is a function which is known to be n-private [8]. In a summation, the effect of one party's input can be thought of as applying a permutation to the sum of the other parties' inputs. We generalize this by defining a permuted sum where we give one of the parties a special role and let her input select an arbitrary permutation to be applied to the sum of the other parties' inputs. All functions which are sums, i.e. can be rewritten as $\sum_{i=1}^{n} f_i(x_i)$, are also permuted sums. In our applications, the sum may be a Boolean sum or over $\mathbb{Z}_3$. We show two example functions which are permuted sums in Figure 3

Definition 6 (Permuted sum). *We say that a function is a* permuted sum *if it can be written as $\pi_{x_i}(\sum_{j \neq i} f_j(x_j))$ where π_x is a permutation. We refer to party i as the* permuter.

0 0 1 1 2 2		0 1 2 | 0 2 1 | 1 0 2
1 2 0 2 0 1		1 2 0 | 2 1 0 | 0 2 1
(a) f		2 0 1 | 1 0 2 | 2 1 0
		(b) g

Fig. 3. Two example permuted sums. In f, party 2 (selecting column) is the permuter selecting one of the 6 permutations. The function $g = \pi_{x_3}(x_1 + x_2)$ where π_1 is the identity permutation, $\pi_2 = (12)$ and $\pi_3 = (01)$.

With these definitions, we are now ready to state and prove our structure lemma:

Lemma 8 (Structure lemma). *For every normalized n-argument function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$, at least one of the following holds:*

- *f has an embedded* OR
- *f is a permuted sum*
- *f is collapsible*

We present protocols for n-privately evaluating permuted sums (Protocol 2) and collapsible functions (Protocol 3) which do not contain an embedded OR. In Theorem 22 we show that if f contains an embedded OR, it also contains a small embedded OR. This, together with Corollary 4 concludes the proof of our Theorem 1.

To prove the structure lemma, we perform a case-analysis based on a property of f we call a link:

Definition 7 (Link, link-free). *We say that an n-argument function has a* link *(*over output α*) in dimension k if there exists inputs $x, y, x \neq y$, and $\boldsymbol{a}$ such that $\alpha = f^{\boldsymbol{a}}_{\{k\}}(x) = f^{\boldsymbol{a}}_{\{k\}}(y)$. We say that f has* links in c dimensions *if there are precisely c distinct k such that f has a link in dimension k. We say that a function is* link-free *if it has no links.*

Lemma 9. *In a corner-free n-argument function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$, if there are links between inputs x and y in dimension k over two distinct outputs, then x and y are redundant for player k.*

Proof. Let f have links over α and β between inputs x and y in dimension k. That is, there exists values $\boldsymbol{a}$, $\boldsymbol{b}$ such that $f^{\boldsymbol{a}}_{\{k\}}(x) = f^{\boldsymbol{a}}_{\{k\}}(y) = \alpha$, and $f^{\boldsymbol{b}}_{\{k\}}(x) = f^{\boldsymbol{b}}_{\{k\}}(y) = \beta$. Suppose that for some $\boldsymbol{c}$ we have $f^{\boldsymbol{c}}_{\{k\}}(x) \neq f^{\boldsymbol{c}}_{\{k\}}(y)$. Then one of $f^{\boldsymbol{c}}_{\{k\}}(x)$ and $f^{\boldsymbol{c}}_{\{k\}}(y)$ equals α or β. If one of them is α then f has an embedded OR with $S_1 = \{k\}$, $S_2 = \{k\}^C$ using inputs (x, y) and $(\boldsymbol{a}, \boldsymbol{c})$. If one is β then f has an embedded OR with $S_1 = \{k\}$, $S_2 = \{k\}^C$ using inputs (x, y) and $(\boldsymbol{b}, \boldsymbol{c})$. □

Looking at the proof of Lemma 9 we begin to see the importance of the small range of the function to the analysis. It also highlights the added complexities

compared to the Boolean case, as for a Boolean function any link implies that two inputs are redundant. From the lemma and its proof follow two corollaries about normalized functions with range $\mathbb{Z}_3$:

Corollary 10. *For a normalized n-argument function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$ with a link over α in dimension k for inputs x, y, for all $\boldsymbol{a}$, we have that $f^{\boldsymbol{a}}_{\{k\}}(x)$ uniquely determines $f^{\boldsymbol{a}}_{\{k\}}(y)$. More specifically, the possible combinations of values are $(\alpha, \alpha); (\beta, \gamma); (\gamma, \beta)$.*

Proof. Follows from the proof of Lemma 9. □

Corollary 11. *A normalized n-argument function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$ cannot have links over α in dimension k for inputs x, y, and x, z.*

Proof. By Corollary 10 the value at x determines the value at both y and z and hence inputs y and z are redundant. □

Analogously to an embedded OR, we introduce notation for the various 2×2 substructures in a function. Apart from the embedded OR, two of them feature prominently in our proofs. Firstly, a 2×2 substructure with one output occurring on the diagonal, and the two other values occurring once each on the opposite diagonal is called Aff_3. Secondly, a 2×2 substructure where one output is on one diagonal, and another is on the other is referred to as an XOR. For the XOR, we also define the type of an XOR as the pair (without order) of outputs in the XOR. All the substructures which can occur (up to symmetries) are depicted in Figure 4. A 2×2 substructure where only one output occurs is called constant, and if we want to emphasize that it is the output α which occurs, we write (α)-constant.

$\alpha\ \alpha$	$\alpha\ \alpha$	$\alpha\ \alpha$	$\alpha\ \alpha$	$\alpha\ \beta$	$\alpha\ \beta$
$\alpha\ \alpha$	$\alpha\ \beta$	$\beta\ \beta$	$\beta\ \gamma$	$\beta\ \alpha$	$\gamma\ \alpha$
(a) Constant	(b) OR	(c) 2-link	(d) Link	(e) XOR	(f) Aff_3

Fig. 4. The six 2×2 substructures.

Definition 8 (Type of an XOR). *If an XOR consists of outputs α and β we say that it is an XOR of type (α, β), denoted (α, β)-XOR. The order of elements is not important, so for functions to $\mathbb{Z}_3$ there are three possible types of XOR: (α, β), (α, γ), (β, γ).*

Our name Aff_3 comes from the fact that it can be expressed as an affine function modulo 3, analogously to the fact that XOR can be expressed as a sum modulo 2. We do not need that it is affine, but we make use of the fact that a function where all subfunctions are of the form Aff_3 can be written as a sum on the form $\sum_{i=1}^{n} f_i(x_i)$ with summation in $\mathbb{Z}_3$.

Lemma 12. *An n-argument corner-free function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$ such that all 2×2 subfunctions are of the form Aff_3 can be expressed as $\sum_{i=1}^{n} f_i(x_i)$ with summation in $\mathbb{Z}_3$.*

Proof. By Lemma 7 we need to verify that (1) holds for all 2×2 subfunctions, which are all of the form Aff_3. For all ways of assigning 0, 1 and 2 (distinctly) to α, β, γ we have that $2\alpha \equiv \beta + \gamma \pmod 3$. As $2 \equiv -1 \pmod 3$ this is equivalent to $\alpha + \beta + \gamma \equiv 0 \pmod 3$. □

In our proof of Lemma 8 we consider the substructures occurring in f. We begin by establishing three preliminary lemmas. The lemmas come into play primarily in cases when f contains links in few dimensions (none or one), and if f has an XOR spanned by dimensions i, j and f is link-free in those dimensions, then $|A_i| = |A_j| = 2$, giving some intuition for the condition on the size of the two inputs in the lemmas. We highlight the proof idea for each of the lemmas and give full proofs in the appendix.

Lemma 13. *Let f be an n-argument corner-free function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$ with i, j such that $|A_i| = |A_j| = 2$ such that for all $\boldsymbol{a}$, $f^{\boldsymbol{a}}_{\{i,j\}}$ is an XOR. If all three types of XOR's occur then there is a dimension k such that the input in dimension k determines the type of XOR.*

Proof (Idea). We show that if no such k exists then f contains an embedded OR. Full proof in Section A.2. □

Lemma 14. *Let f be an n-argument corner-free function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$ with i, j such that $|A_i| = |A_j| = 2$ and an output α such that for all $\boldsymbol{a}$ precisely one diagonal of $f^{\boldsymbol{a}}_{\{i,j\}}$ has two α's. Then f is collapsible.*

Proof (Idea). If f is not collapsible then the collapsed function contains an embedded OR. We show that this implies an embedded OR in f as well. Full proof in Section A.3. □

Lemma 15. *An n-argument corner-free function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$ with i, j such that $|A_i| = |A_j| = 2$ and such that for some $\boldsymbol{a}$, $f^{\boldsymbol{a}}_{\{i,j\}}$ is an Aff_3 and for some $\boldsymbol{b}$, $f^{\boldsymbol{b}}_{\{i,j\}}$ is an XOR is collapsible.*

Proof (Idea). We prove that f fulfills the conditions of Lemma 14. Full proof in Section A.4. □

Our proof of Lemma 8 proceeds in three separate lemmas, depending on whether the function f is link-free (Lemma 16), has links in one dimension (Lemma 17), or if it has links in two or more dimensions (Lemma 18). As the proofs are long and consist mainly of case analysis, we give them in the appendix and simply state the lemmas here.

Lemma 16. *Every n-argument link-free, corner-free function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$ is collapsible or a permuted sum.*

Proof (Idea). Case analysis showing we can apply one of Lemma 13, Lemma 14 and Lemma 15. Full proof in Section A.5 □

Lemma 17. *Every n-argument function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$ with links in 1 dimension and without an embedded* OR *is collapsible or a permuted sum.*

Proof (Idea). Case analysis showing we can apply one of Lemma 13, Lemma 14 and Lemma 15. Full proof in Section A.6. □

Lemma 18. *Every n-argument function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$ with links in 2 or more dimensions and without an embedded* OR *is collapsible.*

Proof (Idea). We show that all links must be over the same output. This gives some implications for the substructures of f which we use to show f must be collapsible. Full proof in Section A.7. □

4 Protocols

With the structure lemma established, we can now turn to the question of n-private protocols for collapsible functions and permuted sums. From the definitions of the two classes, we have two natural and easy protocols. The main problem we need to address in this section is proving the existence of a protocol for collapsible functions. For a function which is collapsible by collapsing β and γ it is clear from the definition that we can n-privately evaluate if the output is α or if it is one of β and γ. The key issue is to prove that we can then proceed with a second step where we can n-privately evaluate whether the output is β or if it is γ.

The construction of this second step relies on the passive model of adversaries and the knowledge that the output of the function is not α. Thus, in our second step we compute a sum which may have different outputs at points where the original function had α's. Such a construction is inherently insecure with active adversaries, as they may switch inputs between the first step of the decomposition and the second and would then learn some information about the other parties' inputs.

In both of our protocols we use a subprotocol by Chor and Kushilevitz [8] for n-private summation over any finite Abelian group. For completeness, we include a description of their protocol as Protocol 1. When used in our protocol for a permuted sum, the summation is either Boolean or in $\mathbb{Z}_3$ depending on the function f (but not on the inputs).

Protocol 1 (Summation [8]). The protocol for summation where party P_i participates with input x_i proceeds as follows:

1. In round $1 \leq i \leq n-2$, party P_i sums all its received messages, $w_i = \sum_{j=1}^{i-1} z_{j,i}$. Then, it chooses random group elements $z_{i,i+1}, z_{i,i+2}, \ldots, z_{i,n-1}$. Finally, it computes $z_{i,n}$ such that $x_i + w_i = \sum_{j=i+1}^{n} z_{i,j}$ and sends $z_{i,j}$ to P_j $(j > i)$.

2. In round $n-1$, party P_{n-1} computes $z_{n-1,n} = x_{n-1} + \sum_{j=1}^{n-2} z_{j,n-1}$ and sends $z_{n-1,n}$ to P_n.
3. In round n, party P_n computes the sum s as $s = x_n + \sum_{j=1}^{n-1} z_{j,n}$.

All sums are computed over some fixed finite Abelian group.

Protocol 2 (Permuted sum). The protocol for evaluating a permuted sum f, where party P_i (without loss of generality we assume the permuter is party n) participates with input x_i proceeds as follows:

1. Use Protocol 1 to privately compute $s = \sum_{j=1}^{n-1} f_j(x_j)$ such that only the permuter learns s.
2. The permuter computes the output as $\pi_{x_n}(s)$ and sends it to the other parties.

The sum is computed modulo 2, or 3, depending on f.

Protocol 3 (Collapsible). The protocol for evaluating a function f collapsible with partition $R' = \{\gamma\}$, where party P_i participates with input x_i proceeds as follows:

1. Use Protocol 1 to compute $s = \sum_{i=1}^{n} f_i(x_i) \pmod 2$, with f_i such that $s = 1$ iff $f(\boldsymbol{x}) = \gamma$
2. If $s = 0$, compute $s' = \sum_{i=1}^{n} g_i(x_i) \pmod 4$, with g_i such that $f(\boldsymbol{x}) = \alpha$ implies $s' = 0$, and $f(\boldsymbol{x}) = \beta$ implies $s' = 2$.

