

Solving a 676-bit Discrete Logarithm Problem in $\text{GF}(3^{6n})$

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Abstract. Pairings on elliptic curves over finite fields are crucial for constructing various cryptographic schemes. The η_T pairing on supersingular curves over $\text{GF}(3^n)$ is particularly popular since it is efficiently implementable. Taking into account the Menezes-Okamoto-Vanstone (MOV) attack, the discrete logarithm problem (DLP) in $\text{GF}(3^{6n})$ becomes a concern for the security of cryptosystems using η_T pairings in this case. In 2006, Joux and Lercier proposed a new variant of the function field sieve in the medium prime case, named JL06-FFS. We have, however, not yet found any practical implementations on JL06-FFS over $\text{GF}(3^{6n})$. Therefore, we first fulfill such an implementation and we successfully set a new record for solving the DLP in $\text{GF}(3^{6n})$, the DLP in $\text{GF}(3^{6 \cdot 71})$ of 676-bit size. In addition, we also compare JL06-FFS and an earlier version, named JL02-FFS, with practical experiments. Our results confirm that the former is several times faster than the latter under certain conditions.

Key words: function field sieve, discrete logarithm problem, pairing-based cryptosystems

1 Introduction

Based on pairings, many novel cryptographic protocols have been successively constructed, such as identity-based encryptions [8], forward-secure cryptosystems, proxy cryptosystems, keyword searchable PKEs [7]. As a result, two requirements arose: efficient pairing computation and security parameter selection.

The η_T pairing [5] on supersingular curves over $\text{GF}(3^n)$ has been efficiently implemented both in software and hardware [6, 13, 14]¹. Along with the increase in computation speed on the η_T pairing, one may ask whether cryptosystems based on the η_T pairing are still secure. It is well known that a discrete logarithm problem (DLP) on supersingular curves over $\text{GF}(q)$ can be converted to a DLP in $\text{GF}(q^m)$ (where q is a prime power and m is not larger than 6) [24]. Therefore, the DLP in $\text{GF}(3^{6n})$ is one of the most important problems in analyzing the cryptosystems constructed with the η_T pairing on supersingular curves over $\text{GF}(3^n)$.

¹ Here, n is a prime number such as $n = 97, 163$ and 193 [25].

The function field sieve (FFS) is the most efficient algorithm for solving the DLP in finite fields of small characteristic. The complexity of the FFS for solving the DLP in $\text{GF}(3^{6n})$ is $L_{3^{6n}}[1/3, c]$ with constant c , where

$$L_{3^{6n}}[1/3, c] = \exp((c + o(1))(\log 3^{6n})^{1/3}(\log \log 3^{6n})^{2/3}).$$

Here $o(1)$ stands for a function that converges to zero as n approaches infinity.

The first FFS was proposed by Adleman [1] in 1994. Five years later, Adleman and Huang proposed an improved FFS (AH-FFS) with $c = (32/9)^{1/3}$ [2]. In 2002, Joux and Lercier proposed a practical improvement of the FFS (JL02-FFS) [16]. Since a definition polynomial of the function field in JL02-FFS can select more flexibly, JL02-FFS is more practical than AH-FFS, though its asymptotic complexity is the same as that of AH-FFS. Furthermore, by using JL02-FFS, Joux and Lercier succeeded in solving the DLP in $\text{GF}(2^{613})$. This refreshed the record for solving the DLP in finite fields of characteristic two with regard to bit size [15]. In 2006, Joux and Lercier proposed another new variant of the FFS (JL06-FFS) [18]. JL06-FFS has the same asymptotic complexity with JL02-FFS for solving the DLP in $\text{GF}(3^{6n})$, where n is a prime number². This work implied that JL06-FFS might be efficient for solving the DLP in extension fields of $\text{GF}(3^6)$ of degree n . However, to our knowledge, there have been no practical experiments. Note that JL02-FFS can also be applied to extension fields of $\text{GF}(3^6)$ of degree n , but [12] showed no advantage using $\text{GF}(3^6)$ as the base field.

Our contributions. We have first conducted experiments on JL06-FFS. In JL06-FFS, $\text{GF}(3^{6n})$ is constructed as extension fields of $\text{GF}(3^6)$ of degree n , and thus the Galois action can be dealt for reducing required relations. By our implementation, we succeeded in solving the DLP in $\text{GF}(3^{6 \cdot 71})$ of 676-bit size with about 33 days computation, which is the new record for solving the DLP in $\text{GF}(3^{6n})$. Our work contributes to the selecting of security parameters. Additionally, we compared JL06-FFS [18] with JL02-FFS [16], and according to the experimental results, we confirmed that JL06-FFS is several times faster than JL02-FFS with $n = 19, 61$.

The rest of the paper is organized as follows. In Section 2, we briefly review the FFS algorithm. In Section 3, we compare JL02-FFS with JL06-FFS according to the polynomial selection method and experimental results. In Section 4, we describe our implementation on how to solve the DLP in $\text{GF}(3^{6 \cdot 71})$ in detail, which is based on JL06-FFS. Concluding remarks are made in Section 5.

2 Outline of Function Field Sieve

In this section, we describe an overview of the FFS [1], which consists of four steps: polynomial selection, collection of relations, linear algebra, and individual logarithm. We particularly deal with the FFS for solving the DLP in extension

² When n is a composite number, this variant may have complexity $L_{3^{6n}}[1/3, 3^{1/3}]$ for solving the DLP in $\text{GF}(3^{6n})$ (When JL06-FFS has complexity $L_{q^m}[1/3, 3^{1/3}]$, we call it JL06-FFS-2). We do not deal with this case in this paper.

fields of $\text{GF}(3^6)$ of degree n and describe the four steps below. For more details, refer to related work as [1, 12, 16, 18].

Throughout this paper, let γ be a generator of the multiplicative group of $\text{GF}(3^{6n})$ and $\alpha \in \langle \gamma \rangle$, then we try to find the smallest positive integer $\log_\gamma \alpha$ such that $\gamma^{\log_\gamma \alpha} = \alpha$, which is called the discrete logarithm.

1. **Polynomial selection:** Select $f \in \text{GF}(3^6)[x]$ such that f is a monic irreducible polynomial of degree n , then $\text{GF}(3^{6n}) \cong \text{GF}(3^6)[x]/(f)$. Next, find a polynomial $H(x, y) \in \text{GF}(3^6)[x, y]$ satisfying the eight conditions proposed by Adleman [1]. Then there is a surjective homomorphism

$$\Phi : \begin{cases} \text{GF}(3^6)[x, y]/(H) & \rightarrow \text{GF}(3^{6n}) \cong \text{GF}(3^6)[x]/(f) \\ y & \mapsto m, \end{cases}$$

where m is in $\text{GF}(3^6)[x]$ such that $H(x, m) \equiv 0 \pmod{f}$. Here we select the smoothness bound B and define a rational factorbase B_R and an algebraic factorbase B_A as follows:

$$\begin{aligned} B_R &= \{\mathfrak{p} \in \text{GF}(3^6)[x] \mid \deg(\mathfrak{p}) \leq B, \mathfrak{p} \text{ is irreducible}\}, \\ B_A &= \{\langle \mathfrak{p}, y - t \rangle \in \text{Div}(\text{GF}(3^6)[x, y]/(H)) \mid \mathfrak{p} \in B_R, t \equiv m \pmod{\mathfrak{p}}\}, \end{aligned}$$

where $\text{Div}(\text{GF}(3^6)[x, y]/(H))$ is the divisor group of $\text{GF}(3^6)[x, y]/(H)$ and $\langle \mathfrak{p}, y - t \rangle$ is a divisor generated by \mathfrak{p} and $y - t$.