The correctness of Protocol 2 follows immediately from the definition of a permuted sum. In Protocol 3, since f is collapsible, the functions f_i exist by the definition of a collapsible function. However, the existence of appropriate g_i is not as straightforward. We prove, constructively, in Lemma 21 that they always exist for corner-free collapsible functions with range $\mathbb{Z}_3$. We stress that the choice of g_i does not depend on the input $\boldsymbol{x}$, but only on the function f.

The privacy of both these protocols is straightforward, and we only sketch the arguments.

Theorem 19. *Protocol 2 is n-private.*

Proof. The subprotocol used for summation was proven to be n-private in [8]. Due to the structure of the function, we see that the permuter, P_n, learns the sum s from $f(\boldsymbol{x})$ and x_n, since $s = \pi_{x_n}^{-1}(f(\boldsymbol{x}))$. □

Theorem 20. *Protocol 3 is n-private.*

Proof. The subprotocol used for summation was proven to be n-private in [8]. When the output is γ then, by the privacy of the summation sub-protocol, the protocol is private. Furthermore, when the output is one of α, β, then the privacy of the composed protocol also follow directly from the privacy of the subprotocols. The first sum only reveals that the output is one of α, β, and then, the condition on g_i is sufficient to guarantee that the sum s' reveals nothing but whether the output is α or β, as with a passive adversary we are guaranteed that s' is either 0 or 2. □

While the privacy is straightforward, the proof that there are functions g_i as required by Protocol 3 is rather involved and we simply state the lemma here and give the proof in the appendix. One may intuitively expect that such functions could simply be Boolean, but it turns out that for some f we do need the full range of $\mathbb{Z}_4$.

Lemma 21. *Protocol 3 can evaluate all corner-free, collapsible functions with range $\mathbb{Z}_3$.*

Proof (Idea). We construct a function g such that $f(\boldsymbol{x}) = \alpha \implies g(\boldsymbol{x}) = 0$ and $f(\boldsymbol{x}) = \beta \implies g(\boldsymbol{x}) = 2$. By case analysis on the induced rectangles in g, we show that g satisfies the conditions of Lemma 7 and hence there are g_i as required by Protocol 3. Full proof in Section A.8. □

5 An Embedded OR Implies a Small Embedded OR

Previously, we have often assumed that functions are free of embedded OR's of *any* degree (i.e., that they are corner-free). However, to be able to apply Corollary 4 we need to show that a sufficiently small embedded OR exists.

For Boolean functions f, if f has an embedded OR of any degree, then it also has an embedded OR of degree 1, as proved in [11], explaining the zero-one nature of Boolean privacy.

It turns out that for functions with range $\mathbb{Z}_3$, similarly to the Boolean case, the presence of a large embedded OR implies that the function also contains a small one. We state the theorem here and give the proof in the appendix.

Theorem 22. *Every n-argument function $f : A_1 \times \ldots A_n \to \mathbb{Z}_3$ that has an embedded* OR *of any degree has an embedded* OR *of degree at most* 3*. Furthermore, every* 4*-argument function $f : A_1 \times A_2 \times A_3 \times A_4 \to \mathbb{Z}_3$ that has an embedded* OR*, also has one of degree at most* 2*.*

Proof (Idea). The basic idea is similar to that used in the proof of Theorem 6. However, while the boolean case is fairly straightforward, our proof results in a fairly extensive case analysis. Full proof in Section A.9. □

6 Proof of the Main Theorem

We now conclude by re-stating our main theorem and presenting the proof.

Theorem 1 (Main theorem). *For every n-argument function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$, f is either n-private, or it is $\lfloor (n-1)/2 \rfloor$-private and not $\lceil n/2 \rceil$-private.*

Proof. If f is corner-free, then by Lemma 8 it is a permuted sum, collapsible, or both. Thus, it can be n-privately computed by Protocol 2 or Protocol 3.

If f is not corner-free, then by Theorem 22 it contains an embedded OR of degree at most $\lceil n/2 \rceil$. Thus, by Corollary 4, f is not $\lceil n/2 \rceil$-private. □

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A Proofs

A.1 Proof of Lemma 7

Lemma 7. *If an n-argument function $f : A_1 \times \ldots \times A_n \to G$, where G is an Abelian group, has the property that for every pair of dimensions j, k and inputs*

$x_1, x_2, y_1, y_2, \boldsymbol{a}$ *the following equivalence holds:*

$$f^{\boldsymbol{a}}_{\{j,k\}}(x_1, y_1) + f^{\boldsymbol{a}}_{\{j,k\}}(x_2, y_2) = f^{\boldsymbol{a}}_{\{j,k\}}(x_1, y_2) + f^{\boldsymbol{a}}_{\{j,k\}}(x_2, y_1), \tag{1}$$

then f *can be rewritten as* $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} f_i(x_i)$.

Proof. Induction on n. Rewrite (1) as

$$f^{\boldsymbol{a}}_{\{j,k\}}(x_1, y_1) - f^{\boldsymbol{a}}_{\{j,k\}}(x_1, y_2) = f^{\boldsymbol{a}}_{\{j,k\}}(x_2, y_1) - f^{\boldsymbol{a}}_{\{j,k\}}(x_2, y_2)$$

with $k = n$. This says that the function

$$g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, y_1) - f(x_1, \ldots, x_{n-1}, y_2)$$

does not change value when you change one arbitrary input in an arbitrary way. This implies that g is a constant function. The lemma now follows by induction on n.

□

A.2 Proof of Lemma 13

Lemma 13. *Let* f *be an* n*-argument corner-free function* $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$ *with* i, j *such that* $|A_i| = |A_j| = 2$ *such that for all* $\boldsymbol{a}$, $f^{\boldsymbol{a}}_{\{i,j\}}$ *is an* XOR*. If all three types of* XOR*'s occur then there is a dimension* k *such that the input in dimension* k *determines the type of* XOR*.*

Proof. We assume that there is an (α, β)-XOR and an (α, γ)-XOR at Hamming distance 1. We denote the dimension by which they differ by k and relabel the inputs in dimension k such that $f^{\boldsymbol{a}}_{\{i,j,k\}}(\cdot, \cdot, 1)$ is an (α, β)-XOR and $f^{\boldsymbol{a}}_{\{i,j,k\}}(\cdot, \cdot, 2)$ is an (α, γ)-XOR.

We proceed to show that there is no $\boldsymbol{b}$ such that $f^{\boldsymbol{b}}_{\{i,j,k\}}(\cdot, \cdot, 1)$ or $f^{\boldsymbol{b}}_{\{i,j,k\}}(\cdot, \cdot, 2)$ is a (β, γ)-XOR. Assume to the contrary that there is a $\boldsymbol{b}$ such that $f^{\boldsymbol{b}}_{\{i,j,k\}}(\cdot, \cdot, 1)$ is a (β, γ)-OR (the case if it occurs at input 2 in dimension k is analogous). We illustrate this case in Figure 5 where we for simplicity show $\boldsymbol{b}$ as differing from $\boldsymbol{a}$ in only one dimension, which is not something we assume in the proof.

$\alpha\ \beta$	$\alpha\ \gamma$
$\beta\ \alpha$	$\gamma\ \alpha$
$\gamma\ \beta$	
$\beta\ \gamma$	

Fig. 5. Illustration of a contradiction in the proof of Lemma 13.

What values can the function take at $f^{\boldsymbol{b}}_{\{i,j,k\}}(1, 1, 2)$? We claim that any output at that position would violate the assumption that f is corner-free. In

Figure 5 we can see why this is true for a simple function (writing α, β or γ anywhere in the missing 2×2 field can be verified to result in an embedded OR). We proceed with three almost identical cases (differing only in the interdependency of the coordinates, the core idea is captured by Figure 5):

Case 1: $f^{\boldsymbol{b}}_{\{i,j,k\}}(1,1,2) = \alpha$. We can find y (equal to 1 or 2) such that $f^{\boldsymbol{a}}_{\{i,j,k\}}(1,y,2) = \alpha$ as $f^{\boldsymbol{a}}_{\{i,j,k\}}(\cdot,\cdot,2)$ is an (α,γ)-XOR. We can also find x such that $f^{\boldsymbol{a}}_{\{i,j,k\}}(x,y,1) = \alpha$ as $f^{\boldsymbol{a}}_{\{i,j,k\}}(\cdot,\cdot,1)$ is an (α,β)-XOR. However, as $f^{\boldsymbol{b}}_{\{i,j,k\}}(\cdot,\cdot,1)$ is a (β,γ)-XOR we are guaranteed that $f^{\boldsymbol{b}}_{\{i,j,k\}}(x,1,1) \neq \alpha$. Thus, f contains an embedded OR with $S_1 = \{i,k\}$ and $S_2 = S_1^C$ using $(1,2);(x,1)$ on S_1 and $y;1$ or j and $\boldsymbol{a};\boldsymbol{b}$ on the rest of S_2.

Case 2: $f^{\boldsymbol{b}}_{\{i,j,k\}}(1,1,2) = \beta$. We can find x (equal to 1 or 2) such that $f^{\boldsymbol{b}}_{\{i,j,k\}}(x,1,1) = \beta$ as $f^{\boldsymbol{b}}_{\{i,j,k\}}(\cdot,\cdot,1)$ is an (β,γ)-XOR. We can also find y such that $f^{\boldsymbol{a}}_{\{i,j,k\}}(x,y,1) = \beta$ as $f^{\boldsymbol{a}}_{\{i,j,k\}}(\cdot,\cdot,1)$ is an (α,β)-XOR. However, as $f^{\boldsymbol{a}}_{\{i,j,k\}}(\cdot,\cdot,2)$ is a (α,γ)-XOR we are guaranteed that $f^{\boldsymbol{a}}_{\{i,j,k\}}(1,y,2) \neq \beta$. Thus, f contains an embedded OR with $S_1 = \{i,k\}$ and $S_2 = S_1^C$ using $(1,2);(x,1)$ on S_1 and $y;1$ or j and $\boldsymbol{a};\boldsymbol{b}$ on the rest of S_2.

Case 3: $f^{\boldsymbol{b}}_{\{i,j,k\}}(1,1,2) = \gamma$. We can find x (equal to 1 or 2) such that $f^{\boldsymbol{b}}_{\{i,j,k\}}(x,1,1) = \gamma$ as $f^{\boldsymbol{b}}_{\{i,j,k\}}(\cdot,\cdot,1)$ is an (β,γ)-XOR. We can also find y such that $f^{\boldsymbol{a}}_{\{i,j,k\}}(1,y,2) = \gamma$ as $f^{\boldsymbol{a}}_{\{i,j,k\}}(\cdot,\cdot,2)$ is an (α,γ)-XOR. However, as $f^{\boldsymbol{a}}_{\{i,j,k\}}(\cdot,\cdot,1)$ is a (α,β)-XOR we are guaranteed that $f^{\boldsymbol{a}}_{\{i,j,k\}}(x,y,1) \neq \gamma$. Thus, f contains an embedded OR with $S_1 = \{i,k\}$ and $S_2 = S_1^C$ using $(1,2);(x,1)$ on S_1 and $y;1$ or j and $\boldsymbol{a};\boldsymbol{b}$ on the rest of S_2.

We now conclude that there is no $\boldsymbol{b}$ such that $f^{\boldsymbol{b}}_{\{i,j,k\}}(\cdot,\cdot,1)$ or $f^{\boldsymbol{b}}_{\{i,j,k\}}(\cdot,\cdot,2)$ is a (β,γ)-XOR. As f has all types of XOR's, there must still be a (β,γ)-XOR in the function. Thus, we see that $|A_k| \geq 3$, and we can assume there is a $\boldsymbol{b}$ such that $f^{\boldsymbol{b}}_{\{i,j,k\}}(\cdot,\cdot,3)$ is a (β,γ)-XOR.

We claim that $f^{\boldsymbol{a}}_{\{i,j,k\}}(\cdot,\cdot,3)$ must be a (β,γ)-XOR. To see this we observe that if it was another type of XOR, then by the same proof that showed that there is no (β,γ)-XOR for $k = 1,2$ we could have shown that there was not (β,γ)-XOR for $k = 3$, but we know that $f^{\boldsymbol{b}}_{\{i,j,k\}}(\cdot,\cdot,3)$ is a (β,γ)-XOR. We now see that for a given x_k, all XOR's must be of the same type as that at $f^{\boldsymbol{a}}_{\{i,j,x_k\}}(\cdot,\cdot,k)$ which concludes our proof. □

A.3 Proof of Lemma 14

Lemma 14. *Let f be an n-argument corner-free function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$ with i,j such that $|A_i| = |A_j| = 2$ and an output α such that for all $\boldsymbol{a}$ precisely one diagonal of $f^{\boldsymbol{a}}_{\{i,j\}}$ has two α's. Then f is collapsible.*

Proof. We claim that f is collapsible by collapsing β and γ. To prove this we show that the collapsed function

$$g(\boldsymbol{x}) = \begin{cases} 1 & \text{if } f(\boldsymbol{x}) \in \{\beta,\gamma\} \\ 0 & \text{if } f(\boldsymbol{x}) = \alpha \end{cases}$$

does not contain an embedded OR of degree 1. Then by Theorem 6 we have that g is corner-free and Theorem 5 implies that the collapsed function can be written as a Boolean sum.

We begin by observing that as each 2×2 plane spanned by dimensions i, j contains exactly one diagonal with α's, each such 2×2 plane contains two α's, as if it had three α's it would be an embedded OR and if it had four α's both diagonals would have two α's. We further make the observation that a pair of neighboring outputs in dimension i (and analogously in j) are such that exactly one of them is α. More formally, for all $\boldsymbol{c}$ if $f^{\boldsymbol{c}}_{\{i\}}(1) = \alpha$ then $f^{\boldsymbol{c}}_{\{i\}}(2) \neq \alpha$ and if $f^{\boldsymbol{c}}_{\{i\}}(1) \neq \alpha$ then $f^{\boldsymbol{c}}_{\{i\}}(2) = \alpha$.

As g is Boolean, by Theorem 6 we know that if g has an embedded OR (of any degree), it also has an embedded OR of degree 1. We assume by contradiction that there is an embedded OR of degree 1 in g. We reorder inputs and dimensions such that the embedded OR is spanned by dimensions $1, 2$ using inputs $(1, 2); (1, 2)$, with other inputs as $\boldsymbol{a}$. We say that $g^{\boldsymbol{a}}_{\{1,2\}}$ is an embedded OR with slight abuse of notation (as $|A_1|$ or $|A_2|$ could be greater than 2). We see that the embedded OR cannot have three 1's as g takes the value 0 where f takes the value α, so an embedded OR with three 0's corresponds to an embedded OR with three α in f, which is corner-free. Thus, the embedded OR must have three 1's.

From our observation we know that each 2×2 plane in g spanned by i, j has two 0's, so there cannot be an OR in g with three 1's spanned by dimensions i, j. Thus, at least one of i and j must be different from both 1 and 2. We assume $i \neq 1, 2$ and reorder inputs such that the embedded OR occurs when $x_i = 1$. Let $\boldsymbol{b}$ be $\boldsymbol{a}$ with the value at x_i removed.

We now consider what values occur at $f^{\boldsymbol{b}}_{\{1,2,i\}}(\cdot, \cdot, 2)$. We know that of the four outputs of $f^{\boldsymbol{b}}_{\{1,2,i\}}(\cdot, \cdot, 1)$ one is α and three are different from α. But by our observation, this implies that of the four outputs of $f^{\boldsymbol{b}}_{\{1,2,i\}}(\cdot, \cdot, 2)$ three are α and one is different from α. This concludes our proof as it shows that an embedded OR in g implies an embedded OR in f which we assumed to be corner-free. □

A.4 Proof of Lemma 15

Lemma 15. *An n-argument corner-free function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$ with i, j such that $|A_i| = |A_j| = 2$ and such that for some $\boldsymbol{a}$, $f^{\boldsymbol{a}}_{\{i,j\}}$ is an Aff_3 and for some $\boldsymbol{b}$, $f^{\boldsymbol{b}}_{\{i,j\}}$ is an XOR is collapsible.*

Proof. Let the output that appears twice in $f^{\boldsymbol{a}}_{\{i,j\}}$ be α, and reorder inputs such that $f^{\boldsymbol{a}}_{\{i,j\}}(1, 1) = f^{\boldsymbol{a}}_{\{i,j\}}(2, 2) = \alpha$.

We now claim that $f^{\boldsymbol{b}}_{\{i,j\}}(1, 2) = f^{\boldsymbol{b}}_{\{i,j\}}(2, 1) = \alpha$. As $f^{\boldsymbol{b}}_{\{i,j\}}$ is an XOR we know that $f^{\boldsymbol{b}}_{\{i,j\}}(1, 2) = f^{\boldsymbol{b}}_{\{i,j\}}(2, 1)$. If $f^{\boldsymbol{b}}_{\{i,j\}}(1, 2) = f^{\boldsymbol{b}}_{\{i,j\}}(2, 1) \in \{\beta, \gamma\}$ then there is an embedded OR with $S_1 = \{i, j\}$ and $S_2 = S_1^C$ as using inputs $(1, 2); (2, 1)$ on S_1 and $\boldsymbol{a}; \boldsymbol{b}$ on S_2. We assume the other diagonal of the XOR consists of β's, i.e. $f^{\boldsymbol{b}}_{\{i,j\}}(1, 1) = f^{\boldsymbol{b}}_{\{i,j\}}(2, 2) = \beta$, and that $f^{\boldsymbol{a}}_{\{i,j\}}(1, 2) = \beta$, $f^{\boldsymbol{a}}_{\{i,j\}}(2, 1) = \gamma$. This is without loss of generality as we can relabel outputs and switch the roles of parties 1 and 2.

$\boldsymbol{\alpha}\ \beta$	$\beta\ \boldsymbol{\alpha}$
$\boldsymbol{\gamma}\ \alpha$	$\alpha\ \boldsymbol{\beta}$
(a) $f^{\boldsymbol{a}}_{\{i,j\}}$	(b) $f^{\boldsymbol{b}}_{\{i,j\}}$

Fig. 6. An XOR and Aff_3 in f. The outputs involved in proving that f has no links in dimension i are highlighted.

We claim that the function f cannot have any links in dimensions i or j. To see this for dimension i, we see that $f^{\boldsymbol{a}}_{\{i,j\}}(1,1) = \alpha$ and $f^{\boldsymbol{a}}_{\{i,j\}}(2,1) = \gamma$ but also $f^{\boldsymbol{b}}_{\{i,j\}}(1,2) = \alpha$ and $f^{\boldsymbol{b}}_{\{i,j\}}(2,2) = \beta$. Thus, the value of $f^{\boldsymbol{c}}_{\{i\}}(2)$ is not a function of the value of $f^{\boldsymbol{c}}_{\{i\}}(1)$ for all $\boldsymbol{c}$ and the contrapositive form of Corollary 10 gives that f cannot have a link between inputs 1 and 2 in dimension i. Similarly for dimension j, we have that $f^{\boldsymbol{a}}_{\{i,j\}}(2,2) = \alpha$ and $f^{\boldsymbol{a}}_{\{i,j\}}(2,1) = \gamma$, but also $f^{\boldsymbol{b}}_{\{i,j\}}(1,2) = \alpha$ and $f^{\boldsymbol{b}}_{\{i,j\}}(1,1) = \beta$. This demonstrates that $f^{\boldsymbol{c}}_{\{j\}}(1)$ is not a function of $f^{\boldsymbol{c}}_{\{j\}}(2)$ for all $\boldsymbol{c}$, and by the contrapositive form of Corollary 10, there is no link between inputs 2 and 1 in dimension j.

We proceed by proving that for all $\boldsymbol{c}$ precisely one of the two diagonals of $f^{\boldsymbol{c}}_{\{i,j\}}$ contains two α's. What are the possible values for $(f^{\boldsymbol{c}}_{\{i,j\}}(1,2), f^{\boldsymbol{c}}_{\{i,j\}}(2,1))$? We proved (when $\boldsymbol{c} = \boldsymbol{b}$ but we made no use of any properties of $\boldsymbol{b}$) that they cannot be (β,β) or (γ,γ). Furthermore, as $f^{\boldsymbol{b}}_{\{i,j\}}(1,2) = f^{\boldsymbol{b}}_{\{i,j\}}(2,1) = \alpha$ it cannot be that precisely one of the values is α, as then f would have an embedded OR with $S_1 = \{i,j\}$ and $S_2 = S_1^C$ using inputs $(1,2);(2,1)$ on S_1 and $\boldsymbol{b};\boldsymbol{c}$ on S_2. Thus the only remaining possibilities are $(\alpha,\alpha);(\beta,\gamma);(\gamma,\beta)$. As f has no links in dimension i or j we see that in the first case neither $f^{\boldsymbol{c}}_{\{i,j\}}(1,1)$ nor $f^{\boldsymbol{c}}_{\{i,j\}}(2,2)$ can equal α. In the two latter cases we have again by the link-freeness in dimensions i and j that $f^{\boldsymbol{c}}_{\{i,j\}}(1,1) = f^{\boldsymbol{c}}_{\{i,j\}}(2,2) = \alpha$. By Lemma 14 we have that f is collapsible as claimed. □

A.5 Proof of Lemma 16

Lemma 16. *Every n-argument link-free, corner-free function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$ is collapsible or a permuted sum.*

Proof. For a link-free and corner-free function, only two possibilities remain for the structure of an induced rectangle: it can either be an XOR or an Aff_3. If f contains an XOR spanned by dimensions i and j, then $|A_i| = |A_j| = 2$ since f is link-free.

We proceed with a case analysis. If f does not contain an XOR, then we select the first case. Otherwise, we pick an arbitrary XOR occurring in f and fix the dimensions i and j spanning it, and denote by $\boldsymbol{a}$ a set of inputs such that $f^{\boldsymbol{a}}_{\{i,j\}}$ is an (α,β)-XOR (if f has an XOR, we can relabel outputs such that there is an (α,β)-XOR). When we have fixed dimensions i,j we select one of the four last cases of our proof based *only* on the 2×2-planes spanned by dimensions i and j.

Case 1: Only Aff_3 (all dimensions). If all induced rectangles of f are of the form Aff_3, then f satisfies the condition of Lemma 12 and is a sum, and thus also a permuted sum.

Case 2: Both XOR and Aff_3 (spanned by i, j). By Lemma 15 we have that$f$ is collapsible.

Case 3: Only XOR, one type of XOR (spanned by i, j). If only one type of XOR's occur, then f is a Boolean corner-free function and by Theorem 5 we have that f is a sum, and thus also a permuted sum.

Case 4: Only XOR, two types of XOR (spanned by i, j). We assume that f contains XOR's of types (α, β) and (α, γ). Then α occurs on exactly one diagonal of all 2×2 planes spanned by dimensions i, j and by Lemma 14 f is collapsible by collapsing β and γ.

Case 5: Only XOR, three types of XOR (spanned by i, j). By Lemma 13 we see that there must be a dimension k such that the input in dimension k determines the type of the XOR. Reorder inputs such that for input 1 in dimension k the 2×2-planes spanned by i and j are (α, β)-XOR's. We let $\boldsymbol{a} = (1)$ and $S_1 = \{k\}^C$ and see that $f^{\boldsymbol{a}}_{\{S_1\}}$ is a Boolean corner-free function. Thus, Theorem 5 implies that $f^{\boldsymbol{a}}_{\{S_1\}}(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) = \sum_{i \neq k} f_i(x_i)$ with the sum computed modulo 2.

We claim that f is a permuted sum with P_k as the permuter and the sum computed modulo 2. To see this, we prove that for all $x_k \in A_k$ and for all inputs $\boldsymbol{b}$ we have $f^{\boldsymbol{b}}_{\{k\}}(x_k) = \pi_{x_k}\{f^{\boldsymbol{b}}_{\{k\}}(1)\}$. As f is link-free, we have that $f^{\boldsymbol{b}}_{\{k\}}(x_k) \neq f^{\boldsymbol{b}}_{\{k\}}(1)$. If x_k is such that the 2×2-planes spanned by dimensions 1 and 2 when the input in dimension k is x_k are (α, β)-XOR then this means that $f^{\boldsymbol{b}}_{\{k\}}(1) = \alpha \implies f^{\boldsymbol{b}}_{\{k\}}(x_k) = \beta$ and $f^{\boldsymbol{b}}_{\{k\}}(1) = \beta \implies f^{\boldsymbol{b}}_{\{k\}}(x_k) = \alpha$. Similarly if the XOR's are (α, γ)-XOR's $f^{\boldsymbol{b}}_{\{k\}}(1) = \alpha \implies f^{\boldsymbol{b}}_{\{k\}}(x_k) = \gamma$, and as the 2×2-planes are XOR's we have $f^{\boldsymbol{b}}_{\{k\}}(1) \neq \alpha \implies f^{\boldsymbol{b}}_{\{k\}}(x_k) = \alpha$. The case for x_k with (β, γ)-XOR's is analogous, concluding the proof. □

A.6 Proof of Lemma 17

Lemma 17. *Every n-argument function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$ with links in 1 dimension and without an embedded* OR *is collapsible or a permuted sum.*

Proof. For convenience of notation, we reorder parties and inputs such that there is a link between inputs 1 and 2 in dimension 1 over output α. We consider the functions $g_m(x_2 \ldots, x_n) = f(m, x_2, \ldots, x_n)$. As f has links only in 1 dimension, each g_m is link-free. From Corollary 10, we have that $g_2 = \pi_2 \circ g_1$ where π_2 is a permutation (transposing β and γ). If there is a link between inputs m and m' in dimension 1 we say there is a link between g_m and $g_{m'}$, with slight abuse of notation.