2. **Collection of relations:** For $r, s \in \text{GF}(3^6)[x]$ of degree not larger than B , find at least $(\#B_R + \#B_A)$ relatively prime pairs (r, s) such that

$$\begin{aligned} rm + s &= \prod_{\mathfrak{p}_i \in B_R} \mathfrak{p}_i^{a_i} \\ \langle ry + s \rangle &= \sum_{\langle \mathfrak{p}_j, t_j \rangle \in B_A} b_j \langle \mathfrak{p}_j, y - t_j \rangle. \end{aligned} \quad (1)$$

Such a pair (r, s) is called a double smooth pair. For each (r, s) , compute the following equations:

$$rm + s, \quad (2)$$

$$(-r)^d H(x, -s/r). \quad (3)$$

Equation (3) is said to be B -smooth if it is factorized into irreducible polynomials of degree not larger than B , and then we have

$$(-r)^d H(x, -s/r) = \prod_{\langle \mathfrak{p}_j, t_j \rangle \in B_A} \mathfrak{p}_j^{b_j}, \quad (4)$$

where t_j is uniquely determined by r, s and \mathfrak{p}_j . Then the b_j in Equation (4) is exactly the same as the one in Equation (1). When both Equations (2) and (3) are B -smooth, a pair (r, s) is a double smooth pair. Eventually, we obtain the following relation:

$$\sum_{\mathfrak{p}_i \in B_R} a_i \log_\gamma \mathfrak{p}_i \equiv \sum_{\langle \mathfrak{p}_j, t_j \rangle \in B_A} b_j \log_\gamma \kappa_j \pmod{(3^{6n} - 1)/(3^6 - 1)}, \quad (5)$$

where

$$\kappa_j = \Phi(\lambda_j)^{1/h}, \quad \langle \lambda_j \rangle = h \langle \mathfrak{p}_j y - t_j \rangle, \quad (6)$$

for the class number h of the quotient field of $\text{GF}(3^6)(x)[y]/(H)$.

3. **Linear algebra:** For the number R of relations, construct an $R \times (\#B_R + \#B_A)$ matrix M from the relations in Equation (5) and $(\#B_R + \#B_A)$ dimensional column vector \mathbf{v} as follows:

$$M = \begin{pmatrix} a_1^{(1)} & \dots & a_{\#B_R}^{(1)} & -b_1^{(1)} & \dots & -b_{\#B_A}^{(1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_1^{(R)} & \dots & a_{\#B_R}^{(R)} & -b_1^{(R)} & \dots & -b_{\#B_A}^{(R)} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} \log_\gamma \mathfrak{p}_1 \\ \vdots \\ \log_\gamma \mathfrak{p}_{\#B_R} \\ \log_\gamma \kappa_1 \\ \vdots \\ \log_\gamma \kappa_{\#B_A} \end{pmatrix}.$$

Then we solve the linear equation

$$M\mathbf{v} \equiv 0 \pmod{(3^{6n} - 1)/(3^6 - 1)}. \quad (7)$$

4. **Individual logarithm:** Find integers e_i, f_j such that

$$\log_\gamma \alpha \equiv \sum_{\mathfrak{p}_i \in B_R} e_i \log_\gamma \mathfrak{p}_i + \sum_{\langle \mathfrak{p}_j, t_j \rangle \in B_A} f_j \log_\gamma \kappa_j \pmod{(3^{6n} - 1)/(3^6 - 1)},$$

then compute the discrete logarithm $\log_\gamma \alpha$. This is done using the special- \mathfrak{q} descent method [16, 18, 19].

3 Comparison of Polynomial Selection on JL02-FFS and JL06-FFS

The two most efficient variants of the FFS for solving the DLP in $\text{GF}(3^{6n})$ are JL02-FFS and JL06-FFS. Although they have the same asymptotic complexity, there is a considerable difference between them in the fixed extension degree for practical use. The time complexities of JL02-FFS and JL06-FFS depend on the size of each sieving area, which is the number of pairs (r, s) , and each size is explained in the following subsections. Note that our comparison is done merely by the size of the sieving area, and the detailed analysis should incorporate the non-integer smoothness bound estimated by Granger [11].

3.1 Polynomial Selection of JL02-FFS and Its Sieving Area

At first we describe an outline of the polynomial selection of JL02-FFS, after that we estimate the size of the sieving area. In order to distinguish from previous section, we set the subindex “02” after the symbols.

Let $H_{02}(x, y)$ of degree d_{02} in y be formed as C_{ab} curves [23]:

$$H_{02}(x, y) = h_{a,0}y^a + h_{0,b}x^b + \sum_{ib+ja < ab} h_{i,j}y^i x^j \quad (h_{i,j} \in \text{GF}(3), h_{a,0}, h_{0,b} \neq 0).$$

Then randomly choose the polynomial $u_1, u_2 \in \text{GF}(3)[x]$ of degree at most $\lceil 6n/d_{02} \rceil$. We try to find an irreducible polynomial $f_{02} = u_2^{d_{02}} H_{02}(x, -u_1/u_2) \in \text{GF}(3)[x]$ of degree $6n$ such that $\gcd(u_2, f_{02}) = 1$, then u_2 is invertible modulo f_{02} . Then, there is a surjective homomorphism

$$\Phi_{02} : \begin{cases} \text{GF}(3)[x, y]/(H_{02}) & \rightarrow \text{GF}(3^{6n}) \cong \text{GF}(3)[x]/(f_{02}) \\ y & \mapsto -u_1/u_2, \end{cases}$$

where $H_{02}(x, y)$ holds $H_{02}(x, -u_1/u_2) \equiv 0 \pmod{f_{02}}$. In this polynomial selection, we need to modify Equation (2) to $su_2 - ru_1$. Note that r and s are chosen in $\text{GF}(3)[x]$ of degree not larger than B_{02} in JL02-FFS, the size of the sieving area in the collection of relation step is

$$3^{B_{02}+1} \cdot 3^{B_{02}+1}. \quad (8)$$

From heuristic analysis in [16], JL02-FFS becomes optimized when we choose the smoothness bound B_{02} as

$$B_{02} = \lceil (4/9)^{1/3} (6n)^{1/3} \log_3(6n)^{2/3} \rceil. \quad (9)$$

and the extension degree d_{02} of $H_{02}(x, y)$ as $d_{02} = \lceil \sqrt{6n/(B_{02}+1)} \rceil$. For example, for $n = 97, 163, 193$, we have $(n, B_{02}) = (97, 21), (163, 26), (193, 28)$.

3.2 Polynomial Selection of JL06-FFS and Its Sieving Area

Next we describe an outline of the polynomial selection of JL06-FFS and estimate the size of the sieving area of JL06-FFS.

For each extension degree n of $\text{GF}(3^6)$, we choose the smallest smoothness bound B_{06} in JL06-FFS satisfying the following condition,

$$(B_{06} + 1) \log(3^6) \geq \sqrt{n/B_{06}} \log(n/B_{06}) \quad (10)$$

For example, for $n = 97, 163, 193$, we have $(n, B_{06}) = (97, 3), (163, 4), (193, 4)$. Next, we choose positive integers d and d' such that $d \approx \sqrt{n/B_{06}}$ and $d' \approx \sqrt{nB_{06}}$, where $dd' \geq n$. After that, we randomly generate $g(y) \in \text{GF}(3^6)[y]$ of degree d and set $H(x, y) = g(y) + x$. Finally, we try to find an irreducible polynomial f in $\text{GF}(3^6)[x]$ of degree n , which divides $H(x, m)$, where $m \in \text{GF}(3^6)[x]$ of degree d' is chosen randomly. In this polynomial selection, each of the leading coefficients of Equations (2) and (3) depends on r , so we avoid obtaining duplicate relations by fixing the leading coefficient of r as a monic polynomial. Therefore, the size of the sieving area in the collection of relations step is at most

$$(3^6)^{B_{06}+1} \cdot (3^6)^{B_{06}}. \quad (11)$$

3.3 Comparison of Sieving Area

We compare JL06-FFS with JL02-FFS with respect to the size of the sieving area in the collection of relations step in three classes of extension degree n : *experimental class* as $\{19, 31, 47, 61\}$, *medium-security class* as $\{97, 163, 193\}$, and *high-security class* as $\{239, 313, 353, 509\}$. Table 1 lists the smoothness bound and size of the sieving area in each variant. For each n , we obtain the smoothness bound B_{02} in Equation (9) and B_{06} in Equation (10), and estimate the size of the sieving area by Equation (8) in JL02-FFS and by Equation (11) in JL06-FFS.

Table 1. Parameters and sieving area

| | n | Polynomial selection in JL02-FFS | | | Polynomial selection in JL06-FFS | | |
|------------------------------|-----|-------------------------------------|----------|-------------------------|-------------------------------------|----------|-------------------------|
| | | $6n$ | B_{02} | Size of sieving area | n | B_{06} | Size of sieving area |
| Experimental class | 19 | 114 | 10 | 3.1×10^{10} | 19 | 1 | 3.9×10^8 |
| | 31 | 186 | 12 | 2.5×10^{12} | 31 | 2 | 2.1×10^{14} |
| | 47 | 282 | 15 | 1.9×10^{15} | 47 | 2 | 2.1×10^{14} |
| | 61 | 366 | 17 | 1.5×10^{17} | 61 | 2 | 2.1×10^{14} |
| Medium- security class | 97 | 582 | 21 | 9.8×10^{20} | 97 | 3 | 1.1×10^{20} |
| | 163 | 978 | 26 | 5.8×10^{25} | 163 | 4 | 5.8×10^{25} |
| | 193 | 1158 | 28 | 4.7×10^{27} | 193 | 4 | 5.8×10^{25} |
| High-security class | 239 | 1434 | 30 | 3.8×10^{29} | 239 | 4 | 5.8×10^{25} |
| | 313 | 1878 | 34 | 2.5×10^{33} | 313 | 5 | 3.1×10^{31} |
| | 353 | 2118 | 36 | 2.0×10^{35} | 353 | 5 | 3.1×10^{31} |
| | 509 | 3054 | 42 | 1.1×10^{41} | 509 | 6 | 1.6×10^{37} |

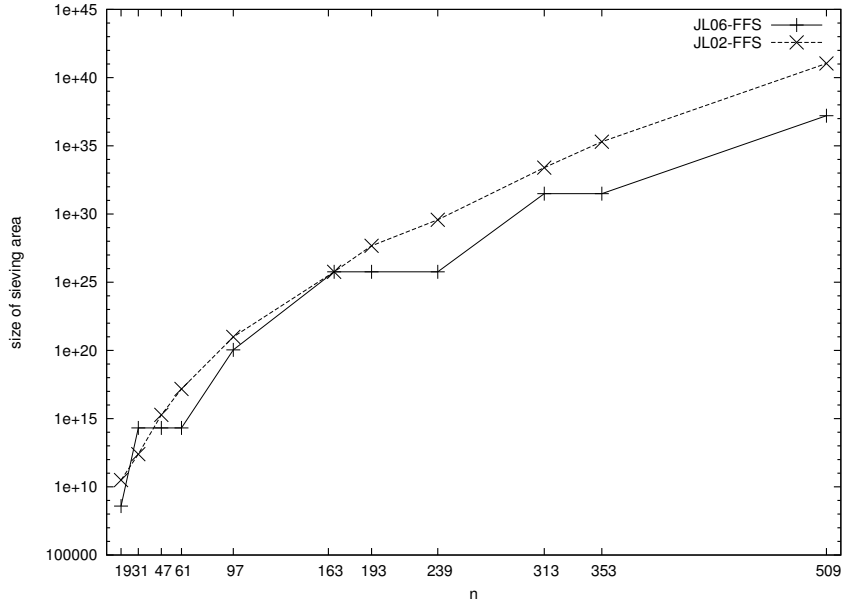


Fig. 1. Size of sieving area over $\text{GF}(3^{6n})$ in JL02-FFS and JL06-FFS

Figure 1 shows the size of the required sieving area over $\text{GF}(3^{6n})$. The sieving area in JL06-FFS is much smaller than that in JL02-FFS when $n \neq 31, 163$. Moreover, the differences between the sieving areas in JL06-FFS and in JL02-FFS increase along with the increase in n . The computational cost in the collection of relations step is closely related to the size of the sieving area, so the collection of relations step in JL06-FFS might be several times faster than that in JL02-FFS.

We have conducted experiments on the collection of relations step in JL02-FFS and JL06-FFS to confirm the difference between their computational costs of that step. Parameters in JL02-FFS and JL06-FFS are listed in Table 2. The curves that we used in our experiments are superelliptic ones, but not C_{ab} curves as [12]. Note that we have only experimented with the experimental class as $n \in \{19, 31, 47, 61\}$, not with medium and high-security classes.

Table 2. Parameters in our experiments

| n | Bit size of $\text{GF}(3^{6n})$ | Experiments with JL02-FFS | | | Experiments with JL06-FFS | | |
|-----|---------------------------------|---------------------------|----------|----------------|---------------------------|----------|-----------|
| | | $6n$ | B_{02} | $H_{02}(x, y)$ | n | B_{06} | $H(x, y)$ |
| 19 | 181 | 114 | 10 | $y^4 + x$ | 19 | 1 | $y^5 + x$ |
| 31 | 295 | 186 | 12 | $y^4 + x$ | 31 | 2 | $y^4 + x$ |
| 47 | 447 | 282 | 15 | $y^4 + x$ | 47 | 2 | $y^5 + x$ |
| 61 | 581 | 366 | 17 | $y^5 + x$ | 61 | 2 | $y^6 + x$ |

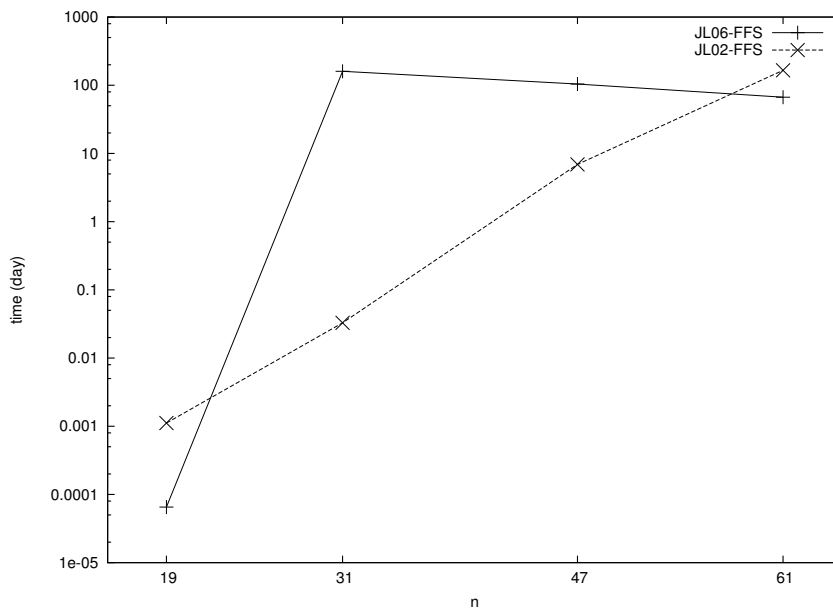


Fig. 2. Estimated time taken to compute entire sieving area in the collection of relations step over $\text{GF}(3^{6n})$ in JL02-FFS and JL06-FFS

In our experiments, we used 96 cores, each of which had the same performance about Intel 2.83GHz Xeon. We implemented the lattice sieve [26] in JL02-FFS as [12, 15, 16]. On the other hand, we implemented the polynomial sieve [10] in JL06-FFS, since we fixed r as a monic polynomial in the collection of relations step and so the lattice sieve might not be efficient. The details of our implementation in JL06-FFS are described in Section 4.

Figure 2 shows the time complexity of JL02-FFS and JL06-FFS to compute the entire sieving area in the collection of relations step in $\text{GF}(3^{6n})$ with $n = 19, 31, 47, 61$, respectively. Note that we estimated the time when the computation lasts over one hour.

When $n = 19, 61$, our implementation on JL06-FFS is faster than that on JL02-FFS, and we confirm that JL06-FFS is more efficient than JL02-FFS for solving the DLP in $\text{GF}(3^{6n})$. In particular, when $n = 61$, our implementation of JL06-FFS takes about 66 days for the collection of relations step, but our implementation of JL02-FFS takes about 165 days for the same step. Therefore, the former is 2.5 times faster than the latter. Accordingly, we expect that JL06-FFS will be efficient for solving the DLP in $\text{GF}(3^{6n})$ for larger n .

4 Solving the DLP in $\text{GF}(3^{6 \cdot 71})$

In this section, we report that the DLP in $\text{GF}(3^{6 \cdot 71})$ of 676-bit size is solved by improving JL06-FFS. In our implementation, we deal with four practical improvements, polynomial sieve, free relation, Galois action, and parallel Lanczos method.