As g_1 is corner-free and link-free, there are only two possible 2×2 structures which can occur: Aff_3 and XOR. If an XOR occurs as a substructure spanned by dimensions i, j, then since g_1 is link-free we must have $|A_i| = |A_j| = 2$. Our proof proceeds in five cases depending on the structures in g_1 (but not on f as

a whole). As in the proof of Lemma 16, in the four cases when g_1 has an XOR we fix dimensions i, j spanning an XOR and then select case based *only* on the 2×2 substructures spanned by dimensions i and j. Our cases are:

1. g_1 without XOR (all dimensions)
2. g_1 with both XOR and Aff_3 (spanned by i, j)
3. g_1 with only XOR's, one type of XOR (spanned by i, j)
4. g_1 with only XOR's, two types of XOR (spanned by i, j)
5. g_1 with only XOR's, three types of XOR (spanned by i, j)

Case 1: g_1 without XOR (all dimensions). We begin with the case that g_1 does not contain an XOR, and thus only consists of Aff_3. As $g_2 = \pi_2 \circ g_1$ the same is true for g_2. As all substructures of g_1 are of the form Aff_3, we can apply Lemma 12 to see that $g_1 = \sum_{k=2}^{n} f_k(x_k)$ where the sum is over $\mathbb{Z}_3$. Thus, if $g_m = \pi_m \circ g_1$ for all m, then f is a permuted sum with P_1 as the permuter. By Corollary 10 we know that if there is a link between g_1 and g_m or g_2 and g_m then $g_m = \pi_m \circ g_1$ as desired. This will be the case for all but a special case (when f is collapsible). The remainder of this proof assumes f is not a permuted sum and, after proving many restrictions on such f, shows that it is then collapsible by collapsing β and γ.

If f is not a permuted sum then there is an m such that g_m cannot be written on the form $g_m = \pi_m \circ g_1$. We claim that in this case, all Aff_3 in g_1 must have two α (the output linking g_1 and g_2). To see this, consider an Aff_3 in g_1 with only one α. Let it be spanned by some dimensions i, j and occur at inputs $\boldsymbol{a}$. We say that ${g_1}^{\boldsymbol{a}}_{\{i,j\}}$ is an Aff_3 (with slight abuse of notation as $|A_i|$ or $|A_j|$ may be greater than 2). As π_2 transposes β and γ, ${g_2}^{\boldsymbol{a}}_{\{i,j\}}$ is also an Aff_3 with one α. But this implies that for g_m not to have a link to either g_1 or g_2 then ${g_m}^{\boldsymbol{a}}_{\{i,j\}}$ would have to take the value α at precisely three points, giving an embedded OR. We illustrate this case in Figure 7. We claim that this also means that in this case there can be no $m' > 2$ such that $g_{m'}$ has a link to g_1 or g_2. By Corollary 11 the link would have to be over an output different from α which results in a situation analogous to an Aff_3 with only one α.

$$\begin{array}{l} \beta\ \alpha \big| \gamma\ \alpha \big| \cdot\ \cdot \\ \gamma\ \beta \big| \beta\ \gamma \big| \cdot\ \cdot \end{array}$$

Fig. 7. A contradiction in the proof of Lemma 17, showing (a part of) g_1 to the left, g_2 in the middle, and g_m to the right.

Fix two distinct dimensions i, j different from 1. We now show that $|A_i| = |A_j| = 2$. For some inputs $\boldsymbol{a}$ we know that ${g_1}^{\boldsymbol{a}}_{\{i,j\}}$ is an Aff_3 with two α (again, with slight abuse of notation as we have not yet shown $|A_i| = |A_j| = 2$). Reorder inputs such that ${g_1}^{\boldsymbol{a}}_{\{i,j\}}(1,1) = {g_1}^{\boldsymbol{a}}_{\{i,j\}}(2,2) = \alpha$ and consider ${g_1}^{\boldsymbol{a}}_{\{i,j\}}(3,1)$. As the 2×2-plane spanned by inputs 2 and 3 in dimension i and inputs 1 and 2 in dimension j is an Aff_3 we must have ${g_1}^{\boldsymbol{a}}_{\{i,j\}}(3,1) = \alpha$. But then g_1 would have a

link over α in dimension i between inputs 1 and 3 violating that g_1 is link-free. Thus, $|A_i| = 2$, and by an analogous argument $|A_j| = 2$.

We are now ready to apply Lemma 14. As the 2×2-planes spanned by dimensions i and j in g_1 and g_2 are Aff_3 with two α's, they have exactly one diagonal with two α's. For g_m with $m > 2$ we see that each 2×2-plane has at least a diagonal with two α's since g_m must have an α where g_1 has a β or a γ as to not have a link to g_1 or g_2 and each 2×2-plane in g_1 has exactly one diagonal with a β and a γ. As g_m is link-free each 2×2-plane cannot have more than one diagonal with two α's. Thus, the conditions of Lemma 14 are fulfilled and we conclude f is collapsible by collapsing β and γ. We conclude the proof of this case by displaying a function f of this form in Figure 8.

$$\begin{array}{c|c|c|c} \alpha\ \beta & \alpha\ \gamma & \beta\ \alpha & \gamma\ \alpha \\ \gamma\ \alpha & \beta\ \alpha & \alpha\ \gamma & \alpha\ \beta \end{array}$$

Fig. 8. A function f where all induced rectangles in g_1 are Aff_3 with two α's and where for $i > 2$, g_i does not have links to g_1 or g_2.

Case 2: g_1 with both XOR and Aff_3 (spanned by i, j). By Lemma 15 we have that f is collapsible.

Case 3: g_1 with only XOR's, one type of XOR's (spanned by i, j). We now consider the case when all 2×2-planes in g_1 spanned by dimensions i, j are XOR's, all of the same type. We assume the type is (α, β)-XOR's, which is without loss of generality as we can reorder dimensions and we know α must occur in g_1 as there is a link between g_1 and g_2 over α. As $|A_i| = |A_j| = 2$, we know from Lemma 15 that if any 2×2 plane spanned by dimensions i, j (in any g_m) is of the form Aff_3, then f is collapsible.

What remains is to analyze the situation when all 2×2-planes in f spanned by dimensions i, j in all g_m are XOR's. If none of the planes contain (β, γ)-XOR's then each plane has a diagonal with two α's and by Lemma 14 f is collapsible. As $g_2 = \pi_2 \circ g_1$ we know that all XOR's in g_2 are (α, γ)-XOR's. If for some m, g_m has an (β, γ)-XOR spanned by dimensions i, j then by Lemma 13 we know that there is a dimension k such that the input in k determines the type of XOR. As there are (α, β)-XOR's when $x_1 = 1$ and (α, γ)-XOR's when $x_1 = 2$ we see that $k = 1$ and f is a permuted sum with party 1 as the permuter.

Case 4: g_1 with only XOR's, two types of XOR's (spanned by i, j). We now proceed to the case where all 2×2-planes spanned by dimensions i, j in g_1 are XOR's, and there are two types of XOR's among them. We assume that g_1 has an (α, β)-XOR which is without loss of generality as we know that g_1 has at least one α.

We now claim that there is no (β, γ)-XOR in f spanned by dimensions i, j. Assume to the contrary that there is a (β, γ)-XOR in f. Then Lemma 13 applies and we know there is a dimension k such that the input in dimension k determines the type of XOR. Let $\boldsymbol{a}$ be inputs such that $f^{\boldsymbol{a}}_{\{1,i,j\}}(1, \cdot, \cdot)$ is an (α, β)-XOR. Then

as $g_1 = \pi_2 \circ g_2$ we see that $f^{\boldsymbol{a}}_{\{1,i,j\}}(2,\cdot,\cdot)$ is an (α,γ)-XOR. As changing the input x_1 (keeping all other inputs fixed) changes the type of XOR we must have $k = 1$. But we have assumed that g_1 contains two types of XOR, so k cannot be 1 and we get a contradiction.

The only remaining possibility is that all the 2×2-planes spanned by dimensions i, j in f are (α,β)-XOR's and (α,γ)-XOR's. Thus for each 2×2-plane there is a diagonal with two α's and by Lemma 14 we know that f is collapsible by collapsing β and γ.

Case 5: g_1 with only XOR's, three types of XOR's (spanned by i, j). We claim that our final case is such that there are no functions to which it applies. In the previous case with g_1 with only XOR's and two types of XOR's we showed that there could be no (β,γ)-XOR in f. Our proof made use of the fact that there were *at least* two types of XOR's in g_1. Thus, there are no corner-free functions with links in one dimension such that g_1 consists of only XOR's with all three types of XOR's. □

A.7 Proof of Lemma 18

Lemma 18. *Every n-argument function $f : A_1 \times \ldots \times A_n \to \mathbb{Z}_3$ with links in 2 or more dimensions and without an embedded* OR *is collapsible.*

Proof. We reorder inputs and dimensions such that there are links between inputs 1 and 2 in both dimensions 1 and 2. We begin by showing that a normalized f cannot have links over two different outputs in two different dimensions. Assume to the contrary that there is a link over output α in dimension 1 for inputs x_1, x_2, and a link over output β in dimension 2 for inputs y_1, y_2.

Let π_α be the permutation transposing β and γ, written $(\alpha\ \gamma\ \beta)$, and π_β be the permutation transposing α and γ, written $(\gamma\ \beta\ \alpha)$. For each $\boldsymbol{a}$ by Corollary 10 we have that $f^{\boldsymbol{a}}_{\{1,2\}}(x_2, y_1) = \pi_\alpha\{f^{\boldsymbol{a}}_{\{1,2\}}(x_1, y_1)\}$ and $f^{\boldsymbol{a}}_{\{1,2\}}(x_2, y_2) = \pi_\beta\{f^{\boldsymbol{a}}_{\{1,2\}}(x_1, y_2)\}$. Corollary 10 also gives that $f^{\boldsymbol{a}}_{\{1,2\}}(x_1, y_2) = \pi_\beta\{f^{\boldsymbol{a}}_{\{1,2\}}(x_1, y_1)\}$ and $f^{\boldsymbol{a}}_{\{1,2\}}(x_2, y_2) = \pi_\alpha\{f^{\boldsymbol{a}}_{\{2,1\}}(x_1, y_2)\}$. However, the permutation $\pi_\alpha \circ \pi_\beta = (\beta\ \gamma\ \alpha)$ and $\pi_\beta \circ \pi_\alpha = (\gamma\ \alpha\ \beta)$, and hence for all $x : \pi_\alpha\{\pi_\beta(x)\} \neq \pi_\beta\{\pi_\alpha(x)\}$ giving a contradiction.

We relabel and reorder inputs of f such that all links in f are over output α and that there are links in both dimensions 1 and 2 with inputs 1 and 2. This is without loss of generality as the fact that the link in dimension 1 has the same inputs as the link in dimension 2 does not affect anything. We illustrate two functions with links in two dimensions with inputs 1 and 2 in Figure 9

What are the possible substructures spanned by inputs 1 and 2 in dimensions 1 and 2 in f? By Corollary 10 we see that an output in such a 2×2 substructure uniquely determines the others and we either get an (α)-constant or a (β,γ)-XOR. We illustrate the possibilities in Figure 10.

It follows from the fact that f cannot have links over outputs other than α, together with Corollary 11 that all dimensions $|A_i| \leq 4$ (in a row, there can be at most two α's, one β and one γ). We now consider the whole planes spanned

$\beta\ \gamma\ \alpha$	$\alpha\ \alpha\ \beta$
$\gamma\ \beta\ \alpha$	$\alpha\ \alpha\ \gamma$
$\alpha\ \alpha\ \beta$	$\gamma\ \beta\ \alpha$
(a) f_1 with two links	(b) f_2 with two links

Fig. 9. Two different functions with links in two dimensions.

$\alpha\ \alpha$	$\beta\ \gamma$	$\gamma\ \beta$
$\alpha\ \alpha$	$\gamma\ \beta$	$\beta\ \gamma$
(a) (α)-constant	(b) (β,γ)-XOR	(c) (β,γ)-XOR

Fig. 10. The three possibilities for the 2×2-plane spanned by inputs 1 and 2 in dimensions 1 and 2.

by dimensions 1 and 2. For every plane, we know that there is a 2×2 square of either the form (α)-constant or (β,γ)-XOR spanned by inputs 1 and 2. What does this imply for the remainder of the plane?