Particularly, by using the polynomial $H(x, y) = y^6 + x$, we only need to find about 1/8 of the originally required relations in the collection of relations step. Furthermore, via the Galois action, the size of the matrix given by the relations is also decreased to 1/6 of the original. To the best of our knowledge, the 676-bit size is currently the record for solving the DLP in $\text{GF}(3^{6n})$.

4.1 Collection of Relations

In the collection of relations step, we collect many double smooth pairs (r, s) . The simple idea for collecting them is factoring Equations (2) and (3) for all pairs (r, s) . This is not practical since we have to factor them about $(3^6)^B \times (3^6)^{B+1}$ times. In order to reduce the number of factorings, we use a sieving method. The idea of sieving is merely factoring Equations (2) and (3) of the pair (r, s) , which has a high probability of becoming a double smooth pair. Such a pair is called a candidate.

The polynomial sieve [10] and the lattice sieve [26] are well-known sieving algorithms. Although the lattice sieve has been implemented in some experiments of the FFS [12, 15, 16], we implemented the polynomial sieve since r is fixed as a monic polynomial by the polynomial sieve in JL06-FFS, whereas neither r nor s is able to be fixed by the lattice sieve.

Polynomial Sieve We describe the polynomial sieve in Equation (2), namely, $rm + s$. Notice that we can also sieve in Equation (3) with the same procedure. Moreover, we discuss the case where s is fixed and omit the details when r is fixed. By fixed s , we can lead r such that $rm + s$ is divisible by $\mathfrak{p} \in B_R$ or its power, where the degree of \mathfrak{p} is not larger than B . Additionally, $rm + s + k\mathfrak{p}$ with $k \in \text{GF}(3^6)[x]$ is also divisible by \mathfrak{p} . Hence, we can obtain all r of degree less than or equal to B such that $rm + s$ is divisible by \mathfrak{p} . After computing such all r for each \mathfrak{p} , we can obtain the pair (r, s) such that $rm + s$ is divisible by some \mathfrak{p} . If the summation of the degree of all \mathfrak{p} , which divide $rm + s$, reaches $\deg(rm + s)$, then $rm + s$ has a high probability of becoming B -smooth and the pair (r, s) becomes a candidate.

In this procedure, the most time-consuming work is to compute $r + k\mathfrak{p}$ for all $k \in \text{GF}(3^6)[x]$ whose degree is not larger than B . In characteristic two, Gordon and McCurley proposed a method using binary gray codes [10] to compute these $r + k\mathfrak{p}$. Using ternary gray codes, we can also compute them efficiently in characteristic three.

In the polynomial sieve, we sieve with all powers of \mathfrak{p} whose degree is not larger than B . Since B is very small, such as 1 or 2 in JL06-FFS, the power of \mathfrak{p} is only \mathfrak{p}^2 when $\deg(\mathfrak{p}) = 1$. Such polynomials are exceptional since there are 3^6 monic irreducible polynomials of degree 1 in $\text{GF}(3^6)[x]$. In this way, we can obtain only candidates each of which generates a relation in Equation (5) (except that r and s are not relatively prime). Thus, we only check the greatest common divisor of r and s , but not the smoothness of Equations (2) and (3) using the B -smooth test [10].

Free Relation By considering how a divisor $\langle \mathfrak{p} \rangle$ in B_R is factorized into divisors in $\text{GF}(3^6)[x, y]/(H)$, namely, obtaining the following congruent expression that

$$H(x, y) \equiv \prod_{i=1}^d (y - t_i) \pmod{\mathfrak{p}},$$

where d is the degree of $H(x, y)$ on y , we can obtain a relation virtually for free, without the sieving procedure. We call such a relation a free relation.

The number of free relations depends on the degree d of $H(x, y)$ on y and the characteristic of the field treated in the FFS. In fact, there are about $\#B_A/d$ free relations in many cases and, furthermore, they increase when the characteristic is small. For example, in the case of $\text{GF}(3^{6n})$ and $H(x, y) = y^6 + x$, there are about $\#B_A/2$ free relations since $y^6 + x$ is generally factored as $(y - t_1)^3(y - t_2)^3$ modulo \mathfrak{p} .

4.2 Linear Algebra

In the linear algebra step, we solve the linear equation depending on the relations. Specifically, we construct a matrix from the relations and reduce it to a much smaller one using the Galois action. After that, we solve the reduced linear

equation modulo $(3^{6n} - 1)/(3^6 - 1)$, by applying the parallel Lanczos method described as [3]. In this section, we describe the Galois action and our ideas about parallel computation of the matrix operation.

Galois Action Here, we consider to reduce unknowns of linear equations, using the Galois action which was presented in [18].

Let M' be the matrix given by the relations, whose row $M'_{(i)}$ means the i -th relation and j -th column $M'^{(j)}$ corresponds to the factorbase \mathbf{p}_j . In order to use the Galois action, we choose the polynomial $f \in \text{GF}(3^6)[x]$ satisfying that all coefficients of f are in $\text{GF}(3)$ and $\deg f = n$, then we construct $\text{GF}(3^{6n})$ as $\text{GF}(3^6)[x]/(f)$. Let ϕ be the Frobenius power such that $\phi(\xi) = \xi^{3^n}$. As ϕ fixes the element x in $\text{GF}(3)[x]/(f)$, we also have $\phi(x) = x$ in $\text{GF}(3^6)[x]/(f)$ by the assumption of f . However, for an element $c \in \text{GF}(3^6) \setminus \text{GF}(3)$, ϕ does not fix c in $\text{GF}(3^6)[x]/(f)$ by the above assumption that n is coprime to 6. The monic irreducible polynomial $\mathbf{p}_j \in B_R$ of degree not larger than B , and we assume that $B = 1$ for convenience. In fact, $\mathbf{p}_j = x + c_j$ where $c_j \in \text{GF}(3^6)$ since $B = 1$, so we have

$$\phi(\mathbf{p}_j) = \phi(x + c_j) = x + \phi(c_j)$$

in $\text{GF}(3^6)[x]/(f)$. If c_j is not in $\text{GF}(3)$, it is clear that $c_j \neq \phi(c_j)$ in $\text{GF}(3^6)[x]/(f)$. This fact implies that there are ordinarily many unknowns of linear equations, which can be rewritten by the other one via the Galois action. Clearly, for such \mathbf{p}_j , there exists $\mathbf{p}_{j'}$ satisfying that

$$\log_\gamma \mathbf{p}_{j'} = \log_\gamma \phi(\mathbf{p}_j) = 3^n \log_\gamma \mathbf{p}_j \quad (12)$$

where $\mathbf{p}_j \neq \mathbf{p}_{j'}$. Therefore, we can remove the j' -th column $M'^{(j')}$ and set the j -th column $M'^{(j)}$ as $M'^{(j)} + 3^n M'^{(j')}$. Then we denote the new matrix M^* as the reduced M' . Notice that this technique is also used for the algebraic factorbase. Consequently, the number of unknowns is about 1/6 of the original; thus, the number of relations is reduced to about 1/6. In our implementation, we do not reduce the factorbase in the sieving phase (the computation is the same as the case without the Galois action). After sieving, we compress obtained relations using rewritable elements of the factorbase via the Galois action as Equation (12), and so the factorbase is reduced to about 1/6. Using this procedure, we almost do not lose the probability of obtaining the relation. Hence, this technique enables us to perform computations for the collection of relations step about 6 times as fast as before, and the linear algebra step can be also done about 6^2 times faster.

Parallel Lanczos method The reduced matrix M^* is reconstructed to optimize first, then we apply the parallel Lanczos method to it. Before explaining the reconstruction, we begin with the explanation of the parallel computation. Assume that there are four nodes written as $N_{1,1}, N_{1,2}, N_{2,1}, N_{2,2}$ and each node has 4 or 8 cores. As the Figure 3, we partition the reconstructed matrix M into four matrices $M_{i,j}$, and each $M_{i,j}$ is allotted to node $N_{i,j}$ respectively. The given

vector \mathbf{v} is also partitioned into $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_j is given to nodes $N_{i,j}, N_{i',j}$ where $i \neq i'$. Moreover, $M_{i,j}$ is partitioned into L matrices A_ℓ when $N_{i,j}$ has L cores.

Fig. 3. Partitioning M into four matrices $M_{i,j}$ and $M_{i',j}$ into L matrices A_ℓ .

$$M\mathbf{v} = \left(\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & M_{2,2} \end{array} \right) \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}. \quad M_{i,j}\mathbf{v}_j := A\mathbf{v}_j = \begin{pmatrix} \frac{A_1}{\hline} \\ \frac{A_2}{\hline} \\ \vdots \\ \frac{A_L}{\hline} \end{pmatrix} \mathbf{v}_j.$$

We now give the notation of the Lanczos method. The Lanczos method can operate only a symmetric matrix; however, the given matrix M is usually non-symmetric. Therefore, we try to solve the linear equation of the form $M^T M\mathbf{v} = \boldsymbol{\alpha}$, where \mathbf{v} is an unknown column vector consisting of the logarithms of the factorbase and $\boldsymbol{\alpha}$ is the given column vector. Note that computing $M^T M$ is not efficient, so we compute the vector $\mathbf{u} = M\mathbf{v}$ and $M^T \mathbf{u}$. For more details about this computation is in [22].

After partitioning M , we perform a parallel computation for $\mathbf{u} := M\mathbf{v}$ and $\mathbf{w} := M^T \mathbf{u}$ with $M_{i,j}$. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}_1$, and \mathbf{u}_2 be the partitioned vectors such that $\mathbf{v} = \mathbf{v}_1 \oplus \mathbf{v}_2$ and $\mathbf{u} = \mathbf{u}_1 \oplus \mathbf{u}_2$. From Algorithm 1, we obtain the partitioned vector \mathbf{w}_i such that $\mathbf{w} = \mathbf{w}_i \oplus \mathbf{w}_{i'}$ in node $N_{i,j}$, where $i \in \{1, 2\}$ and $i' = 3 - i$. The symbol j' also means that $j' = 3 - j$ for $j \in \{1, 2\}$.

Algorithm 1 (Computation with node $N_{i,j}$.)

Input : the partitioned matrix $M_{i,j}$ and the partitioned vector \mathbf{v}_j .

Output : the partitioned vector \mathbf{w}_j such that $\mathbf{w}_1 \oplus \mathbf{w}_2 = M^T M\mathbf{v}$, where j is equal to 1 or 2.

[Step for computation of $\mathbf{u} := M\mathbf{v}$]

1. $\mathbf{u}_{i,j} := M_{i,j}\mathbf{v}_j$.
2. Give $\mathbf{u}_{i,j}$ to Node $N_{i,j'}$ and receive $\mathbf{u}_{i,j'}$ from $N_{i,j'}$.
3. $\mathbf{u}_i := \mathbf{u}_{i,j} + \mathbf{u}_{i,j'}$.

[Step for computation of $\mathbf{w} := M^T \mathbf{u}$]

4. $\mathbf{w}_{i,j} := M_{i,j}^T \mathbf{u}_i$.
5. Give $\mathbf{w}_{i,j}$ to Node $N_{i',j}$ and receive $\mathbf{w}_{i',j}$ from $N_{i',j}$.
6. $\mathbf{w}_j := \mathbf{w}_{i,j} + \mathbf{w}_{i',j}$.

Lines 4, 5, and 6 describe the computation of $M^T \mathbf{u}$. Note that in each node $N_{i,j}$, by regarding the column of $M_{i,j}$ as the row of $M_{j,i}^T$, we do not have to trade $M_{i,j}$ with $M_{j,i}^T$, namely, we can cut unnecessary operations.

We have discussed the parallel computations among nodes, and now we move on to the parallel computations among cores in one node. Here, A_ℓ denotes the partitioned matrix of $M_{i,j}$ such that $M_{i,j} = \bigoplus_{\ell=1}^L A_\ell$. From Algorithm 2, we can easily obtain $A_\ell \mathbf{v}_j$, and then we set the new vector $\mathbf{u}_{i,j} = (A_1 \mathbf{v}_j, \dots, A_L \mathbf{v}_j)^T$, where L is the number of cores in the same node. Similarly, we can easily obtain $A_\ell^T \mathbf{u}_i$ and compute $\mathbf{w}_{i,j} = \sum_{\ell=1}^L A_\ell^T \mathbf{u}_i$ by using Algorithm 3.

Algorithm 2 (Parallel computation of $M_{i,j}v_j$ among L cores in the same node.)
 Input : the partitioned matrix $A := M_{i,j}$ whose size is $s \times t$ and the partitioned t -vector v_j .

Output : the partitioned vector $u_{i,j}$ such that $u_{i,j} = Av_j$.

1. Compute $b_\ell := A_\ell v_j$ for $\ell = 1$ to $\ell = L$ in parallel.
2. $u_{i,j} = \bigoplus_{\ell=1}^L b_\ell$.

Algorithm 3 (Parallel computation of $M_{i,j}^T u_i$ among L cores in the same node.)
 Input : the partitioned matrix $A := M_{i,j}$ whose size is $s \times t$ and the partitioned s -vector u_i .

Output : the partitioned vector $w_{i,j}$ such that $w_{i,j} = A^T u_i$.

1. Compute $c_\ell := A_\ell^T u_i$ for $\ell = 1$ to $\ell = L$ in parallel.
2. $w_{i,j} = \sum_{\ell=1}^L c_\ell$.

From the parallel computations of $M_{i,j}v_j$ and so on, we obtain the vector $M^T M v$ from Algorithm 1 and 2. Therefore, we need to reconstruct M so that each node has the balanced calculation amount of computing $M_{i,j}v_j$ and so on. It is clear that the calculation amount depends on the number of non-zero elements in the allotted matrix, and the distribution of non-zero elements in M is not uniformity. In fact, the number of non-zero elements in a column of M is not balanced, but that in a row is balanced. Thus, we reconstruct the new matrix M so that the number of non-zero elements in $M_{1,1}$ and $M_{2,1}$ is almost equal to that in $M_{1,2}$ and $M_{2,2}$ by sorting columns of M^* defined in the section of the Galois action. We perform a similar strategy as above for the parallel computation among cores in the same node, namely, A is partitioned into 4 or 8 smaller matrices A_ℓ so that each A_ℓ has almost the same number of non-zero elements.

4.3 Computation Results

In this section, we describe our computation results of the 676-bit DLP in $\text{GF}(3^{6 \cdot 71})$, which contains a multiplicative subgroup whose order is a 112-bit prime. We construct $\text{GF}(3^6)$ as $\text{GF}(3)[z]/(z^6 + 2z + 2)$ and define a mapping $\psi : \mathbb{Z} \rightarrow \text{GF}(3^6)[x]$, such that $\psi^{-1} : z \mapsto 3, x \mapsto 3^6$, in order to represent the element in $\text{GF}(3^6)[x]$.

In the polynomial selection step, we set $H(x, y) = y^6 + x$ in order to use the Galois action. Moreover, we select $m \in \text{GF}(3^6)[x]$ such that all its coefficients are in $\text{GF}(3)$ to construct f whose coefficients are also in $\text{GF}(3)$. By an easy computation, we obtain proper m and f as follows,

$$\begin{aligned}
 m &= \psi (0x456bc\ 60e76c11\ 1e679735\ c929fc55) \\
 f &= \psi (\quad 0x9\ 2d3e5daf\ 5ac01130\ 4e6909f7\ 09cc8833\ baa757d3 \\
 &\quad 17dc6f99\ 9c8b98b5\ ab8baa01\ d68ec151\ aec39e2e\ ed081c79 \\
 &\quad d851066b\ 3ffb2a4f\ a3e19c1e\ cef46675\ 0918a26d\ 9c7cacd4 \\
 &\quad 8d74ccfe\ 2c1d3b79\ e81e6138\ ab06aef4).
 \end{aligned}$$

Then, $\text{GF}(3^{6n})$ is constructed as $\text{GF}(3^6)[x]/(f)$. When we set the smoothness bound $B = 2$, there are 266,085 elements in the rational factorbase and 265,721 elements in the algebraic factorbase, so we need to collect at least 531,806 relations. However, the size of the sieving area when $B = 2$ is too small to collect enough relations.