We claim that there is no link in dimension 1 between inputs 1 and 3. This follows since there are no links in f over β or γ and since there is a link between inputs 1 and 2 in dimension α by Corollary 11 inputs 1 and 3 in dimension 1 cannot have a link over α. This means that $f^{\boldsymbol{a}}_{\{1\}}(1) = \alpha \iff f^{\boldsymbol{a}}_{\{1\}}(3) \neq \alpha$ for all $\boldsymbol{a}$. Analogously, $f^{\boldsymbol{a}}_{\{1\}}(1) = \alpha \iff f^{\boldsymbol{a}}_{\{1\}}(4) \neq \alpha$ for all $\boldsymbol{a}$. Thus, we see that $f^{\boldsymbol{a}}_{\{1\}}(3) = \alpha \iff f^{\boldsymbol{a}}_{\{1\}}(4) = \alpha$ for all $\boldsymbol{a}$. Since f has a link over α in dimension 1 and 2, by Corollary 10 we have that $f^{\boldsymbol{a}}_{\{1\}}(1) = \alpha \iff f^{\boldsymbol{a}}_{\{1\}}(4) = \alpha$ for all $\boldsymbol{a}$. An identical argument applies for dimension 2.

Recalling the notation $\not\alpha$ used to denote an output which is either β or γ, we see that there are only two possibilities for the planes spanned by dimensions 1 and 2. We illustrate these for the maximal case when $|A_1| = |A_2| = 4$ in Figure 11.

$\alpha\ \alpha\ \not\alpha\ \not\alpha$	$\not\alpha\ \not\alpha\ \alpha\ \alpha$
$\alpha\ \alpha\ \not\alpha\ \not\alpha$	$\not\alpha\ \not\alpha\ \alpha\ \alpha$
$\not\alpha\ \not\alpha\ \alpha\ \alpha$	$\alpha\ \alpha\ \not\alpha\ \not\alpha$
$\not\alpha\ \not\alpha\ \alpha\ \alpha$	$\alpha\ \alpha\ \not\alpha\ \not\alpha$

Fig. 11. The two possible structures for planes spanned by dimensions 1 and 2.

We claim that f is collapsible by collapsing β and γ and define the collapsed function

$$g(\boldsymbol{x}) = \begin{cases} 1 & \text{if } f(\boldsymbol{x}) \in \{\beta,\gamma\} \\ 0 & \text{if } f(\boldsymbol{x}) = \alpha \end{cases} .$$

We show that g does not contain an embedded OR of degree 1. Then by Theorem 6 g is corner-free and from Theorem 5 we can then conclude that the collapsed function can be written as a Boolean sum.

Assume by contradiction that there is an embedded OR of degree 1 in g. We then show that there is an embedded OR in f, which is a contradiction since f is corner-free. If there is an embedded OR in g with three 0's, then that corresponds to an embedded OR in f with three α's, so we assume g has an embedded OR with three 1's.

We begin by showing that the embedded OR is not over dimension 1 or 2. By the structure of f we have that for a pair of inputs x_1, x_2 in dimension 1 we either have $f^{\boldsymbol{a}}_{\{1\}}(x_1) = \alpha \iff f^{\boldsymbol{a}}_{\{1\}}(x_2) = \alpha$ or $f^{\boldsymbol{a}}_{\{1\}}(x_1) = \alpha \iff f^{\boldsymbol{a}}_{\{1\}}(x_2) \neq \alpha$ for all $\boldsymbol{a}$. From this it follows that a 2×2-plane spanned by dimension 1 must have an even number of α's, so in particular it cannot have one α. The case for dimension 2 is analogous.

We assume there is an embedded OR in g with three 1's in dimensions j and k with inputs x_1, x_2 and y_1, y_2 and consider the planes spanned by dimensions 1 and 2. We let x_1 and y_1 be the inputs of the 0 in the embedded OR in g. We consider the case when the embedded OR occurs at input 1 in both dimensions 1 and 2, the case for when it occurs at some other input in dimensions 1 and 2 is completely analogous. Let the inputs where the embedded OR occurs be $\boldsymbol{a}$. If $|A_1| > 2$ (or $|A_2| > 2$) then since $f^{\boldsymbol{b}}_{\{1\}}(1) = \alpha \iff f^{\boldsymbol{b}}_{\{1\}}(3) \neq \alpha$ for all $\boldsymbol{b}$ we see that of the four outputs $f^{\boldsymbol{a}}_{\{1,j,k\}}(3, x_1, y_1), f^{\boldsymbol{a}}_{\{1,j,k\}}(3, x_1, y_2), f^{\boldsymbol{a}}_{\{1,j,k\}}(3, x_2, y_1)$, and $f^{\boldsymbol{a}}_{\{1,j,k\}}(3, x_2, y_2)$ precisely three must be α's, showing an embedded OR in f.

Finally, we let $|A_1| = |A_2| = 2$. We know that of the considered 2×2-planes spanned by dimension 1 and 2, three are of the form (β, γ)-XOR and one is (α)-constant. We claim that there is an embedded OR in f of degree 2 with $S_1 = \{1, j\}$ and $S_2 = \{2, k\}$ using inputs $(1, x_1); (2, x_2)$ on S_1 and $(1, y_1); (2, y_2)$ on S_2. We illustrate this in Figure 12, and then give a formal proof.

$$\begin{array}{cc|cc} \alpha & \alpha & \beta & \gamma \\ \alpha & \alpha & \gamma & \beta \\ \hline \beta & \gamma & \gamma & \beta \\ \gamma & \beta & \beta & \gamma \end{array}$$

Fig. 12. An embedded OR in f with dimensions 1 and 2 spanning the 2×2 planes with dimension j over the horizontal line and dimension k over the vertical.

We now verify that we have an embedded OR in f as claimed. We know that $f^{\boldsymbol{a}}_{\{1,2,j,k\}}(1, 1, x_1, y_1) = \alpha$ and all of $f^{\boldsymbol{a}}_{\{1,2,j,k\}}(1, 2, x_1, y_2)$, $f^{\boldsymbol{a}}_{\{1,2,j,k\}}(2, 1, x_2, y_1)$, and $f^{\boldsymbol{a}}_{\{1,2,j,k\}}(2, 2, x_2, y_2)$ are different from α. As f has no links over outputs β or γ $f^{\boldsymbol{a}}_{\{1,2,j,k\}}(1, 2, x_1, y_2) \neq f^{\boldsymbol{a}}_{\{1,2,j,k\}}(2, 2, x_1, y_2)$ and $f^{\boldsymbol{a}}_{\{1,2,j,k\}}(2, 2, x_1, y_2) \neq f^{\boldsymbol{a}}_{\{1,2,j,k\}}(2, 2, x_2, y_2)$, which implies $f^{\boldsymbol{a}}_{\{1,2,j,k\}}(1, 2, x_1, y_2) = f^{\boldsymbol{a}}_{\{1,2,j,k\}}(2, 2, x_2, y_2)$. Analogously we see that $f^{\boldsymbol{a}}_{\{1,2,j,k\}}(2, 2, x_2, y_2) = f^{\boldsymbol{a}}_{\{1,2,j,k\}}(2, 1, x_2, y_1)$ showing

that we do have an embedded OR as claimed. Thus, we conclude the proof that if f is not collapsible by collapsing β and γ, then f is not corner-free. □

A.8 Proof of Lemma 21

Lemma 21. *Protocol 3 can evaluate all corner-free, collapsible functions with range $\mathbb{Z}_3$.*

Proof. Let f be corner-free and collapsible by collapsing α and β. We prove that there are functions g_i such that $f(x_1, \ldots, x_n) = \alpha \implies \sum_{i=1}^{n} g_i(x_i) = 0$ and $f(x_1, \ldots, x_n) = \beta \implies \sum_{i=1}^{n} g_i(x_i) = 2$ where the sum is computed modulo 4. One may wonder why the sum is not modulo 2. At the end of the proof, we show that whether the sum must be modulo 4 or if it could be modulo 2 is a property of the function f. In Figure 13 we show a function such that the sum must be modulo 4.

One viewpoint is that we are given a partially filled function table for a function g with $g(x_1, \ldots, x_n) = 0$ where $f(x_1, \ldots, x_n) = \alpha$, $g(x_1, \ldots, x_n) = 2$ where $f(x_1, \ldots, x_n) = \beta$, and a blank where $f(x_1, \ldots, x_n) = \gamma$. Our proof goes by "filling in the blanks" such that the conditions of Lemma 7 are satisfied which implies that there are g_i with the desired properties. We illustrate the partially filled function table in Figure 13.

$\alpha\ \beta\ \gamma \mid \gamma\ \gamma\ \alpha$	$0\ 2\ \cdot \mid \cdot\ \cdot\ 0$	$0\ 2\ 1 \mid 3\ 1\ 0$
$\beta\ \alpha\ \gamma \mid \gamma\ \gamma\ \beta$	$2\ 0\ \cdot \mid \cdot\ \cdot\ 2$	$2\ 0\ 3 \mid 1\ 3\ 2$
$\gamma\ \gamma\ \alpha \mid \beta\ \alpha\ \gamma$	$\cdot\ \cdot\ 0 \mid 2\ 0\ \cdot$	$3\ 1\ 0 \mid 2\ 0\ 3$
(a) Collapsible f	(b) Partial g	(c) g

Fig. 13. An example collapsible f, the corresponding partially filled function table g, and a completely defined g satisfying the conditions of Lemma 7.

We refer to the values defined in the partially filled g as *given*. The condition for Lemma 7 to apply is that all induced rectangles of g satisfy (1). Our proof strategy is to describe a procedure by which we fill in the blanks and then prove that this results in (1) holding for all induced rectangles in g by case analysis. We assign terms of the form x, $x + 2$, $-x$, and $-x + 2$ to the blanks and prove that for at least one of $x = 0$ and $x = 1$ the conditions of Lemma 7 are satisfied. The case $x = 0$ corresponds to when the g_i could have been Boolean, and $x = 1$ to when we need the full range of $\mathbb{Z}_4$.

We begin by observing that if g is completely specified (i.e., the output γ never occurs in f) then g is a boolean corner-free function and by Theorem 5 we can find $g'_i(x_i)$ such that $g(x_1, \ldots, x_n) = \sum_{i=1}^{n} 2g_i(x_i)$ where the computation is modulo 4. For the degenerate case when g is given as only blanks (i.e., $f(x_1, \ldots, x_n) = \gamma$ for all inputs) then we can simply let g be constant 0. Thus, we proceed with the non-degenerate case when g contains both given values and blanks.

We claim that there is no link in f over output α or β, implying that there are no links in the partially filled g over output 0 or 2. Assume to the contrary that there is a link over α (the case with a link over β is analogous) in dimension k for inputs x, y with other inputs as $\boldsymbol{a}$. Consider the pair of values $f^{\boldsymbol{b}}_{\{k\}}(x), f^{\boldsymbol{b}}_{\{k\}}(y)$ for some $\boldsymbol{b} \neq \boldsymbol{a}$. Since f is normalized it follows from Lemma 9 they cannot be (β, β) or (γ, γ). Since f is corner-free, the pair cannot contain exactly one α. Furthermore, as f is collapsible by collapsing α and β the pair cannot contain exactly one β either, as then the collapsed function has an embedded OR. Thus, the only remaining option is (α, α), and this must hold for all $\boldsymbol{b} \neq \boldsymbol{a}$, showing inputs x and y are redundant in dimension k which contradicts that f is normalized.

As f does not have any links over α or β and by Corollary 11 can have at most one link over γ in a dimension, we see that for all $i, |A_i| \leq 4$ (at most one α, one β and two γ). This means that for each row in dimension 1 (a subfunction on the form $g^{\boldsymbol{a}}_{\{1\}}$ for fixed $\boldsymbol{a}$) the function g can have at most one 0, one 2, and two blanks.

Consider the "first" row, by which we mean $g^{\mathbf{1}}_{\{1\}}$ where $\mathbf{1} = (1, 1, \ldots, 1)$. Without loss of generality we reorder dimensions and inputs such that $g^{\mathbf{1}}_{\{1\}}(1)$ is given and $g^{\mathbf{1}}_{\{1\}}(2)$ is blank. We let $d_1 = g^{\mathbf{1}}_{\{1\}}(1)$. We fill in the blank at $g^{\mathbf{1}}_{\{1\}}(2)$ with x, and the second blank (if there is one) with $x + 2$.

We proceed to make an observation on the pattern of blank outputs in g. As f is collapsible by collapsing α and β, by the definition of collapsible we have that there exist functions $f_i(x_i)$ such that $\sum_{i=1}^{n} f_i(x_i) = 0 \implies f(x_1, \ldots, x_n) = \gamma \implies g(x_1, \ldots, x_n)$ is blank. In particular, this implies that for all rows in g, there are only two possible patterns for where the blanks are, and the two patterns are complementary. For every dimension i (we already did this for dimension 1) such that f_i is not constant, we reorder the inputs such that $f_i(1) \neq f_i(2)$ (which means that the pattern of blanks changes between input 1 and 2).