We settle this problem by using the Galois action, since we can considerably reduce the number of required elements in the factorbase described in Section 4.2. In fact, we need only 88,674 relations, and so this number is about 1/6 the number of the originally required relations.

Moreover, we deal with free relations which are obtained without sieving. If we choose $H(x, y)$ as $y^6 + x$, then it is fortunately factored as $(y - t_1)^3(y - t_2)^3 \pmod{\mathfrak{p}}$ for most of elements \mathfrak{p} in the factorbase, and so there are 132,860 ($\approx \#B_A/2$) free relations. Even if we delete many duplicates which are produced by using the Galois action, 22,155 free relations remain. Thus, we only have to find at least 66,519 relations in the collection of relations step, and this number is about 1/8 that of the originally required relations.

In the collection of relations step, we use the polynomial sieve described in Section 4.1 and compute relations using five nodes, each consisting of Intel Quad-Core Xeon E5440 (2.83 GHz) \times 2 CPUs with 16-GB RAM, one node consisting of Intel Quad-Core Xeon X5355 (2.66 GHz) \times 2 CPUs with 16-GB RAM, and twelve nodes, each consisting of Intel Quad-Core Xeon L5420 (2.33 GHz) \times 1 CPU with 4-GB RAM, total of 96 cores. In 18 days of computation, after removing duplicates, we found 66,646 relations. Thus, we obtained a total of 88,801 relations, which are enough to solve the linear equation in Equation (7).

The linear equation constructed from the relations has to be solved modulo $(3^{6 \cdot 71} - 1)/(3^6 - 1)$; however, the Lanczos method may fail when the modulus has a small prime factor. Therefore, we work modulo the factor N_i of $(3^{6 \cdot 71} - 1)/(3^6 - 1)$,

$$\begin{aligned} N_1 &= (3^{2 \cdot 71} + 3^{71} + 1)/(13 \cdot 5113), \\ N_2 &= (3^{2 \cdot 71} - 3^{71} + 1)/(7 \cdot 210019 \cdot 49682251 \cdot 55126531), \\ N_3 &= (3^{71} + 1)/(2^2 \cdot 853 \cdot 2131 \cdot 82219), \\ N_4 &= (3^{71} - 1)/2. \end{aligned}$$

where every prime factor of N_i is larger than 30 bits and N_i is relatively prime to each other.

We use a cluster with four nodes, each consisting of Intel Quad-Core Xeon E5440 (2.83 GHz) \times 2 CPUs with 16-GB RAM, and three clusters with four nodes, each consisting of Intel Quad-Core Xeon L5420 (2.33 GHz) \times 1 CPU with 4-GB RAM. With about 12 hours computation, we solve the linear equation modulo N_i via the parallel Lanczos method with the four nodes described in Section 4.2 on each cluster. With the Chinese remainder theorem and the Galois action of ϕ , we solved discrete logarithms of the elements in the factorbase modulo $N = \prod_{i=1}^4 N_i$. Some examples of the relation and discrete logarithms of elements in factorbase are given in Appendix.

In the individual logarithm step, our target of computing the logarithm is the element

$$\begin{aligned}\pi(x) &= \psi(\lfloor \pi \times 10^{202} \rfloor) \\ &= (z^4 + z^3 + 2z^2 + 1)x^{70} + \cdots + (z^5 + 2z^4 + 2z^3 + z^2 + 2)\end{aligned}$$

in basis $\gamma = \psi(0x456)$. The complete value of $\pi(x)$, γ and f is written in Appendix. We choose the representation of $\pi(x)$ as a product of elements of degree at most 7 as follows:

$$\gamma^\tau \pi(x) \equiv z_1/z_2 \pmod{f}, \text{ where}$$

$$\begin{aligned}z_1 &= \psi(0x333) \times \psi(0x345) \times \psi(0x427) \times \psi(0x43b) \times \psi(0x4c3) \\ &\quad \times \psi(0xd909\ 66c7e3ec) \times \psi(0x293996d\ cc380672) \\ &\quad \times \psi(0x3ff378e\ 3d4659d0) \times \psi(0x6\ 27d6c281\ 0a0fc5a2) \\ &\quad \times \psi(0x8\ f4797e29\ a9ec3b4a), \\ z_2 &= \psi(0x318) \times \psi(0x45\ 4c6fbfd4) \times \psi(0x54\ c69e6f97) \\ &\quad \times \psi(0x1686d\ 42782189) \times \psi(0x3cf67a5\ 84055cd8) \\ &\quad \times \psi(0x8\ f68ab2e2\ 5d2bc04f) \times \psi(0xb\ cc56922c\ f651b383), \\ \tau &= \quad 0x2\ 0f822e8c\ ac48792a\ e2aea337\ c9002b49\ bbf1b864 \\ &\quad 43a6111b\ 24c5593d\ e44daf43\ e26de26e\ 1f85f982\ 1ba485b3 \\ &\quad beda74bd\ f782626d\ 6cd38bb2\ 8f829867\ 5dc04adc\ f8741c24,\end{aligned}$$

and z_1, z_2 are 7-smooth. Then, we compute the logarithms of z_1 and z_2 in basis γ using the special- q descent technique [16, 18]. With about 14 days computation using five nodes, each consisting of Intel Quad-Core Xeon E5440 (2.83 GHz) \times 2 CPUs with 16-GB RAM, and one node consisting of Intel Quad-Core Xeon X5355 (2.66 GHz) \times 2 CPUs with 16-GB RAM, we compute the logarithms,

$$\begin{aligned}\log_\gamma z_1 &\equiv \quad 0x3\ fc71c577\ 10be8e3f\ e7af0fba\ e00e711f\ 0ad6dd50 \\ &\quad 38fb8f26\ c0fad3b\ 448cab2f\ 67671247\ 285f9e95\ dc501717 \\ &\quad d9def844\ a75f9e58\ f04a9bd2\ 3a5d0fdb\ 8f8ebb9f\ fea4deea, \\ \log_\gamma z_2 &\equiv \quad 0x4\ 82febaec\ ae4382e0\ e651f577\ 09df4e7d\ 99d99d34 \\ &\quad 03db5d5e\ 521c4e2b\ da89ec33\ 6c9d45d6\ 2dd1f982\ 2f198fb2 \\ &\quad 6c069414\ 3b0b1544\ ece8e4b1\ 5304872f\ 6ff261fd\ 03b271c7.\end{aligned}$$

modulo N , and so we obtain $\log_\gamma \pi(x) \pmod{N}$.

The logarithm in multiplicative subgroups of less than 30 bits are computed using the Pollard's ρ method in a minute. Using the Pohlig-Hellman method, we compute the logarithm $\log_\gamma \pi(x)$:

$$\begin{aligned}\log_\gamma \pi(x) &= \quad 0x8\ 78b54797\ 2fb6ff9b\ 57add5d5\ 11f69de6\ a3853f98 \\ &\quad 68d53cc0\ 5b531076\ 2872ac6a\ 320874bf\ ba6d66d6\ 8e5e245f \\ &\quad 39778f02\ 31ae791a\ acbab8c7\ 5ee6850c\ 9f5df0e5\ f6b8ab0b \\ &\quad 95d8bdb1\ aea95b1f\ bad82465\ 25590f66\end{aligned}$$

Table 3. Records for solving the DLP in finite fields

| Finite Fields | $\text{GF}(p)$ | $\text{GF}(2^n)$ | $\text{GF}(p^3)$ | $\text{GF}(p^{30})$ | $\text{GF}(3^{6n})$ |
|-------------------------|------------------------|--|-------------------------------------|-------------------------------------|---|
| Reference | [21] | [15] | [20] | [18] | This Work |
| Date | Feb. 5, 2007 | Sep. 22, 2005 | Aug. 23, 2006 | Nov. 9, 2005 | Dec. 9, 2009 |
| Algorithm | NFS* | JL02-FFS | JLSV06-NFS [†] | JL06-FFS-2 [‡] | JL06-FFS |
| Collection of Relations | Many CPUs [¶] | 4 nodes of 16 Itanium 2 (1.3GHz) | 16 Alpha processors (1.15GHz) | 16 Alpha processors (1.15GHz) | Xeon (2.83GHz) 96 cores in total |
| Linear Algebra | 12–24 Xeon (3.2GHz) | 4 nodes of 16 Itanium 2 (1.3GHz) | 16 Alpha processors (1.15GHz) | 16 Alpha processors (1.15GHz) | Xeon (2.83GHz) 80 cores in total |
| Timing | 33 days | 17 days | 19 days | 12 hours | 33 days |
| Bit Size | 532 | 613 | 394 | 556 | 676 |

*NFS: Number Field Sieve [9, 17]. [†]JLSV06-NFS: NFS in the medium prime case [20].