We proceed to fill in g. For each row in dimension 1, we apply (1) as if it was at Hamming distance 1 from the first row (even though most rows are not). In more detail, for a row with the same pattern of blanks as the first row, denote the given value at $x_1 = 1$ by c_1. A blank position for $x_1 = i$ is filled with the value $c_1 - d_1 + d_i$ where d_i is value we filled in at $x_1 = i$ on the first row (and thus, either $d_i = x$ or $d_i = x + 2$). On a row with a pattern opposite of that of the first row, we fill in a blank at $x_1 = i$ with $d_i + c_2 - x$, where c_2 is the value given at $x_1 = 2$ of that row (recall that the value at $x_1 = 2$ on the first row is x). As all arithmetic is modulo 4, we make use of the equality $2 \equiv -2$, and thus the terms we fill in blanks with are: $x, x + 2, -x$, and $-x + 2$, as previously claimed. We show how g from Figure 13 is filled in in Figure 14 (but we have not reordered inputs such that the second value on the first row is a blank).

We now proceed to show that the conditions of Lemma 7 are fulfilled by either $x = 0$ or $x = 1$. Following our observation on the pattern of blanks, we see that every induced rectangle in g must have an even number of terms with x's. We now proceed with a case analysis for each induced rectangle in g, with five cases. We prove that for the first four of the cases, no condition is imposed

$$\begin{array}{ccc|ccc} 0 & 2 & x & -x & (-x+2) & 0 \\ 2 & 0 & (x+2) & (-x+2) & -x & 2 \\ -x & (-x+2) & 0 & 2 & 0 & (x+2) \end{array}$$

Fig. 14. Filling in f from Figure 13.

upon x. The presence of rectangles in the final case determines if $x = 0$ or $x = 1$ fulfills the conditions of Lemma 7, and if no such rectangle is present in g, the both choices for x work. We remark that our construction is such that if $x = y$ satisfies the conditions of Lemma 7, then so does $x = y + 2$, however, we do not formally prove this. We also remark that for $x = 0$ all blanks will be assigned the value 0 or 2, and for $x = 1$ all blanks will be assigned the value 1 or 3.

Our five cases for an induced rectangle in g are:

1. Four given values
2. Four terms with x's
3. Two terms with x's which are neighbors
4. Two terms with x's on a diagonal, opposite signs on the x terms
5. Two terms with x's on a diagonal, same signs on the x terms

Case 1: Four given values. First, consider an induced rectangle consisting only of values 0 and 2. The condition that f is corner-free implies that (1) is satisfied for such a rectangle.

Case 2: Four terms with x's. Next, consider an induced rectangle consisting only of terms involving x's. Due to how we assigned the terms, two terms with x's which are neighbors (Hamming distance 1) must have the same sign, and precisely one of them must have a $+2$ term. Thus, in an induced rectangle, two of the terms contain a $+2$ term, (1) becomes one of $x + 2 + x + 2 \equiv x + x$, or $-x + 2 - x + 2 \equiv -x - x$, both of which are tautologies.

Case 3: Two terms with x's which are neighbors. As the third case, consider an induced rectangle with two terms involving x's, such that the two terms involving x's are neighbors. Then, by the fact that there are no links over outputs 0 and 2, and that neighboring x's have the same sign and differ by 2, we see that the equation must be one of $x+2+2 \equiv x+0$, $x+2+0 \equiv x+2$, $-x+2+2 \equiv -x+0$, and $-x + 2 + 0 \equiv -x + 2$, which are all tautologies.

Case 4: Two terms with x's on a diagonal, opposite signs on the x terms. In our fourth case, we consider an induced rectangle with two non-neighboring terms involving x where the two x's have opposite signs. We note that, for a fixed input to x_1, all x's have the same sign, so such a rectangle must involve the dimension 1. Let one of the rows contain the terms $x + c_1$ and c_2, and the second contain c_3, and $-x + c_4$, with c_1, c_2, c_3, c_4 being constants. Let $x + d_1$ be the value in the first row at the same position as the $x + c_1$ and c_3 terms, and let d_2 be the value in the first row at the same position as the c_2 and $-x + c_4$ terms. We now see from the procedure used to assign the x terms that $c_1 \equiv d_1 + c_2 - d_2$, and $c_4 \equiv d_2 + c_3 - d_1$. Thus, $x + c_1 - x + c_4 \equiv d_1 + c_2 - d_2 + d_2 + c_3 - d_1 \equiv c_2 + c_3$, as required to satisfy (1).

Case 5: Two terms with x's on a diagonal, same signs on the x terms. Finally, we proceed to the last, and most complicated, case, with an induced rectangle with two non-neighboring terms involving x, such that both x's have the same sign. For this case, we'll need the following observation:

For every dimension j, there is a pair of constants, $p_{j,1}$ and $p_{j,2} \in 0, 2$ such that, for every $\boldsymbol{a}$ such that $g^{\boldsymbol{a}}_{\{1,j\}}(1,1), g^{\boldsymbol{a}}_{\{1,j\}}(2,2) \in \{0,2\}$:

$$g^{\boldsymbol{a}}_{\{1,j\}}(1,1) = g^{\boldsymbol{a}}_{\{1,j\}}(2,2) + p_{j,1}\,, \tag{3}$$

and for every $\boldsymbol{a}$ such that $g^{\boldsymbol{a}}_{\{1,j\}}(2,1), g^{\boldsymbol{a}}_{\{1,j\}}(1,2) \in \{0,2\}$:

$$g^{\boldsymbol{a}}_{\{1,j\}}(2,1) = g^{\boldsymbol{a}}_{\{1,j\}}(1,2) + p_{j,2}\,. \tag{4}$$

The observation follows from the fact that f does not contain an embedded OR. This, since, if there are two vectors of inputs $\boldsymbol{a}$, $\boldsymbol{b}$ such that

$$\begin{array}{l} g^{\boldsymbol{a}}_{\{1,j\}}(1,1) - g^{\boldsymbol{a}}_{\{1,j\}}(2,2) \neq \\ g^{\boldsymbol{b}}_{\{1,j\}}(1,1) - g^{\boldsymbol{b}}_{\{1,j\}}(2,2)\ \ , \end{array}$$

then an odd number of the above values must be 0, and the remaining must be 2. Thus, f has an embedded OR with partition $S_1 = \{i,j\}$ and $S_2 = S_1^C$ using inputs $(1,1);(2,2)$ and $\boldsymbol{a};\boldsymbol{b}$. The case for $p_{j,2}$ is analogous.

Furthermore, we claim that for any pair of dimensions j,k, it must be the case that $p_{j,1} + p_{j,2} + p_{k,1} + p_{k,2} \equiv 0 \pmod 4$, and thus either for all j we have $p_{j,1} + p_{j,2} \equiv 0$, or for all j, $p_{j,1} + p_{j,2} \equiv 2$. Fix a pair of dimensions j,k and inputs $\boldsymbol{a}$. We begin by noting that if j is such that f_j is a constant function, and thus the pattern of blanks does not depend on x_j, then $p_{j,1}$ and $p_{j,2}$ are undefined, and analogously for k. We reordered inputs such that $f_1(1) \neq f_1(2)$, implying that exactly one of $f^{\boldsymbol{a}}_{\{1,j,k\}}(1,1,1)$ and $f^{\boldsymbol{a}}_{\{j,k\}}(2,1,1)$ is α or β and thus that the value on one of those positions in g is given.

Consider the case when $g^{\boldsymbol{a}}_{\{1,j,k\}}(1,1,1)$ is given (the case when $f^{\boldsymbol{a}}_{\{1,j,k\}}(2,1,1)$ is given is analogous) and let $c = g^{\boldsymbol{a}}_{\{1,j,k\}}(1,1,1)$. Since $f_1(1) \neq f_1(2)$ and $f_j(1) \neq f_j(2)$, we claim that $g^{\boldsymbol{a}}_{\{1,j,k\}}(2,2,1)$ is given. This follows since $f_1(1) + f_j(1) + f_k(1) \equiv f_1(2) + f_j(2) + f_k(1) \pmod 2$. Then by (3) we see that $c \equiv g^{\boldsymbol{a}}_{\{1,j,k\}}(2,2,1) + p_{j,1}$. Analogously in dimension k, (4) implies $g^{\boldsymbol{a}}_{\{1,j,k\}}(2,2,1) \equiv g^{\boldsymbol{a}}_{\{1,j,k\}}(1,2,2) + p_{k,2}$ and thus $c \equiv g^{\boldsymbol{a}}_{\{1,j,k\}}(1,2,2) + p_{k,2} + p_{j,1}$. Continuing, (4) gives $g^{\boldsymbol{a}}_{\{1,j,k\}}(2,1,2) \equiv g^{\boldsymbol{a}}_{\{1,j,k\}}(1,2,2) + p_{j,2}$. As all values involved are 0 or 2 and we are working modulo 4, we can apply the equivalence $y \equiv -y$ and rewrite as $g^{\boldsymbol{a}}_{\{1,j,k\}}(1,2,2) \equiv g^{\boldsymbol{a}}_{\{1,j,k\}}(2,1,2) + p_{j,2}$ so $c \equiv g^{\boldsymbol{a}}_{\{1,j,k\}}(2,1,2) + p_{j,2} + p_{k,2} + p_{j,1}$. And, finally, by (3) we have $g^{\boldsymbol{a}}_{\{1,j,k\}}(1,1,1) \equiv g^{\boldsymbol{a}}_{\{1,j,k\}}(1,2,2) + p_{k,1}$ which after rewriting results in $c \equiv g^{\boldsymbol{a}}_{\{1,j,k\}}(1,1,1) + p_{j,1} + p_{j,2} + p_{k,1} + p_{k,2}$. Since we defined $c = g^{\boldsymbol{a}}_{\{1,j,k\}}(1,1,1)$ we see that $p_{j,1} + p_{j,2} + p_{k,1} + p_{k,2} \equiv 0$. We illustrate the proof in Figure 15.

With these observations, we are ready to complete the discussion of the final case of this proof. Recall that we reordered inputs such that the first two positions of the first row are d_1, and x.

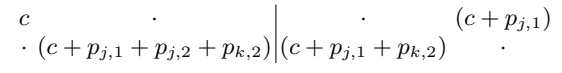

Fig. 15. Proving that $p_{j,1} + p_{j,2} + p_{k,1} + p_{k,2} \equiv 0$. Dimensions 1 and k span the 2×2 planes.

By our construction, a rectangle with two terms involving x's with the same sign on a diagonal occurs only when the input x_1 is fixed. Furthermore, we see that if the rectangle considered is spanned by dimensions i, j with inputs a_1, a_2 in dimension i and b_1, b_2 in dimension j then $f_i(a_1) \neq f_i(a_2)$ and $f_j(b_1) \neq f_j(b_2)$, as otherwise the rectangle does not have a diagonal with terms involving x's.

We further claim that we can restrict our attention to $x_i \in \{1, 2\}$ for all i. Recall that we reordered inputs such that for all dimensions affecting the pattern of blanks it changes between input 1 and 2. By the fact that f has no links over output α and β there are no links over given values 0 or 2 in g. This, together with our construction implies that for every dimension k, if a_1, a_2 are inputs such that $f_k(a_1) = f_k(a_2)$ (or, equivalently, $g^{a_1}_{\{k\}}$ has the same pattern of blanks as $g^{a_2}_{\{k\}}$) we have for all inputs $\boldsymbol{b}$ that $g^{\boldsymbol{b}}_{\{k\}}(a_1) \equiv g^{\boldsymbol{b}}_{\{k\}}(a_2) + 2 \pmod 4$. Considering the structure of (1) we see that it holds for $x_k = a_1$ iff it holds for $x_k = a_2$.

Recall that we reordered inputs such that $d_1 = g(\mathbf{1})$ is given. We now consider an induced rectangle spanned by inputs 1 and 2 in both dimensions j and k with $x_1 = 1$ with others inputs $\boldsymbol{a}$. We have $g^{\boldsymbol{a}}_{\{1,j,k\}}(1,1,1) = -x + c_1$, $g^{\boldsymbol{a}}_{\{1,j,k\}}(1,1,2) = c_2$, $g^{\boldsymbol{a}}_{\{1,j,k\}}(1,2,1) = c_3$, and $g^{\boldsymbol{a}}_{\{1,j,k\}}(1,2,2) = -x + c_4$. We know that $g^{\boldsymbol{a}}_{\{1,j,k\}}(2,1,1)$ must be given, and we denote it by c_5. By our construction, $c_1 = d_1 + c_5$ and reordering, we see that $c_5 = c_1 - d_1$. Analogously $g^{\boldsymbol{a}}_{\{1,j,k\}}(2,2,2) = c_4 - d_1$. Furthermore, by (4) we have that $c_1 - d_1 \equiv c_2 + p_{j,2}$ and by (3) we have that $c_3 \equiv c_4 - d_1 + p_{j,1}$. As $c_1, c_2, c_3, c_4, d_1, p_{j,1}, p_{j,2} \in \{0, 2\}$ we can use the identities $y \equiv -y$ and $2y \equiv 0$ with them. We need to show $-x + c_1 - x + c_4 \equiv c_2 + c_3$. Rewriting gives $2x \equiv c_1 + c_2 + c_3 + c_4 \equiv 2c_2 + 2c_3 + 2d_1 + p_{j,1} + p_{j,2} \equiv p_{j,1} + p_{j,2}$. As $p_{j,1} + p_{j,2}$ is a constant depending on f which is either 0 or 2, we see that $x = 0$ or $x = 1$ satisfies all equations on this form.