[‡]See footnote ² on page 2. [¶]There are no detailed descriptions of computational resources in [21].

and completely solve the DLP in $\text{GF}(3^{6 \cdot 71})$ of 676-bit.

4.4 For Larger Extension Degrees

We have solved the DLP in $\text{GF}(3^{6n})$ for n in the experimental class, where the smoothness bound B (i.e., B_{06}) is less than or equal to 2 (ref. Table 1). Note that the size of the sieving area increases $(3^6)^2$ -fold if the smoothness bound B increases by one (see Equation (11)). However, we expect that, if we set $B = 3$, the DLP in $\text{GF}(3^{6 \cdot 97})$ might be computed for several years by using dozens of our computational resources through various techniques such as large prime variation, block sieving and sieving via bucket sort [29, 4], and SIMD implementation.

5 Concluding Remarks

In this study, we implemented a new variant of the FFS in $\text{GF}(3^{6n})$ (n is a prime), proposed by Joux and Lercier in 2006 [18], and compared it with the earlier variant, which was also proposed by Joux and Lercier in 2002 [16] with practical experiments. In solving the DLP in $\text{GF}(3^{6n})$, these two variants of the FFS have the same asymptotic complexity, but we expected the new variant to be more efficient than the earlier one in some extension degrees n . From our experimental results, we confirmed this forecast when the extension degree $n = 19, 61$. Moreover, with our implementations, we succeeded in solving the DLP in $\text{GF}(3^{6 \cdot 71})$ of 676-bit size with about 33 days computation.

We have experimented with the DLP in $\text{GF}(3^{6n})$ required for pairing-based cryptosystems. The security of pairing-based cryptosystems relies on the difficulty of the DLP in various finite fields, for example, $\text{GF}(2^{4n})$ and $\text{GF}(p^{12})$. Table 3 presents the current records for solving the DLP in various finite fields. All the DLPs used for pairing-based cryptosystems have not examined yet. It is

an open problem to analyze the hardness of the DLP with practical key sizes in such finite fields.

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Appendix A: Some Solutions of the DLP in $\text{GF}(3^{6 \cdot 71})$

We present some solutions (discrete logarithms) in factorbase used in our implementation for solving the DLP in $\text{GF}(3^{6 \cdot 71})$. We have found 66,646 relations satisfying Equation (5). We give one of them as an example,

$$\sum_{i=0}^6 \log_{\gamma} \mathbf{p}_i \equiv \sum_{j=0}^4 3 \log_{\gamma} \kappa_j \pmod{(3^{6 \cdot 71} - 1)/(3^6 - 1)}, \quad (13)$$

where each \mathbf{p}_i is in rational factorbase,

$$\begin{aligned} \mathbf{p}_0 &= \psi(0x2d9), \mathbf{p}_1 = \psi(0x90581), \mathbf{p}_2 = \psi(0x9ea2b), \mathbf{p}_3 = \psi(0xb1a07), \\ \mathbf{p}_4 &= \psi(0xb942e), \mathbf{p}_5 = \psi(0xcada1), \mathbf{p}_6 = \psi(0xd6d36), \end{aligned}$$

and each κ_j corresponding to an element in algebraic factorbase by Equation (6) is given as follows,

$$\begin{aligned} \kappa_0 &: \langle \psi(0x3c3), y - \psi(0x175) \rangle, \kappa_1 : \langle \psi(0x3c4), y - \psi(0x200) \rangle, \\ \kappa_2 &: \langle \psi(0x533), y - \psi(0x258) \rangle, \kappa_3 : \langle \psi(0xda9c2), y - \psi(0x4cc58) \rangle, \\ \kappa_4 &: \langle \psi(0xed6e4), y - \psi(0x387b6) \rangle. \end{aligned}$$

By the Galois action, we have

$$\mathbf{p}_6 \equiv \mathbf{p}_3^{3^{3 \cdot 71}} \pmod{f}, \quad \kappa_4 \equiv \kappa_3^{3^{3 \cdot 71}} \pmod{f},$$

and so we can remove \mathbf{p}_6 and κ_4 from unknowns of Equation (7).

Let N be the product of prime factors of $(3^{6 \cdot 71} - 1)$, where those prime factors are not larger than 55, 126, 531 (Note that N is a 602-bit integer). Equation (13) also holds modulo N instead of $(3^{6 \cdot 71} - 1)/(3^6 - 1)$, and so we obtain the following solutions of Equation (13) modulo N except for $\log_{\gamma} \mathbf{p}_6$ and $\log_{\gamma} \kappa_4$, after performing the linear algebra step:

$$\begin{aligned} \log_{\gamma} \mathbf{p}_0 &\equiv && 0x8\ 9e0c0faa\ 4190baa5\ c885e3b7\ 308ae498\ eb2d4a03 \\ &&& 0dfab3d9\ 16437d96\ bfd4e2b9\ 014f5402\ 90aa2f83\ 7b9cc76b \\ &&& 16ae97ef\ dcc9c319\ 670f0f9c\ 47e8ea96\ 4754cfbf\ 1529c311, \\ \log_{\gamma} \mathbf{p}_1 &\equiv && 0x2\ e8b84752\ 70de651a\ b03ae702\ e3268e86\ 77179013 \\ &&& 0c9edab5\ 31d2ac5b\ 2a23da92\ 2e8352c5\ 321832bf\ ff36a8d5 \\ &&& 2d16c9e5\ ae47c6fc\ 2ba7a1c5\ cc990233\ 34c3d6da\ 25e08d52, \\ \log_{\gamma} \mathbf{p}_2 &\equiv && 0x7\ b565cae8\ 39dc8d83\ 415b0b9e\ 164c7b55\ 6e57ad98 \\ &&& 80b8f232\ 7cf30ebe\ 972ac1fb\ 2d1133be\ 5cdd9604\ c9ea6e83 \\ &&& c1c8c9f3\ 2f9fa4c6\ 51d65ded\ 33d2e4c7\ 8ff8d162\ 3a5408c9, \end{aligned}$$

$$\begin{aligned} \log_\gamma p_3 &\equiv && 0x6 ae81aef6 7c0fddcf 7c23e69e c3f18e07 bf546751 \\ &&& 8df9d1ad 78113a85 9a2578c8 36764402 2598160b 5c055ed4 \\ &&& 7d412a42 17c987c0 14aafff7 03ef6fa4 c6771dfd 150b88f2, \\ \log_\gamma p_4 &\equiv && 0x7 2e418546 92ba2b75 8d0831df 1d5ca5c0 f6d8a05d \\ &&& 0528c97d 16c4f782 d9b59ce7 d55deefe bf85390a 23113680 \\ &&& b184d203 d1d3b6a4 e9d9263a 8544acd7 5afc9974 78a4498a, \\ \log_\gamma p_5 &\equiv && 0x1 c35f26bf 717ed338 cfd71243 b86c024b 98b18342 \\ &&& 4710450a d9aaf2e3 557ce5ed debbc870 0fc840f2 19aca778 \\ &&& 2ba931a2 cdd2cb53 a2dafcaa 28a5176e a378bf8c 9a6cd33c, \end{aligned}$$