The case for $x_1 = 2$ is almost identical. Consider an induced rectangle spanned by inputs 1 and 2 in both dimensions j and k with $x_1 = 2$ and others inputs as $\boldsymbol{a}$. We have $g^{\boldsymbol{a}}_{\{1,j,k\}}(2,1,1) = x + c_1$, $g^{\boldsymbol{a}}_{\{1,j,k\}}(2,1,2) = c_2$, $g^{\boldsymbol{a}}_{\{1,j,k\}}(2,2,1) = c_3$, and $g^{\boldsymbol{a}}_{\{1,j,k\}}(2,2,2) = x + c_4$. We know that $g^{\boldsymbol{a}}_{\{1,j,k\}}(2,1,1)$ must be given, and we denote it by c_5. We have that $c_1 \equiv c_5 - d_1$ by our construction, so $c_5 \equiv c_1 + d_1$. Analogously, $g^{\boldsymbol{a}}_{\{1,j,k\}}(2,2,1) = c_4 + d_1$. Furthermore, by (3) we have that $c_1 + d_1 \equiv c_2 + p_{j,1}$ and by (4) we have that $c_3 \equiv c_4 + d_1 + p_{j,2}$. As $c_1, c_2, c_3, c_4, d_1, p_{j,1}, p_{j,2} \in \{0, 2\}$ we can use the identities $y \equiv -y$ and $2y \equiv 0$ with them. We need to show $x + c_1 + x + c_4 \equiv c_2 + c_3$. Rewriting gives $2x \equiv c_1 + c_2 + c_3 + c_4 \equiv 2c_2 + 2c_3 + 2d_1 + p_{j,1} + p_{j,2} \equiv p_{j,1} + p_{j,2}$. As $p_{j,1} + p_{j,2}$ is a constant depending on f which is either 0 or 2, we see that $x = 0$ or $x = 1$ satisfies all equations on this form.

□

A.9 Proof of Theorem 22

Theorem 22. *Every n-argument function $f : A_1 \times \dots A_n \to \mathbb{Z}_3$ that has an embedded* OR *of any degree has an embedded* OR *of degree at most* 3. *Furthermore, every* 4*-argument function $f : A_1 \times A_2 \times A_3 \times A_4 \to \mathbb{Z}_3$ that has an embedded* OR*, also has one of degree at most* 2.

Proof. Let f have an embedded OR of degree $k \geq 3$, corresponding to a partition X_1, X_2, A, with $|X_1| \geq |X_2|$. We reorder inputs such that $X_1 = \{x_1, x_2, \dots, x_k\}$. We then show that we can find a new partition X_1', X_2', A', either such that $|X_1'| < |X_1|$ and $X_2 = X_2'$, or such that $|X_1'| \leq 2$, and $|X_2'| \leq |X_2| + 1$, also corresponding to an embedded OR. From this, the theorem follows.

We relabel outputs such that the embedded OR contains three α's and one β and let $\boldsymbol{a}, \boldsymbol{b}$ be inputs for X_1, and $\boldsymbol{c}, \boldsymbol{d}$ for X_2 realizing the embedded OR. Thus, there exists $\boldsymbol{e}$ such that $f^{\boldsymbol{e}}_{\{X_1,X_2\}}(\boldsymbol{a}, \boldsymbol{c}) = f^{\boldsymbol{e}}_{\{X_1,X_2\}}(\boldsymbol{b}, \boldsymbol{c}) = f^{\boldsymbol{e}}_{\{X_1,X_2\}}(\boldsymbol{a}, \boldsymbol{d}) = \alpha$ and $f^{\boldsymbol{e}}_{\{X_1,X_2\}}(\boldsymbol{b}, \boldsymbol{d}) = \beta$. For $i \leq k$, let a_i be the element in $\boldsymbol{a}$ that is used for input x_i, and b_i the corresponding element in $\boldsymbol{b}$. We write a vector with a single element x as (x).

We now consider what happens if we take $\boldsymbol{a}$ and replace some values with the corresponding values from from $\boldsymbol{b}$. To simplify the discussion, we define for $S \subseteq X_1$ the function $g(S, \boldsymbol{x}) = f^{\boldsymbol{e}}_{\{X_1,X_2\}}(\boldsymbol{a}_{S=\boldsymbol{b}}, \boldsymbol{x})$ where $\boldsymbol{a}_{S=\boldsymbol{b}}$ denotes $\boldsymbol{a}$ with values as indicated by the subset S replaced by the corresponding values from $\boldsymbol{b}$. We proceed with a case analysis depending on what values $g(S, \boldsymbol{c})$ and $g(S, \boldsymbol{d})$ takes for nonempty $S \subset X_1$. In each case, we assume the previously discussed cases do not apply. For instance, in case 3 we assume that there is no S such that $g(S, \boldsymbol{c}) = \alpha$. We proceed with the following cases:

1. $g(S, \boldsymbol{c}) = \alpha$ or $g(S, \boldsymbol{d}) = \alpha$ for some S
2. $g(S, \boldsymbol{c}) = \beta$ and $g(S, \boldsymbol{d}) = \beta$ for some S
3. $g(S, \boldsymbol{c}) = \gamma$ and $g(S, \boldsymbol{d}) = \gamma$ for some S
4. The pair $g(S, \boldsymbol{c}), g(S, \boldsymbol{d}) \in \{(\beta, \gamma), (\gamma, \beta)\}$ for all S

Case 1: $g(S, \boldsymbol{c}) = \alpha$ or $g(S, \boldsymbol{d}) = \alpha$ for some S. If $g(S, \boldsymbol{c}) \neq g(S, \boldsymbol{d})$ then we claim that there is an embedded OR with $X_1' = S$ and $X_2' = X_2$. Assume $g(S, \boldsymbol{c}) = \alpha$ and let $\boldsymbol{e}'$ be $\boldsymbol{e}$ extended with the elements from $\boldsymbol{a}$ for arguments in $X_1' \backslash X_1$, let $\boldsymbol{a}'$ be the elements of $\boldsymbol{a}$ in $X_1' \cap X_1$, and let $\boldsymbol{b}'$ be the elements of $\boldsymbol{b}$ in $X_1' \cap X_1$. Then we can verify that

$$\begin{aligned}
f^{\boldsymbol{e}'}_{\{X_1',X_2'\}}(\boldsymbol{a}', \boldsymbol{c}) &= f^{\boldsymbol{e}}_{\{X_1,X_2\}}(\boldsymbol{a}, \boldsymbol{c}) &&= \alpha \\
f^{\boldsymbol{e}'}_{\{X_1',X_2'\}}(\boldsymbol{a}', \boldsymbol{d}) &= f^{\boldsymbol{e}}_{\{X_1,X_2\}}(\boldsymbol{a}, \boldsymbol{d}) &&= \alpha \\
f^{\boldsymbol{e}'}_{\{X_1',X_2'\}}(\boldsymbol{b}', \boldsymbol{c}) &= g(S, \boldsymbol{c}) &&= \alpha \\
f^{\boldsymbol{e}'}_{\{X_1',X_2'\}}(\boldsymbol{b}', \boldsymbol{d}) &= g(S, \boldsymbol{d}) &&\neq \alpha\,.
\end{aligned}$$

If $g(S, \boldsymbol{c}) = g(S, \boldsymbol{d}) = \alpha$ then we claim there is an embedded OR with $X_1' = X_1 \backslash S$ and $X_2' = X_2$. Let $\boldsymbol{e}'$ be $\boldsymbol{e}$ extended with the elements from $\boldsymbol{b}$ for arguments

in $X_1 \backslash X_1'$, let $\boldsymbol{a}'$ be the elements of $\boldsymbol{a}$ in $X_1' \cap X_1$, and let $\boldsymbol{b}'$ be the elements of $\boldsymbol{b}$ in $X_1' \cap X_1$. Then we can verify that

$$\begin{aligned} f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{a}', \boldsymbol{c}) &= g(S, \boldsymbol{c}) &&= \alpha \\ f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{a}', \boldsymbol{d}) &= g(S, \boldsymbol{d}) &&= \alpha \\ f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{b}', \boldsymbol{c}) &= f^{\boldsymbol{e}}_{\{X_1, X_2\}}(\boldsymbol{b}, \boldsymbol{c}) &&= \alpha \\ f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{b}', \boldsymbol{d}) &= f^{\boldsymbol{e}}_{\{X_1, X_2\}}(\boldsymbol{b}, \boldsymbol{d}) &&= \beta\,. \end{aligned}$$

Case 2: $g(S, \boldsymbol{c}) = \beta$ and $g(S, \boldsymbol{d}) = \beta$ for some S. This case is analogous to the case when $g(S, \boldsymbol{c}) = g(S, \boldsymbol{d}) = \alpha$.

Case 3: $g(S, \boldsymbol{c}) = \gamma$ and $g(S, \boldsymbol{d}) = \gamma$ for some S. We assume the previous cases did not apply to g, and thus that for all non-empty $S' \subset X_1$ we have that the pair $g(S', \boldsymbol{c}), g(S', \boldsymbol{d}) \in \{(\beta, \gamma), (\gamma, \beta), (\gamma, \gamma)\}$. We begin with the case that for all non-empty $S' \subset X_1$ we have that $g(S', \boldsymbol{c}) = g(S', \boldsymbol{d}) = \gamma$. Then we claim there is an embedded OR with $X_1' = \{x_1\}$ and $X_2' = X_2 \cup \{x_2\}$. Let $\boldsymbol{e}'$ be $\boldsymbol{e}$ extended with the elements from $\boldsymbol{a}$ for arguments in $X_1 \backslash \{x_1, x_2\}$, let $\boldsymbol{a}' = (a_1)$ and $\boldsymbol{b}' = (b_1)$, let $\boldsymbol{c}'$ be $\boldsymbol{c}$ extended with a_j, and let $\boldsymbol{d}'$ be $\boldsymbol{d}$ extended with b_j. Then $f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{a}', \boldsymbol{c}') = f^{\boldsymbol{e}}_{\{X_1, X_2\}}(\boldsymbol{a}, \boldsymbol{c}) = \alpha$ and $f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{a}', \boldsymbol{d}') = f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{b}', \boldsymbol{c}') = f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{b}', \boldsymbol{d}') = \gamma$. We illustrate this case in Figure 16.

$S =$	$\boldsymbol{x} = \boldsymbol{c}$	$\boldsymbol{x} = \boldsymbol{d}$
$\emptyset$	$\boldsymbol{\alpha}$	α
$\{x_1\}$	γ	$\boldsymbol{\gamma}$
$\{x_2\}$	$\boldsymbol{\gamma}$	γ
$\{x_1, x_2\}$	γ	$\boldsymbol{\gamma}$

Fig. 16. The function $g(S, \boldsymbol{x})$ when for all $S' : g(S', \boldsymbol{c}) = g(S', \boldsymbol{d}) = \gamma$ with an embedded OR marked by bold symbols.

We now consider the case when for some S', the pair $g(S', \boldsymbol{c}), g(S', \boldsymbol{d}) \in \{(\beta, \gamma), (\gamma, \beta)\}$. Then there is $S_1, S_2 \subset X_1$ differing in exactly one element (which we denote x_i) such that $g(S_1, \boldsymbol{c}) = g(S_1, \boldsymbol{d}) = \gamma$ and the pair $g(S_2, \boldsymbol{c}), g(S_2, \boldsymbol{d}) \in \{(\beta, \gamma), (\gamma, \beta)\}$. From this we see that there is an embedded OR with $X_1' = \{x_i\}$ and $X_2' = X_2$.

Case 4: The pair $g(S, \boldsymbol{c}), g(S, \boldsymbol{d}) \in \{(\beta, \gamma), (\gamma, \beta)\}$ for all S.

This case is the most complicated, and we proceed with two sub-cases. In our analysis, we define the function

$$h(y_1, \ldots, y_k) = \begin{cases} 0 & \text{if } g(S, \boldsymbol{d}) = \beta \\ 1 & \text{if } g(S, \boldsymbol{d}) = \gamma \end{cases}$$

where $S = \{x_i : y_i = 1\}$. We remark that $h(0, \ldots, 0)$ is left undefined (we define its value later) and $h(1, \ldots, 1) = 1$ as $g(X_1, \boldsymbol{d}) = \beta$. We claim that if there is an

embedded OR in h, it corresponds to an embedded OR in f. As h is a Boolean function, we know from Theorem 6 that if there is an embedded OR in h, then there is one of degree 1.

Consider an embedded OR in h between inputs y_i and y_j. Then we know that there is an embedded OR with β and γ with corners at $f^{\boldsymbol{e}'}_{\{i,j\}}(a_i, a_j)$, $f^{\boldsymbol{e}'}_{\{i,j\}}(a_i, b_j)$, $f^{\boldsymbol{e}'}_{\{i,j\}}(b_i, a_j)$, and $f^{\boldsymbol{e}'}_{\{i,j\}}(b_i, a_j)$ where $\boldsymbol{e}'$ is $\boldsymbol{e}$ extended with the values of $\boldsymbol{d}$ on S_2, the values of $\boldsymbol{a}$ for $\{x_i : y_i = 0\}$ and the values of $\boldsymbol{b}$ for $\{x_i : y_i = 1\}$ in the embedded OR.

We now return to $h(0, \ldots, 0)$ and define it to either of 0 and 1 that results in f remaining corner-free. We analyze the case when this fails and both $h(0, \ldots, 0) = 0$ and $h(0, \ldots, 0) = 1$ creates an embedded OR (but h was corner-free with $h(0, \ldots, 0)$ undefined) and claim that there is then a small embedded OR in f.

To see this, we consider what values h takes when exactly one of its inputs is non-zero, or, equivalently, the values of $g(S, \boldsymbol{d})$ for singleton sets S which is more notationally convenient to discuss. We proceed in two cases depending on whether $g(S, \boldsymbol{d})$ has the same output for all singleton sets or not. The embedded OR prove exists turn out to be identical to those which we prove to exist in the case when h is independent of one or more inputs.

Consider the case when there is $x_i, x_j \in X_1$ such that $g(\{x_i\}, \boldsymbol{d}) \neq g(\{x_j\}, \boldsymbol{d})$. Choose x_i such that $g(\{x_i\}, \boldsymbol{d}) = g(\{x_i, x_j\}, \boldsymbol{d})$. Then we claim there is an embedded OR with $X'_1 = \{x_i\}$ and $X'_2 = X_2 \cup \{x_j\}$. For $\boldsymbol{a}' = (a_i)$, $\boldsymbol{b}' = (b_i)$, $\boldsymbol{c}'$ as $\boldsymbol{c}$ extended with a_j, $\boldsymbol{d}'$ as $\boldsymbol{d}$ extended with b_j and $\boldsymbol{e}'$ as $\boldsymbol{e}$ extended with values from $\boldsymbol{a}$ where $X_1 \backslash \{x_i, x_j\}$. Then we have $f^{\boldsymbol{e}'}_{\{X'_1, X'_2\}}(\boldsymbol{a}', \boldsymbol{c}') = \alpha$. $f^{\boldsymbol{e}'}_{\{X'_1, X'_2\}}(\boldsymbol{a}', \boldsymbol{d}') = f^{\boldsymbol{e}'}_{\{X'_1, X'_2\}}(\boldsymbol{b}', \boldsymbol{c}') = f^{\boldsymbol{e}'}_{\{X'_1, X'_2\}}(\boldsymbol{b}', \boldsymbol{d}') \neq \alpha$. We illustrate this in Figure 17(a).

Consider the case when for all $x_i, x_j \in X_1$ we have $g(\{x_i\}, \boldsymbol{d}) = g(\{x_j\}, \boldsymbol{d})$. Then the fact that both assigning $h(0, \ldots, 0) = 0$ and assigning $h(0, \ldots, 0) = 1$ implies that there is a pair x_i, x_j such that $g(\{x_i, x_j\}, \boldsymbol{d}) = g(\{x_i\}, \boldsymbol{d})$, and that there is a pair x_k, x_l such that $g(\{x_k, x_l\}, \boldsymbol{d}) \neq g(\{x_k\}, \boldsymbol{d})$. We claim that there is an embedded OR with $X'_1 = \{x_i\}$ and $X'_2 = \{x_j\}$. Let $\boldsymbol{e}$ be $\boldsymbol{e}$ extended with all elements from $\boldsymbol{d}$ and elements from $\boldsymbol{a}$ where $X_1 \backslash \{x_i, x_j\}$. We can verify that $f^{\boldsymbol{e}'}_{\{i,j\}}(a_i, a_j) = \alpha$ and $f^{\boldsymbol{e}'}_{\{i,j\}}(b_i, a_j) = f^{\boldsymbol{e}'}_{\{i,j\}}(a_i, b_j) = f^{\boldsymbol{e}'}_{\{i,j\}}(b_i, b_j) \neq \alpha$. We illustrate this in Figure 17(b).

If h is corner-free, then Theorem 5 gives that $h(y_1, \ldots, y_k) = \sum_{i=1}^{k} f_k(y_k)$ modulo 2. We proceed in two cases, depending on whether there is an i such that h is independent of y_i, or if h depends on all its inputs.

Case 4.1: There is an i such that h is independent of y_i.

If h is independent of two variables y_i, y_j then we claim there is an embedded OR with $X'_1 = \{x_i\}$ and $X'_2 = \{x_j\}$. We can verify that $f^{\boldsymbol{e}'}_{\{i,j\}}(a_i, a_j) = \alpha$ and $f^{\boldsymbol{e}'}_{\{i,j\}}(a_i, b_j) = f^{\boldsymbol{e}'}_{\{i,j\}}(b_i, a_j) = f^{\boldsymbol{e}'}_{\{i,j\}}(b_i, b_j) \neq \alpha$ for $\boldsymbol{e}'$ as $\boldsymbol{e}$ extended with the values from $\boldsymbol{c}$ on X_2 and the values from $\boldsymbol{a}$ on $X_1 \backslash \{x_i, x_j\}$.

If h is independent of one variable y_i then we claim there is an embedded OR with $X'_1 = \{x_i\}$ and $X'_2 = X_2 \cup \{x_j\}$. We can verify that $f^{\boldsymbol{e}'}_{\{i,X'_2\}}(a_i, \boldsymbol{c}') = \alpha$ and

$S =$	$\boldsymbol{x} = \boldsymbol{c}$	$\boldsymbol{x} = \boldsymbol{d}$
$\emptyset$	$\boldsymbol{\alpha}$	α
$\{x_i\}$	β	$\boldsymbol{\gamma}$
$\{x_j\}$	$\boldsymbol{\gamma}$	β
$\{x_i, x_j\}$	β	$\boldsymbol{\gamma}$

(a) f_1

$S =$	$\boldsymbol{x} = \boldsymbol{c}$	$\boldsymbol{x} = \boldsymbol{d}$
$\emptyset$	$\boldsymbol{\alpha}$	α
$\{x_i\}$	$\boldsymbol{\beta}$	γ
$\{x_j\}$	$\boldsymbol{\beta}$	γ
$\{x_i, x_j\}$	$\boldsymbol{\beta}$	γ

(b) f_2

Fig. 17. The two cases when assigning any value to $h(0, \ldots, 0)$ creates an embedded OR in h. Identical to the cases when h is independent on one or more inputs.

$f^{\boldsymbol{e}'}_{\{i,X_2'j\}}(a_i, \boldsymbol{d}') = f^{\boldsymbol{e}'}_{\{i,X_2'\}}(b_i, \boldsymbol{c}') = f^{\boldsymbol{e}'}_{\{i,X_2'j\}}(b_i, \boldsymbol{d}') \neq \alpha$ for $\boldsymbol{e}'$ as $\boldsymbol{e}$ extended with the values from $\boldsymbol{a}$ on $X_1 \backslash \{x_i\}$, with $\boldsymbol{c}'$ as $\boldsymbol{c}$ extended with a_j on x_j, and with $\boldsymbol{d}'$ as $\boldsymbol{d}$ extended with b_j on x_j.

As the embedded OR in these two cases are identical to the ones when assigning a value to $h(0, \ldots, 0)$ creates an embedded OR in h, we refer to Figure 17 for illustrations of the cases.

Case 4.2: h depends on all its inputs.

This last case requires careful analysis. We pick three variables $x_1, x_2, x_3 \in X_1$. As $|X_1| \geq 3$ we need to distinguish between when $|X_1| = 3$ and when $|X_1 > 3$. We show all three cases in Figure 18.

We begin with $|X_1| > 3$. In this case, there is an embedded OR with $X_1' = \{x_1, x_2\}$ and $X_2' = X_2 \cup \{x_3\}$. Define $\boldsymbol{a}' = \{a_1, a_2\}$ and $\boldsymbol{b}' = \{b_1, b_2\}$. Let $\boldsymbol{c}' = \boldsymbol{c}$ extended by $\{a_3\}$ and $\boldsymbol{d}' = \boldsymbol{d}$ extended by $\{b_3\}$. Let $\boldsymbol{e}'$ be $\boldsymbol{e}$ extended by the values from $\boldsymbol{a}$ for $X_1 \backslash \{x_1, x_2, x_3\}$. We verify that $f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{a}', \boldsymbol{c}') = \alpha$ and $f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{a}', \boldsymbol{d}') = f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{b}', \boldsymbol{c}') = f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{b}', \boldsymbol{d}') \neq \alpha$.

When $|X_1| = 3$ we get two cases depending on whether $g(\{x_1\}, \boldsymbol{c}) = \beta$ or $g(\{x_1\}, \boldsymbol{c}) = \gamma$. In the latter case, we claim that the exact same embedded OR as we used for $|X_1| > 3$ exists in f with $f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{a}', \boldsymbol{d}') = f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{b}', \boldsymbol{c}') = f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{b}', \boldsymbol{d}') = \beta$.

Finally, when $|X_1| = 3$ and $g(\{x_1\}, \boldsymbol{c}) = \beta$, we claim there is an embedded OR with $X_1' = \{x_1\}$ and $X_2' = X_2 \cup \{x_2\}$. We let $\boldsymbol{a}' = (b_1)$, $\boldsymbol{b}' = (a_1)$, let $\boldsymbol{c}'$ be $\boldsymbol{c}$ extended with a_2, $\boldsymbol{d}'$ be $\boldsymbol{d}$ extended with b_2 and $\boldsymbol{e}'$ be $\boldsymbol{e}$ extended with b_3. We can verify $f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{a}', \boldsymbol{c}') = \gamma$ and $f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{a}', \boldsymbol{d}') = f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{b}', \boldsymbol{c}') = f^{\boldsymbol{e}'}_{\{X_1', X_2'\}}(\boldsymbol{b}', \boldsymbol{d}') = \beta$.

□

$S =$	$\boldsymbol{x = c}$	$\boldsymbol{x = d}$
$\emptyset$	$\boldsymbol{\alpha}$	α
$\{x_1\}$	β	γ
$\{x_2\}$	β	γ
$\{x_3\}$	β	$\boldsymbol{\gamma}$
$\{x_1, x_2\}$	$\boldsymbol{\gamma}$	β
$\{x_1, x_3\}$	γ	β
$\{x_2, x_3\}$	γ	β
$\{x_1, x_2, x_3\}$	β	$\boldsymbol{\gamma}$

(a) f_1

$S =$	$\boldsymbol{x = c}$	$\boldsymbol{x = d}$
$\emptyset$	$\boldsymbol{\alpha}$	α
$\{x_1\}$	γ	β
$\{x_2\}$	γ	β
$\{x_3\}$	γ	$\boldsymbol{\beta}$
$\{x_1, x_2\}$	$\boldsymbol{\beta}$	γ
$\{x_1, x_3\}$	β	γ
$\{x_2, x_3\}$	β	γ
$\{x_1, x_2, x_3\}$	α	$\boldsymbol{\beta}$

(b) f_2

$S =$	$\boldsymbol{x = c}$	$\boldsymbol{x = d}$
$\emptyset$	α	α
$\{x_1\}$	β	γ
$\{x_2\}$	β	γ
$\{x_3\}$	$\boldsymbol{\beta}$	γ
$\{x_1, x_2\}$	γ	β
$\{x_1, x_3\}$	γ	$\boldsymbol{\beta}$
$\{x_2, x_3\}$	$\boldsymbol{\gamma}$	β
$\{x_1, x_2, x_3\}$	α	$\boldsymbol{\gamma}$
X_1	α	$\boldsymbol{\beta}$

(c) f_3

Fig. 18. The three cases of Case 4.2. In f_1, $|X_1| > 3$. In f_2, $|X_1| = 3$ and $g(\{x_1\}, \boldsymbol{c}) = \beta$, and in f_3, $|X_1| = 3$ and $g(\{x_1\}, \boldsymbol{c}) = \gamma$.