and,

$$\begin{aligned} \log_\gamma \kappa_0 &\equiv && 0x 92671082 6cf3288f 1c83edcf 66fb9041 9bb2239c \\ &&& 10cd8445 820d975e 6f9730fb f4ca3005 279a500d b2fc0f60 \\ &&& b4425edb 65991a31 629d54e7 84ae64b6 080828b3 0fc6ba0b, \\ \log_\gamma \kappa_1 &\equiv && 0x7 06c2cfc d 7fb4f7c8 386ea65b c0c259c3 f14888ec \\ &&& ccda75ee 77ddddf4 065a7da6 981af728 98699166 c52484c6 \\ &&& 73bbefbd a4660135 1244b297 42f3cf76 fdab7cad 3d01e8a1, \\ \log_\gamma \kappa_2 &\equiv && 0x5 4623bf43 0ede6e43 bbe3cb8b a79c1400 97f7ac1e \\ &&& 2320c70e 5a700159 4460b073 e5c670c5 d19921ea 59f4f9c6 \\ &&& 41ce8203 28edb204 94bd322f 3551d5ee 472cf59b d58d0bd0, \\ \log_\gamma \kappa_3 &\equiv && 0x6 e063f01c 43624c96 30712701 2223edf3 95ddfdc2 \\ &&& aa1dd9f6 dd3636ef 12d9260f 555a2101 c0e94fe5 9a524c5b \\ &&& c2c1d768 1499d7b6 41b71d4f b13566b3 b39794c5 90ff78cb, \end{aligned}$$

mod N . Finally, by the Galois action, we obtain that

$$\begin{aligned} \log_\gamma p_6 &\equiv 3^{3 \cdot 71} \log_\gamma p_3 \equiv \\ &&& 0x7 3deb8075 ee684576 073761e2 974c4eba 72df97ce \\ &&& 299f9e46 87ae3f70 b6cd8b50 1c65ccb3 e9ed8f80 08387efe \\ &&& 9326eea8 7302c1a5 1f0671b5 22e32949 81250923 9b072989, \\ \log_\gamma \kappa_4 &\equiv 3^{3 \cdot 71} \log_\gamma \kappa_3 \equiv \\ &&& 0x4 0473a949 4056ac7c 76677e6f a284977a 2a2e539f \\ &&& 751d5e0b ee628ca8 63e7f732 a02886c2 0711d445 0006c79a \\ &&& 778c6fbf abb923e7 e89deb8d 0c7f5508 2d797bd2 2414eaa1, \end{aligned}$$

mod N .

Appendix B: Complete Value of Elements

Table 4 presents the complete value of a target element $\pi(x)$, a generator γ , and an irreducible polynomial f used in Section 4.3.

Table 4. Complete Value of $\pi(x)$, γ and f

| deg. of term | $\pi(x)$ | γ | f |
|--------------|------------------------------------|------------------------|-----|
| 71 | | | 1 |
| 70 | $z^4 + z^3 + 2z^2 + 1$ | | 0 |
| 69 | $2z^5 + z^3 + z + 1$ | | 0 |
| 68 | $z^4 + z^3 + 2z + 1$ | | 0 |
| 67 | $2z^4 + 2z^3$ | | 0 |
| 66 | $2z^3 + 2$ | | 0 |
| 65 | $z^4 + 2z^3 + z^2 + 2$ | | 0 |
| 64 | $z^4 + z^3 + z^2 + 2z + 1$ | | 0 |
| 63 | z^3 | | 0 |
| 62 | $z^4 + 2z^2 + z$ | | 0 |
| 61 | $2z^3 + z^2 + 1$ | | 0 |
| 60 | $2z^5 + 2z^4 + 2z^3 + z^2 + z + 1$ | | 0 |
| 59 | $z^5 + 2z^3 + z^2 + z$ | | 0 |
| 58 | $2z^3 + 2z^2 + z$ | | 0 |
| 57 | $2z^5 + 2z^3 + 2z^2 + z + 2$ | | 0 |
| 56 | $z^4 + 2z^3 + z^2 + 1$ | | 0 |
| 55 | $z^5 + z^4 + z^3 + z^2$ | | 0 |
| 54 | $2z^4 + z + 2$ | | 0 |
| 53 | $z^3 + 2z^2 + 2z + 2$ | | 0 |
| 52 | $2z^5 + 2z^4 + 2z^2$ | | 0 |
| 51 | $2z^5 + z^4 + z^3 + 2z^2 + 2$ | | 0 |
| 50 | $2z^5 + z^4 + z^2 + 2z$ | | 0 |
| 49 | $2z^5 + 2z^3 + z^2 + z$ | | 0 |
| 48 | $2z^4 + z^3 + z^2 + 2z + 1$ | | 0 |
| 47 | $z^3 + z^2 + 2z + 2$ | | 0 |
| 46 | $2z^5 + z^4 + z^3 + z^2 + 2z + 1$ | | 0 |
| 45 | $2z^5 + z^4 + z^2 + 2$ | | 0 |
| 44 | $2z^5 + 2z^4 + 2z^3 + z + 2$ | | 0 |
| 43 | $2z^4 + 2z^3 + 2z$ | | 0 |
| 42 | $z^5 + 2z^4 + z^2 + z + 2$ | | 0 |
| 41 | $z^5 + 2z^3 + 2z^2 + z + 1$ | | 0 |
| 40 | $z^5 + z^4 + 2z^2 + z + 2$ | | 0 |
| 39 | $z^5 + z^4 + 2z^2 + 1$ | | 0 |
| 38 | $2z^4 + 2z^3 + 2z + 1$ | | 0 |
| 37 | $z^5 + 2z^4 + z^2 + 2z + 1$ | | 0 |
| 36 | $z^5 + z^4 + z^3 + 2z + 2$ | | 0 |
| 35 | 1 | | 0 |
| 34 | $z^4 + 2z^3 + z^2 + 2z + 1$ | | 0 |
| 33 | $2z^4 + 2z^3 + z + 2$ | | 0 |
| 32 | $2z^4 + 2z^2 + 1$ | | 0 |
| 31 | $2z^5 + z^4 + z^3 + z$ | | 0 |
| 30 | $z^4 + z^3 + 2z^2$ | | 0 |
| 29 | $2z^5 + z^4 + 2z^3 + z^2 + 2z + 2$ | | 0 |
| 28 | $2z^5 + z^3 + z^2 + 2z + 2$ | | 0 |
| 27 | $z^4 + 2z^2 + z + 1$ | | 0 |
| 26 | $2z^5 + 2z^3 + z + 1$ | | 0 |
| 25 | $z^3 + z^2 + 1$ | | 0 |
| 24 | $2z^5 + z^4 + 2z^3 + z^2 + 2z + 1$ | | 0 |
| 23 | $2z^4 + z^3 + 2z^2 + z$ | | 0 |
| 22 | $2z^5 + z^3 + 2z^2 + z$ | | 0 |
| 21 | $2z^4 + 2z^3 + z^2 + 2z + 1$ | | 0 |
| 20 | $z^5 + 2z^4 + 2z^3 + 2z^2$ | | 0 |
| 19 | $z^5 + z^4 + 2$ | | 0 |
| 18 | $2z^5 + 2z^4 + 2z^3 + 2z^2 + 2z$ | | 0 |
| 17 | $z^4 + 2z^3 + 2z^2 + 2z + 1$ | | 0 |
| 16 | $z^5 + 2z^3 + 2$ | | 0 |
| 15 | $z^5 + 2z^4 + 2$ | | 0 |
| 14 | $2z^5 + z^3 + 2z^2 + z$ | | 0 |
| 13 | $2z^5 + 2z^4 + z^3 + 2$ | | 0 |
| 12 | $z^4 + z^3 + 2z + 2$ | | 0 |
| 11 | $z^2 + 2$ | | 0 |
| 10 | $z^3 + z + 2$ | | 0 |
| 9 | $2z^4 + z^3 + 2z^2 + 2$ | | 0 |
| 8 | $2z^3 + 2z^3 + z + 1$ | | 0 |
| 7 | $2z^5 + 2z^4 + z + 2$ | | 0 |
| 6 | $2z^5 + z^4 + z^3 + z^2$ | | 0 |
| 5 | $2z^5 + z^4 + 2z^3 + 2$ | | 0 |
| 4 | $2z^4 + z^3 + z^2 + 2z + 1$ | | 0 |
| 3 | $z^5 + 2z + 2$ | | 0 |
| 2 | $z^5 + 2z^4 + 2z^3 + 2z^2 + z + 2$ | | 0 |
| 1 | $z^5 + 2z^4 + z^2 + 2z + 1$ | | 1 |
| 0 | $z^5 + 2z^4 + 2z^3 + z^2 + 2$ | $z^5 + z^4 + 2z^3 + z$ | 1 |